Introduction to Quantum Field Theory - II (Theory of Scalar Fields)
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Module - 5
Lecture - 10
S Matrix

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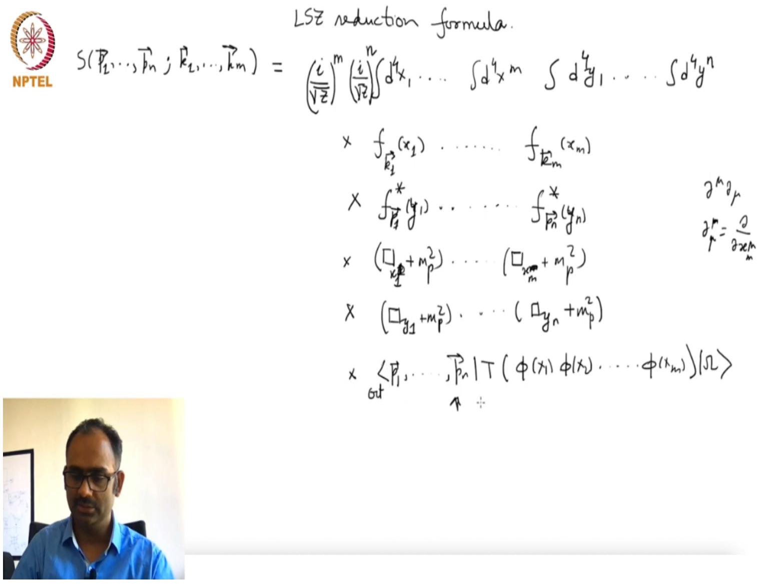

So, let us recall what we have been doing so far. Our eventual aim is to describe scattering of particles and tell what happens in future after these particles have interacted, and towards that goal we have been trying to figure out S matrix elements which are basically the inner products of in and out states. And we had made significant progress in that direction and found that we can write S matrix element which we write as S of this p 1 to p n and k 1 to k m.

So, in general, m and n will be different. We showed that this is equal to i over square root of z and you have m such factors, and then you have integral d 4 x 1 to d 4 x m, then f of k 1 x 1 times f of k 2 x 2 and so forth to f of k m x m times box 1 plus m p square. This is basically del mu del mu where the derivative is with respect to x 1, and then other factors box m plus m p square times p 1 to p n that is the out state, then time ordered product of these fields phi at x 1, phi at x 2, so and so forth phi at x m and here you have the vacuum.

We could write our expression in terms of fields phi and could completely get rid of the in state and instead we have now vacuum. Now, if you were to repeat similar steps for the out state, this one here, then you are going to get the following, and that is something you should check. So, you are going to get in addition to this factor i over square root of z to the m, you are going to get n such factors also. So, you will get this.

Then you are going to get f of p 1 x 1, but this time it will be complex conjugate; p n x n and the complex conjugation here, times you will have these differential operators also for, just like you have box 1 plus m p square, let me put a label x here because now we have both; I made a small mistake. So, here you will have corresponding integration variables which we will write as d 4 y 1, d 4 y n and these will have arguments y 1 to y n.

And these derivatives which are with respect to x 1 and x 2 and so forth, I will explicitly write them as box with subscript x 1 and this one is box with subscript x m meaning this differential operator del mu del mu, this del mu is basically del. So, this is this, that is the box. So, here it is the lower del x m mu. This is the derivative operator here. So, that is what we have in here.

And when we are getting rid of this out state, we will have just like these differential operators. We will have box y 1 plus m p square. So, now, this box which is again del mu del mu, the derivative is with respect to y 1 mu and we have several of them up to box y n plus m p square; and of course, this will get changed to; so, this I will erase and let us see whether I can erase.

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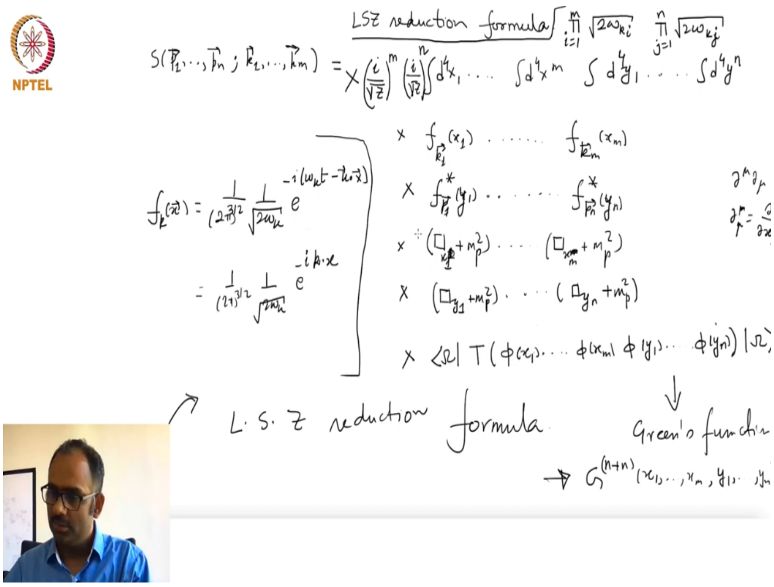

And then, what will you have? We had time ordered product of fields where the number of fields was equal to the number of labels k 1, the number of labels we had in the in state, which was m labels, but now we are going to get in addition to those, n more fields coming from the out state and we will have this time ordered product of phi x 1 up to phi x m that we already had, and now we will also have phi of y 1 to phi of y n and we will have this vacuum expectation value of this time ordered product.

So, this is our LSZ reduction formula; and we have missed these factors of; here we had square root of 2 omega k 1 for x 1 and similarly for other variables. So, I should multiply this somewhere; I should add it here, 2 omega k i product i equal to 1 to m times j equal to 1 to n 2 omega k j. So, this also should be multiplied, so, I will put a cross here so that equation starts from this place. I am sorry that this is not looking so nice, but this is what it is.

So, that is our LSZ reduction formula. Now, you see that these objects, this vacuum expectation value of this time order product, that is what is a Green function is, right. We have talked in detail about this and perturbative expansion of this object in the previous course, part 1 of this lecture on quantum field theory. And you may recall that we can also write this using momentum space variables.

And we saw perturbation expansion of this, how to draw Feynman diagrams; this is what led us to Feynman diagrams and you should go back to the previous course and see what we learnt about all this; but I am not going to go back to that discussion and I will assume that you already know what these objects are and I am going to proceed with that. So, let me just write down here, these are our correlation functions or Green's functions.

This is what we used to call; this one will be G n + m x 1 to x m, then y 1 to y n; that was the notation which we used last time, but standard notation actually. So, now my plan is to further manipulate this and put it in a nice form. And before I do that, I will also just remind you what f of k is. This was 1 over 2 pi 3 over 2, and then you have 1 over 2 omega k in the square root. That was the definition.

So, you see here that you have; let us look at x variables; let us look at x 1. You have d 4 x 1 that is the measure. Then you have f k of x 1. Now, in f k of x 1, if you look at the expression here f k of x, the dependence on x is fairly simple or this is just e to the -i k dot x. This thing is also, you can write this as -i k dot x which is k mu x mu and zeroth component is omega, omega k, that is the energy.

So, as far as x dependency is concerned in f k, that is simple; that is an exponential. Integrating d 4, integrating this exponential over this variable x 1, that is simple; that will give you a delta function. The only problem is that you have this differential operator acting on this Green's function which is also a function of x 1 but what I am going to do is turn this into a simple expression so that I can integrate easily over x 1 and x 2 and so forth.

The idea is that instead of working with this Green's function in the coordinate space, I will go to Fourier space. So, I will Fourier transform this function with respect to these variables; and once I Fourier transform, the x dependence will become simple; it will be a simple exponential function. Then, taking this derivative operator and acting on a simple exponential function is easy, it will give you back that exponential function times some constants; and then I can just integrate out over x 1 or any of the x i's or y i's.

So, that is the plan. And and the reason being simple, because, if I go to Fourier space, the x dependence trivially factors out as an exponential and that is why I am going to go to Fourier space. So, let us see what happens when we do that. So, I will define the Fourier transform.

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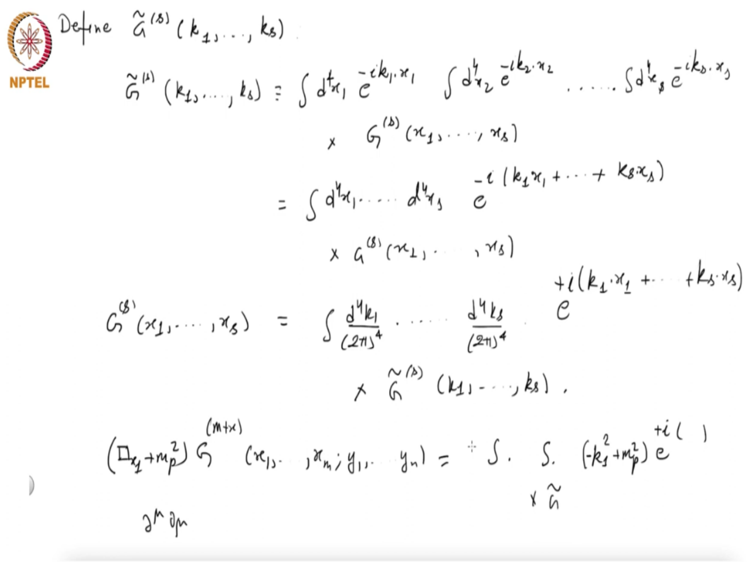

So, let us define G tilde. So, let us say I have a function G tilde which depends on; let me first write, then I will say; it will be easier. So, I am defining the Fourier transform of G tilde of x 1, x 2 so and so forth up to x s. So, G tilde of s k s is defined to be d 4 x 1 e to the -i k 1 dot x 1 integral d 4 x 2 e to the -i k 2 dot x 2 so and so forth, integral d 4 x s e to the -i k s dot x s, and this Fourier transform is being done on G s x 1 to x s.

So, given a correlation function or a Green's function which is a function of these x 1 to x s coordinates, I do a Fourier transform of each of these; I mean, I do a Fourier transform or multi-dimensional Fourier transform and the conjugate variables corresponding to x 1 to x s are k 1 to k s, and that is the general Fourier transform that you will write which is same as d 4 x 1 to d 4 x s times e to the -i k 1 x 1 plus so and so forth k x dot x s k 1 dot x acting on G s x 1 to x s.

Now, you see, this is what I was saying. If I now write G s x 1 to x s in terms of G tilde where it depends on these conjugate variables k 1 to k s, then the x dependence will be simple. So, I invert this and write G s. I am just writing the inverse Fourier transform now. This is d 4 k 1 over 2 pi to the 4 so and so forth d 4 k s over 2 pi to the 4, then you have e to the +i k 1 dot x 1 so and so forth k s dot x s times G tilde s k 1 to k s.

So, you see now, if I go back and substitute in here; this is G; substitute in this expression the Fourier transform or not the Fourier transform but this expression where G tilde is the Fourier transform of G s, then the x dependence is simple; it is of the exponential form. So, that is what I want to do. Another thing is that; just a second; I think I have written here itself. And another thing is that these differential operators act simply on the exponentials.

So, for this thing, let us call this as G n + m and these variables. And I do the similar thing, so, I will have all these exponentials, both with x, x and y, and then I take derivatives. So, let us look at first this one, box x 1 plus m p square acting on the x dependence coming from here. So, if you do so; let us look at box x 1 plus m p square acting on G m + n and then you have x m and y 1 to y n. This will give you what?

This will give you back; all these integrals are still there. I will not write them in full, but you have all these integrals. Then this exponential will give you k 1 minus k 1 square plus m p square times again the same exponential times G tilde. Why? Because the box x 1 is basically del mu del mu with respect to x 1 variable. So, when this x del mu x on e to the i k 1 dot x 1, it pulls down i k 1 mu.

Then you again act with another del mu. It again pulls out i k 1 mu; one mu will be up, one mu will be down, and i times i gives you -1, and k mu k mu is k square. So, you get -k square plus m p square when you take this, the differential operator and act on G. So, if you do this same thing for each of the coordinates x, all of these x coordinates and all of the y coordinates, then you will get the following result. So, instead of writing that, let me just collect all the terms.

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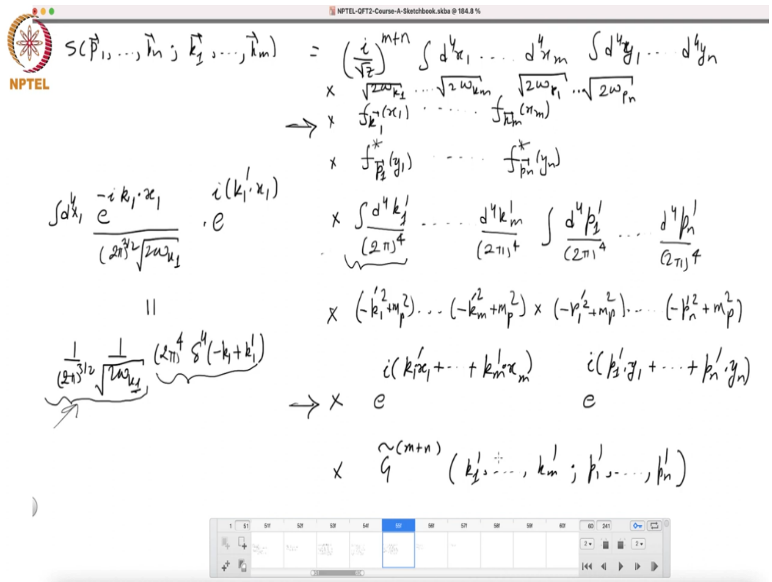

So, what we will do is I will write one more time this thing. Let us write it again. It will be easier to see what is coming from where that way. So, S of p 1 to p n, k 1 to k m is equal to i over square root of z m + n, then d 4 x 1, d 4 x m, d 4 x y 1, d 4 y n. Then you have the f k's; f k m x m times f p 1 y 1 to f p n y n, then all these differential operators. I will write it in the following manner.

So, the variables which I will use is when I am transforming this G, so, corresponding to x 1, I will write k 1 prime; and for x m, I will write k m prime; and for y 1, I will write p 1 prime; I am saying about the conjugate variables; for y n, I will write p n prime. So, you will get the following. This times d 4 k 1 prime over 2 pi to the 4 d 4 k m prime over 2 pi to the 4. This is coming from here.

I am substituting G in this expression, this one, this G in terms of Fourier transform, and this brings this integration measures. I am just putting a label prime on this. That is all I am doing. And then similarly for the y, y 1 to y m, I will have d 4 p 1 prime d 4 p n prime over 2 pi to the 4 and then these exponentials. So, you will have; let me write here; e to the i summation over i k i prime dot x i, or if you wish, let me write it like this, k 1 prime x 1 plus so and so forth k m prime x m, and then you have e to the i p 1 prime dot y 1 p n prime dot y n.

And then finally, G tilde that is the Fourier transform, and the arguments are k 1 prime to k m prime p 1 prime to p n prime. Did we miss something? Yes, we did and that is these differential operators. I have written everything else except for these differential operators. Now, when these differential operators act on this G, they pull out these factors, such factors, minus k 1 square plus m p square, minus k 2 square plus m p square.

So, that is what is left, and I should write here minus k 1 square plus m p square, minus k m square plus m p square times minus; and that is prime, k 1 prime, k m prime; minus p 1 prime square plus m p square so and so forth minus p n prime square plus m p square; and again I have forgotten to write down 2 omega k 1, 2 omega k m, square root of 2 omega k m, then square root of 2 omega p 1 p n.

So, that is the result when you express things in terms of G tilde. Now let us settle x integral and then y integral of course, and that is easy. Remember, let us see where all the x dependence is. So, x dependence is in here, then in here. These are the only 2 places where you have x dependence. So, let us write down what you get for d 4 x 1. So, you will get integral d 4 x 1, then you get e to the; what was it? e to the -i k dot x. So, f k x 1 is e to the -i k 1 dot x 1; just a second, is this correct, f k x 1 f k x? Let us check.

I want to see what is expression of f k of x 1. Anyhow, I will check the sign and then I will tell you in the next video, but let me tell you what I was about to do or what I am going to do. There is some discrepancy in signs compared to my notes, but let us proceed anyway. So, this will have 1 over 2 pi 3 halves and 2 omega k 1 that is f k 1 x 1, and then you have another x dependence coming from here, times e to the i k 1 prime dot x 1.

That is all for x 1. Now, this one you can integrate out and it will give you 1 over 2 pi 3 halves 1 over 2 omega k 1. And this exponential, this will give you, upon integrating, it will give you 2 pi to the 4 delta 4 of minus k 1 plus k 1 prime. That is what you will get. And pen in this issue of sign which I am facing, you are going to get this; that is the delta function you are going to get.

And then similarly, you will get for each of the x variables and also for the y variables. And once you have gotten a delta function integrating over k 1 prime and k 2 prime and p 1 prime and p 2 prime, that will be easy. So, that will be the next step; but before I do that, I want to check my sign and then we will continue in the next video. So, I checked and I do not find any problem with signs actually. So, this is fine, let us proceed.

So, I was doing the x integral. So, I have collected all the factors which contain x 1 and the result is that you have an integral over d 4 x 1 e to the -i k 1 dot x 1 over this factor coming from f k 1. This is what I wrote. And then you have this exponential coming from here, e to the i k 1 prime dot x 1. So, you will get these factors which are in the denominator, and when you integrate over x 1, you generate a delta function with a factor of 2 pi to the 4.

So, that is 2 pi to the 4 delta 4 and k 1 prime minus k 1 which means that this delta function hits only when k 1 prime is equal to k 1. So, this dummy variable k 1 prime, when it equals the variable k 1 which is what is present in this S matrix element, then it contributes, otherwise its contribution is 0. And as you can see that you will get similar factors for each of the x 1.

So, for example, for x 2, I will have exactly the same thing with omega k 1 replaced by omega k 2, and this will have delta 4 minus k 2 plus k 2 prime. So, that is one thing. So, we have taken care of integrals over x 1. So, I will write down on the next sheet.

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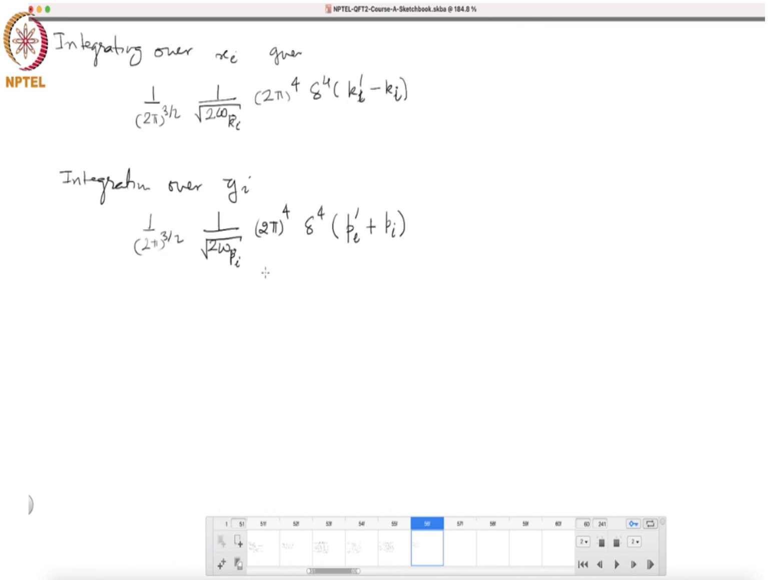

So, each of the or rather integrating over x i gives 1 over 2 pi 3 halves times 1 over 2 omega k i in the square roots and then you have this factor 2 pi to 4 delta 4; I will write it as k 1 prime minus k 1. These are 4 vectors. Now let us look at y integrals, integration over y i. Let us go back. So, here you have; there is again a small mistake here. So, I should show you that we had the p 1's; f p 1, they had a complex conjugation, which is what is missing here.

So, I will put this complex conjugation here. Now, because of complex conjugation, f p 1 would have instead of e to the -i p 1 dot x y 1, it will have e to the i p 1 dot y 1. That is the difference with respect to the previous case, for the case of x 1. This sign, instead of being minus will be a plus; other factors are going to be the same. And then you have e to the i p 1 prime dot y 1. So, the sign of this factor in the case of y 1 does not change.

So, what will we get is, there will be a change of sign here. So, instead of minus p 1 here, we will have p 1. So, for y 1, I will get 1 over 2 pi 3 halves 1 over 2 omega p 1 in the square root and then 2 pi to the 4 delta 4 minus p 1 plus p 1 prime. That is what you are going to get. So, you will get 1 over 2 pi 3 halves and then; I have changed this nib of this pen and I think I should make it bolder brush. Let us hope this is better.

Then 2 pi to the 4 delta 4 and then we have; sorry, here it should have been i p i prime plus p i which means when p i prime is equal to minus p i then this delta function hits; and this delta function hits when k i prime is equal to k i. So, now we can substitute all this into our formula for the S matrix and obtain the following. So, what happens, we will have these factors i over square root of z m + n, then you have all these root 2 omega k i, but then that root 2 omega k i, that will get cancelled.

See, each of these f k i, they are bringing a factor of 1 over root 2 omega k i. So, that gets cancelled with this root 2 omega k i because of the normalisations that we chose which brought these factors. So, each of these root 2 omega k i's are going to disappear. There will be no square root of 2 omega k i in this. Then all the x and y integrals are gone; these f's are gone; integral over d 4 k 1, they will be left; and then of course these objects are there. So, let me write down and then let us check, maybe on the next sheet.

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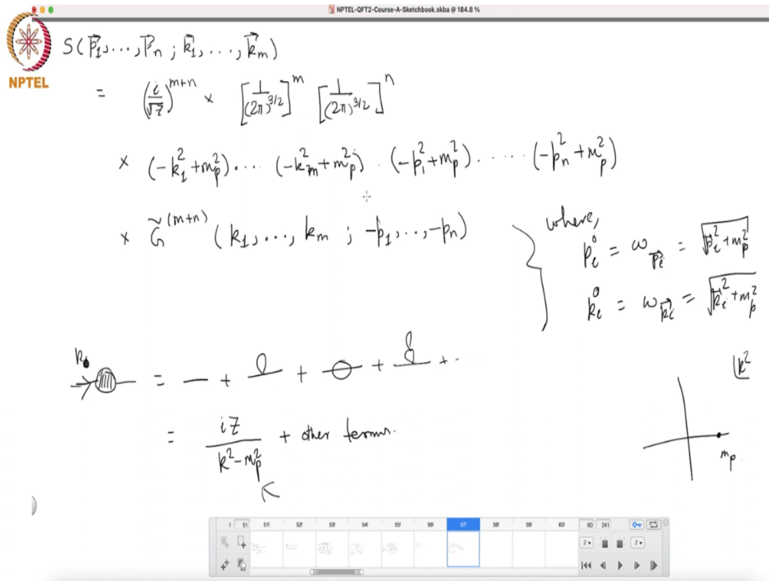

So, I found that S p 1 to p n and then k 1 to k m that is equal to square root of z m + n. Then we will have each of these x integrals brings 1 over 2 pi 3 halves. So, we will have m such factors coming from x i integrals and n such factors coming from p i integrals. So, let me keep that. So, I have taken care of these 2 pi 3 halves and also these square root of 2 omega k i's or rather from here; I should look in here; these have been taken care of.

Now I will have 2 pi to the 4 times these delta functions coming and I should take this and; see, these pi to the 4; let me go slow. So, here, these factors have been taken care of, and let us look at this. I will integrate over this over d 4 k 1 prime. So, look at this. This has d 4 k 1 prime over 2 pi to the 4. So, that 2 pi to the 4 cancels that this one and you are left with only this delta function. So, you have d 4 k 1 prime times the delta function and then the k 1 prime appears here in this factor minus k 1 prime square plus m p square and then the other place is here, G tilde m + n, here, k 1 prime.

So, what I should do is, all the 2 pi factors are gone; all I have to do is turn k 1 prime into k 1 because that is what this delta function does. So, here I will have minus k 1 square plus m p square instead of minus k 1 prime square plus m p square. Here it will be, this is anyway already taken care of; this will become k 1. And when I do for p 1, I should again the same thing, all these 2 pi to the 4 factors are gone, but this time the delta function force is p 1 prime to be equal to minus p 1.

So, what will happen is, here, instead of p 1 prime, I will have minus p 1. So, these first m labels get changed to k 1, k 2 so and so forth k m, and these n labels get changed to minus p 1, minus p 2 so and so forth to minus p n. So, that is the replacement that we need to make. So, I am going to write this result. This is fine. Then I have product of; did I write like that? No. So, I will write it as; let us go back. What was that? Minus k 1 square plus m p square.

So, minus k 1 square plus m p square times other terms and the last term is, last factor is k m square plus m p square. Then I have minus p 1 prime square plus m p square, but p 1 prime I should replace by minus p 1, but when I square it, of course the sign change does not matter. So, I get minus p 1 square plus m p square so and so forth and the last factor is minus p n square plus m p square.

Then what is left is G tilde and here you will have k 1 up to k m and then minus p 1 up to minus p n where remember that these energies p 1 p i 0 is same as; p i, the zeroth component of p is just the energy which is; and similarly for k i 0. That is the constraint you have on these, on this momenta, 4 momenta. So, this is all good. Not here; this I should not put; it continues in the next line. So, now let us look at these factors.

So, if it looks like when these momenta p 1 to p n and k 1 to k m, they are on shell; on shell meaning they satisfy; let us say for k 1, k 1 square is m p square that is the condition of being on shell; then this factor is 0. Similarly, all these factors will be 0 when the corresponding momenta are on shell. So, looks like that the S matrix is 0 for that which of course, it cannot be true that there is no transition probability; the transition probability from in state with all these momenta to an out state with all the p i's is 0.

That cannot happen because of course, you know that scattering will happen. So, clearly, these zeros, they should be killed by some poles coming from here. That is the only way you can save this, otherwise it will not work, otherwise it will give you 0. So, the way it works is the following. I will prove it in detail later, but I will tell you what happens. If you look at a two-point function in an interacting theory, let us say there is momentum k 1, then this two-point function which of course has all these diagrams in there which we have done previously with; let me draw a few.

So, for example, this is this plus this. I am just drawing in 5, 4 theory. This and many other diagrams, infinite of them. These are all diagrams which have only 2 external points, this and this. And when you look at this function, when you look at this two-point Green's function, then its behaviour is given by this. Let us call it k, not k 1, plus other terms that does not bother us right now.

So, thing is that, see, if k square is very close to the physical mass square, meaning if the momentum is close to being on shell, then the claim is that the leading behaviour of this two-point function is given by this term; other terms are not that similar, meaning you have a pole in the two-point function and pole is at the physical mass. So, if you are looking at this object in the k square plane, so, if you are looking at the complex plane of k square, then there is a pole at the physical mass.

So, whatever that physical mass of particles is, there is a pole here. So, this is the dominant behaviour and this is what is going to save us that actually these poles will get cancelled against these zeros which you see here. And I will show you in the next video how that happens; but for now, this I will assume and later in the video after once we have done this S matrix formula completely, I will show you how to obtain this result explicitly. So, see you in the next video.