Introduction to Quantum Field Theory

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Lecture 9 : Quantization of Klein-Gordon Theory continued (2)

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1 Free Real Scalar Field Theory

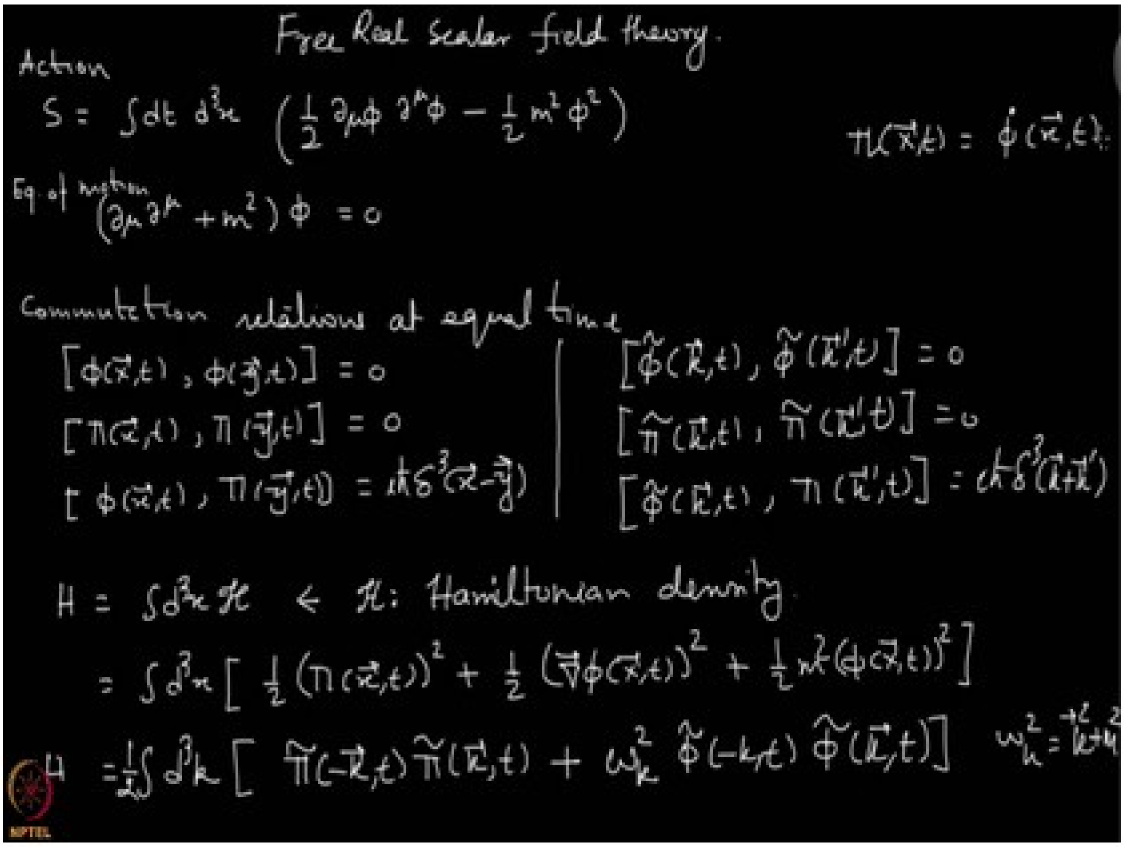


Figure 1: Refer Slide Time: 00:15

Let us continue our discussion of free real scalar field theory. So, I have summarized some of the results that we already have. So, here is the action and then this is our equation of motion which you have seen several times and then the commutation relations are given here. So, if phi commutes with phi, pi commutes with pi and the commutator of phi and pi is given by h bar delta cube x - y and these computation relations are at the same time, equal time.

We also introduce the Fourier variables last time for the Fourier transforms of the fields phi and pi and we also saw that they satisfy the following commutation relations. This commutator vanishes similarly the 2 pi's if we have a commutator of 2 pi's they will also that will also vanish and you get a non-vanishing contribution when you have a commutator of phi with pi and remember you get ih bar delta q k + k prime.

The action,

$$S = \int dt\, d^3x \left[\frac{1}{2}\partial_\mu\phi(\vec{x},t)\partial^\mu\phi(\vec{x},t) - \frac{1}{2}m^2\phi(\vec{x},t)\right] \tag{1}$$

The equation of motion

$$(\partial_\mu\partial^\mu + m^2)\phi(\vec{x},t) = 0 \tag{2}$$

Commutation relations at equal time

$$\begin{aligned} [\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)] &= 0 \\ [\hat{\pi}(\vec{x},t), \hat{\Pi}(\vec{y},t)] &= 0 \\ [\hat{\phi}(\vec{x},t), \hat{\Pi}(\vec{y},t)] &= \imath\hbar\delta^3(\vec{x}-\vec{y}) \end{aligned} \tag{3}$$

Other commutation relations

$$\begin{aligned} [\tilde{\phi}(\vec{k},t), \tilde{\phi}(\vec{k}',t)] &= 0 \\ [\tilde{\Pi}(\vec{k},t), \tilde{\Pi}(\vec{k}',t)] &= 0 \\ [\tilde{\phi}(\vec{k},t), \tilde{\Pi}(\vec{k}',t)] &= \imath\hbar\delta^3(\vec{k}+\vec{k}') \end{aligned} \tag{4}$$

Then we also wrote down the Hamiltonian density where the Hamiltonian let me write as d cube x is curly h where curly h is called Hamiltonian density and that we found to be the following half pi x t squared plus half read into phi x t squared plus half m squared this whole square. We also wrote down I believe check this expression we also wrote down the Hamiltonian in the phi tilde and pi tilde. So, let me the write that one also.

And you have of course the overall factor half tilde but you have a -kt times phi tilde of kt and then here we had a term which was omega k square phi tilde -kt and phi tilde kt where omega k square is k squared + m square also remember that we had, right here the momentum density is phi dot that is good.

$$H = \int d^3x \mathcal{H}$$

Where $\mathcal{H}$ is the Hamiltonian density

$$H = \int d^3x \left[\frac{1}{2}(\Pi(\vec{x},t))^2 + \frac{1}{2}(\vec{\nabla}\phi(\vec{x},t))^2 + \frac{1}{2}m^2(\phi(\vec{x},t))^2\right] \tag{5}$$

$$H = \int d^3k \left[\frac{1}{2}\tilde{\Pi}(-\vec{k},t)\tilde{\Pi}(\vec{k},t) + \omega_k^2\tilde{\phi}(-\vec{k},t)\tilde{\phi}(\vec{k},t)\right] \tag{6}$$

Where

$$\omega_k^2 = \vec{k}^2 + m^2 \tag{7}$$

Now we are taking phi to be real and that should have a bearing on the the phi tilde. So, let us find that out. So, phi is real. So, if I look at the phi tilde kt which is just d cube x e to the -ik dot x phi of xt. So, let me take the complex conjugate of both the sides. So, phi is right now being treated as as a classical field not as an operator at the moment. So, if I take complex conjugate on both sides.

So, I get phi tilde star kt d is I forgot 2 pi over 3 by 2 here this when you take complex conjugation will give you a +ik dot x and because phi is real nothing happens to it, it remains phi. So, you have e to the ik dot x phi. Now x is dummy, yes, so I can see x is being integrated over m and remember when I am writing d cube x what I really mean is this it is a shorthand notation but the full thing is this.

The lower limit is minus infinity and the upper limit is plus infinity. So, anyway let me change from x to -x sorry that is not what I want to do I am sorry. So, what I want to do is I want to change k to -k. So, I get phi tilde phi tilde star -kt is; so, you get a minus sign here which makes the phi tilde k. So, you get phi tilde of kt or phi tilde star I am just changing -k to k here and that will make phi tilde of -kt and if you are going to quantize then you know that star is going to be replaced by a dagger.

So, complex conjugation becomes Hermitian conjugation and then you will have phi tilde dagger kt is equal to phi tilde -kt. So, that is the condition you get because phi zero. So, the phi dagger of k is related to phi of -k and similarly if you the same thing if instead of phi you had a pi then you will get the same relation that is good.

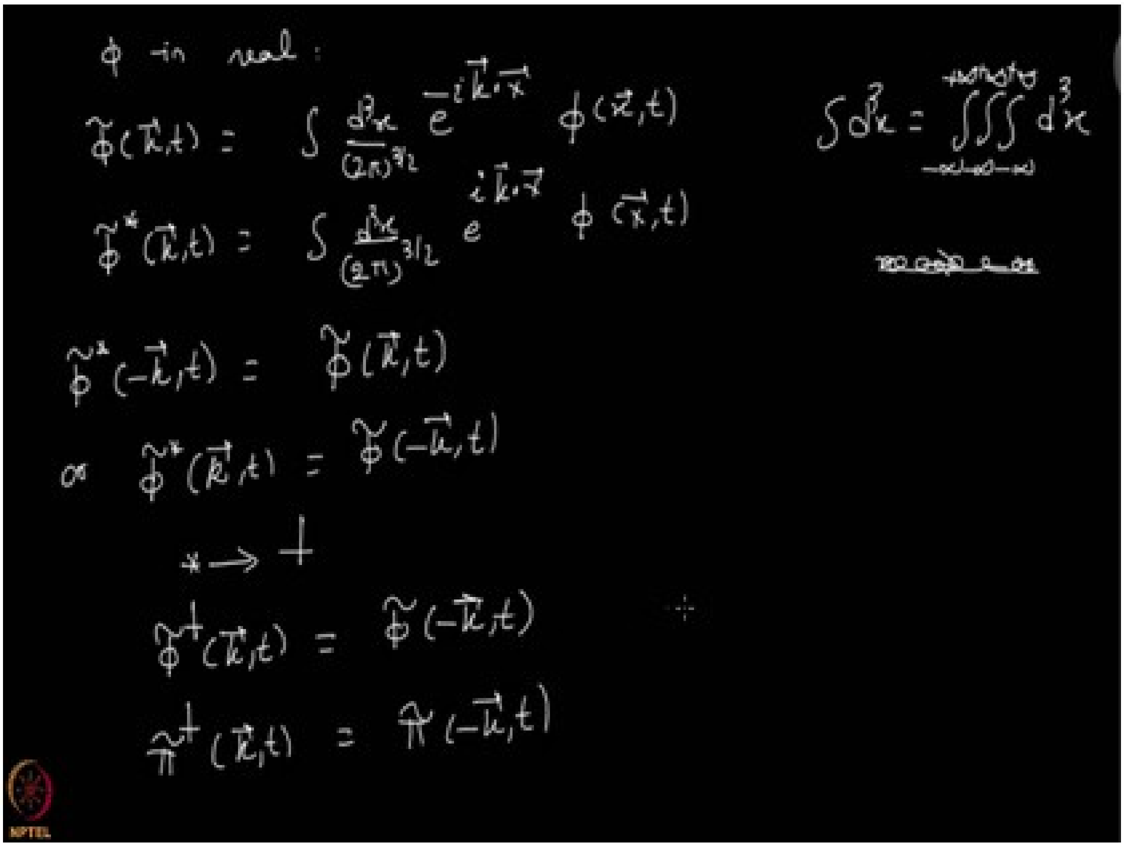


Figure 2: Refer Slide Time: 04:01

$$\tilde{\phi}(\vec{k},t) = \int \frac{d^3x}{(2\pi)^{3/2}}\, e^{-\iota \vec{k}\cdot\vec{x}}\phi(\vec{x},t) \tag{8}$$

$$\tilde{\phi}^*(\vec{k},t) = \int \frac{d^3x}{(2\pi)^{3/2}}\, e^{\iota \vec{k}\cdot\vec{x}}\phi(\vec{x},t) \tag{9}$$

Where

$$\int d^3x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x$$

$$\begin{aligned} \tilde{\phi}^*(-\vec{k},t) &= \tilde{\phi}(\vec{k},t) \\ \tilde{\phi}^*(\vec{k},t) &= \tilde{\phi}(-\vec{k},t) \end{aligned} \tag{10}$$

Such that

$$* \to \dagger$$

$$\begin{aligned} \tilde{\phi}^\dagger(\vec{k},t) &= \tilde{\phi}(-\vec{k},t) \\ \tilde{\Pi}^\dagger(\vec{k},t) &= \tilde{\Pi}(-\vec{k},t) \end{aligned} \tag{11}$$

Now what I will do is I will introduce some new variables and you will immediately realize why these are so, useful. So, I define a kt to be 1 over root 2 omega k square root of omega k and omega k we have already seen that is k square + m square. So, omega k omega k square is k square + m square. So, I have um. So, this is the definition I am giving k - half. So, it is a square root the denominator -k sorry that is good.

Now if I take a dagger of this. So, right now in this equation I am treating phi tilde and pi tilde as operators. So, if I take a machine conjugate of this a dagger I put a dagger on that I get a dagger which will be this. So, omega k is real so it remains omega k phi tilde becomes phi dagger of kt but phi dagger of kt is related to -k but anyway it does not matter I can still keep it like this i goes to -i this is real. So, nothing happens and this becomes this.

So, that is the definition I have introduced. Now it is easy to show the following and I will leave this as an exercise for you; show that if you take this operator a k and a take a k prime if you find the commutation relation there it will vanish. Similarly if you look at a dagger kt a dagger k prime t that commutator will also vanish and if you take a kt and a dagger of k prime t then of course it will not vanish and it will give you actually h bar delta cube k - k prime.

So, notice there is not a the iota is not here. So, the i is missing we have an h bar delta cube k - k prime proof is simple that you can easily do the only thing you would require in proving this is using the fact that pi tilde and pi tilde dagger kt is phi tilde of - kt that you will need to use and similarly for phi tilde beyond this not much is needed kt phi tilde - k. So, if you use this, these 2 results then you will be able to show this that is nice. Another exercise so, first you write down the phi tilde and pi tilde in terms of a and a daggers that is trivial because you have a very simple relation here you can just invert it. Again if you do so, you will get the following and that is not really an exercise I am just writing down the result here. So, you get this a kt plus a dagger of -kt and i 2 over omega k - a dagger of -kt please check the results I hope everything is correct here.

Now one thing I want to do is use this and write the field phi not the pi tilde the phi of xt in terms of a and a daggers that is what I want to do. So, let me do that. So, I want to. Now express phi and pi in a and a dagger that is what I want to do. So let us see phi xt will be what you remember that is the it is related to phi tilde via Fourier transform and this will be e to the ik dot x and phi tilde of k. Now I should write phi tilde of kt.

Now phi tilde of kt I can use this expression for that and substitute this, this will give me maybe I should use line above integral d q k over 2 pi cube then e to the i k dot x and here you have 2 omega k. So, that is this factor which comes in the denominator and you have these 2 a kt + a dagger of - kt I can also slightly modify this what I will do is this one is straightforward there is nothing to be done here.

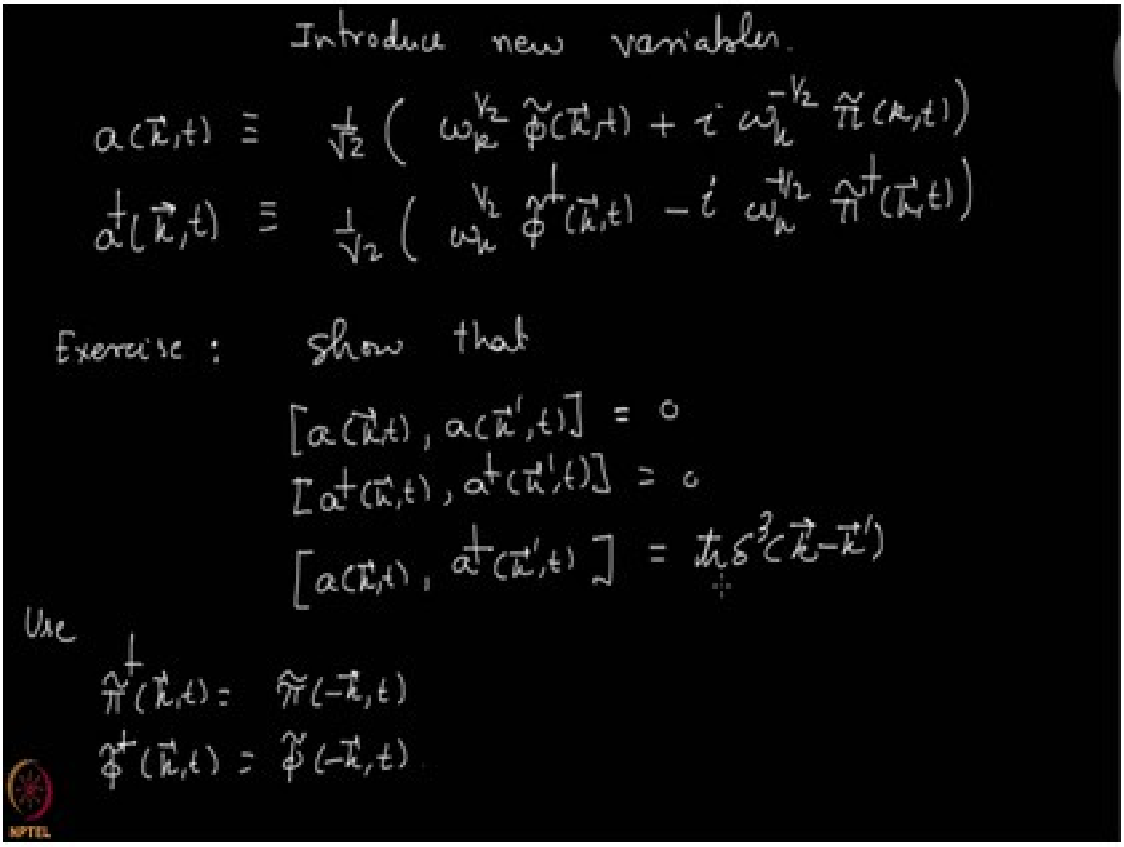


Figure 3: Refer Slide Time: 08:04

Introducing new variables

$$a(\vec{k},t) \equiv \frac{1}{\sqrt{2}}\Big(\omega^{k/2}\,\tilde{\phi}(\vec{k},t) + \iota\,\omega^{-k/2}\,\tilde{\Pi}(\vec{k},t)\Big) \tag{12}$$

$$a^{\dagger}(\vec{k},t) \equiv \frac{1}{\sqrt{2}}\Big(\omega^{k/2}\,\tilde{\phi}^{\dagger}(\vec{k},t) - \iota\,\omega^{-k/2}\,\tilde{\Pi}^{\dagger}(\vec{k},t)\Big) \tag{13}$$

Exercise : Show that

$$[a(\vec{k},t),\, a(\vec{k}',t)] = 0$$

$$[a^\dagger(\vec{k},t),\, a^\dagger(\vec{k}',t)] = 0 \tag{14}$$

$$[a(\vec{k},t),\, a^\dagger(\vec{k}',t)] = \hbar\delta^3(\vec{k}-\vec{k}') \tag{15}$$

Use

$$\tilde{\Pi}^\dagger(\vec{k},t) \;=\; \tilde{\Pi}(-\vec{k},t) \tag{16}$$

$$\tilde{\phi}^\dagger(\vec{k},t) \;=\; \tilde{\phi}(-\vec{k},t) \tag{17}$$

Exercise: Write $\tilde{\phi}(\vec{k},t)$, $\tilde{\Pi}(\vec{k},t)$ in terms of $a(\vec{k},t)$ and $a^\dagger(\vec{k},t)$

$$\begin{aligned}
\sqrt{2\omega_k}\,\tilde{\phi}(\vec{k},t) &= a(\vec{k},t) + a^\dagger(-\vec{k},t) \\
\iota\sqrt{\frac{2}{\omega_k}}\,\tilde{\Pi}(\vec{k},t) &= a(\vec{k},t) - a^\dagger(-\vec{k},t)
\end{aligned} \tag{18}$$

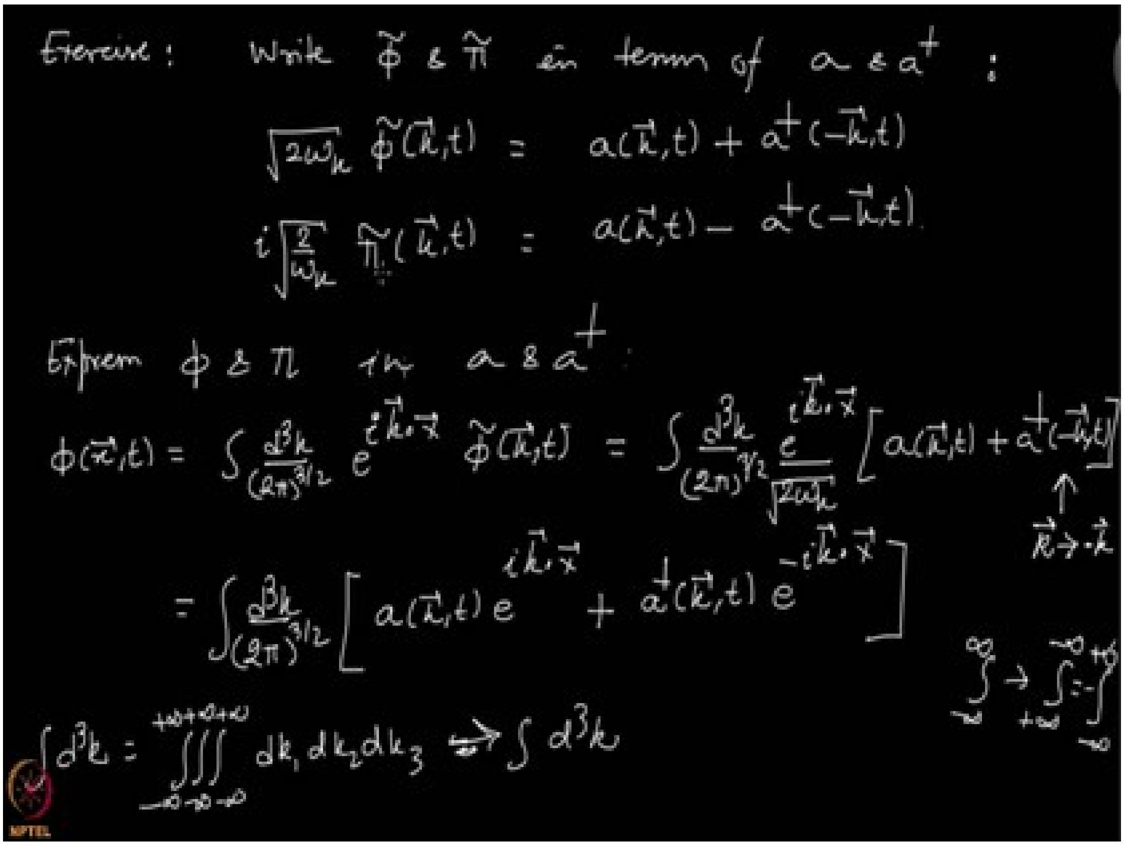


Figure 4: Refer Slide Time: 12:47

Express $\phi(\vec{x},t)$ and $\Pi(\vec{k},t)$ in terms of $a(\vec{k},t)$, $a^\dagger(\vec{k},t)$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{\iota\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{k},t) \tag{19}$$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{e^{\iota\vec{k}\cdot\vec{x}}}{\sqrt{2\omega_k}}[a(\vec{k},t) + a^\dagger(-\vec{k},t)] \tag{20}$$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}}[a(\vec{k},t)e^{\iota\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k},t)e^{-\iota\vec{k}\cdot\vec{x}}] \tag{21}$$

And this one I think I am yeah fine no problem. So, here in this expression what I will do is I will take k and make it -k. So, I will do the change of variable from k to -k that I can do because you have an integral over k. So, that is a dummy variable I can do that integral do that change of variable. If I do that change of variable this becomes a dagger of kt. But here you get e to the - ik dot x and what happens here, here actually nothing happens because of the following.

Let me first write down and then tell you why d cube k remains d q k. So, the result will be d cube k over 2 pi 3 over 2. I had missed this then you have a k of t e to the i k dot x which is fine plus I have changed the variables from -k to k and this one. So, you can split this integral into 2 pieces right. So, each piece is in has its own w variable k. So, in that one in the second one I am doing the change of variable and you get d cube k over 2 pi cube I will show you why. So, this is here and then a dagger k t e to the - i k dot x right that is what you will you will get.

Now let me show you why this has to work. So, if you look at d cube k that is just this. The 3 components of k are k 1 k 2 k 3 and the integration runs from minus infinity to plus infinity and when I am saying that I will change the variable from k to - k then my k 1 goes to − k 1 that is fine but then when I do. So, then the integration limits also change. So, this goes from infinity to my minus sorry plus infinity to minus infinity right.

So, when k going to -k the lower limit minus infinity becomes infinity and upper limit infinity becomes minus infinity. So, that is one thing you get which is you can write as minus of minus infinity to plus infinity. So you get 1 - n from here. So, these are 3. So, you get 3 minus signs when you put it back in this form from minus infinity to plus infinity and the measure dk 1 dk 2 dk 3 these are 3 and each of them is going to become -dk 1 -dk2 -dk3. So, you get one minus sign here because there are 3 such vectors.

So, one minus from here and 1 minus overall 1 minus from here makes it plus. So, this remains d cube d cube k under this change of variables and that is why I had not bothered about this and only made the changes in here. So, that is the expression of phi let me also give you the result of pi.

Similarly as before,

$$\int d^3k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1dk_2dk_3$$

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}}[a(\vec{k},t)e^{\iota\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k},t)e^{-\iota\vec{k}\cdot\vec{x}}] \tag{22}$$

$$\Pi(\vec{x},t) = -\iota\int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{2}{\omega_k}}[a(\vec{k},t)e^{\iota\vec{k}\cdot\vec{x}} - a^\dagger(\vec{k},t)e^{-\iota\vec{k}\cdot\vec{x}}] \tag{23}$$

In fact I want to nicely record them both of them here we will use it later. So, phi xt is d cube k over 2 pi bar 3 by 2 and then I had 1 over 2 omega k in the square root let us check that was missed here this is, this vector. So, you have this one then a kt ik dot x and then plus a dagger kt – ik dot x and you can also calculate for pi and you will get the following -i because when this guy goes on that side it becomes a -i and then you will have omega k.

So, this thing when goes here it will give you omega k over 2 the square root. So, you get omega k over 2 square root and everything else will be the same except for the minus sign. So, this will do exactly what it did for phi and there will be a minus sign. So, you will get let me check yes this is correct. So, these are our phi's and pi in terms of a's and a daggers this will be utilized later. I will give you one simple exercise to do show that so, what you should do is take the Hamiltonian expression which is already given in terms of phi and pi and partial like this one.

So, take this expression and substitute phi tilde and pi tilde which are expressed in terms of a and a daggers I think here somewhere yeah. So, phi tilde and pi tilde you already have in terms of a and a daggers substitute that in the expression of the Hamiltonian and show that you get the following d cube k omega k a kt a dagger kt +s a dagger kt a kt this will be very straightforward exercise. And what I can do is I can write down this in slightly different form.

So, I will use a dagger a. So, what I want to do is I want to keep a to the right and a dagger to the left. So, this term is good but this term is opposite. So, I use the commutation relation interchangeable and that will pick up a delta function because commutation relation will give you a delta function and which will give us the following omega k because it becomes twice a dagger kt a kt. So, that factor of 2 kills this vector of 2 and plus you will get an infinite contribution because of the delta function.

So, this half is coming from here d cube k omega k and commutation relation gives a delta cube of zero because let me show you here you have delta cube of k - k prime. If both k and k prime are same then you have delta cube of zero. So, that is what you get here in delta cube zero and you are integrating over the entire volume of in the k space. So, this comes out this sits outside it is a delta cube zero sitting outside and you are integrating d cube k omega k and omega k is k square plus m square square root that does not provide you any damping when k becomes large. So, this integral is really going to blow up.

So, it is a infinite quantity because the integration limits are from minus infinity to plus infinity. So, this is a divergent piece but that is fine it does not bother us that much because see Hamiltonian when acts on some state it will give energy of that state provided that state is an Eigen state of the Hamiltonian but then absolute energies do not have much meaning at least in quantum field theory what the things which we are doing.

So, you can always shift the zero of energy which is equivalent to saying that I can drop this piece I do not have to worry about this term. So, we will drop this one and write Hamiltonian to be d cube k omega k a dagger kt a kt. Let me emphasize again I think it is clear but let me say again see this is an infinite amount of I mean at an infinity but what really matters is differences between energies. So, if only differences matter then you can choose zero of energy to be wherever you like and that freedom gives you the freedom to drop this space. So, we can drop this piece because where to choose the zero of energy is up to us. So, once we are chosen for the entire system we are we are fine. So, that is infinite constant and I have this as the Hamiltonian. Now I want to do one more thing.

Exercise: Show that

$$H = \frac{1}{2}\int d^3k\,\omega_{\vec{k}}\{a(\vec{k},t)a^\dagger(\vec{k},t) + a^\dagger(\vec{k},t)a(\vec{k},t)\} \tag{24}$$

$$H = \int d^3k\,\omega_{\vec{k}}\,a^\dagger(\vec{k},t)a(\vec{k},t) + \frac{1}{2}\int d^3k\,\omega_{\vec{k}}\delta^3(0) \tag{25}$$

Let me say this you already know that energy by itself or energy is not a Lorentz invariant

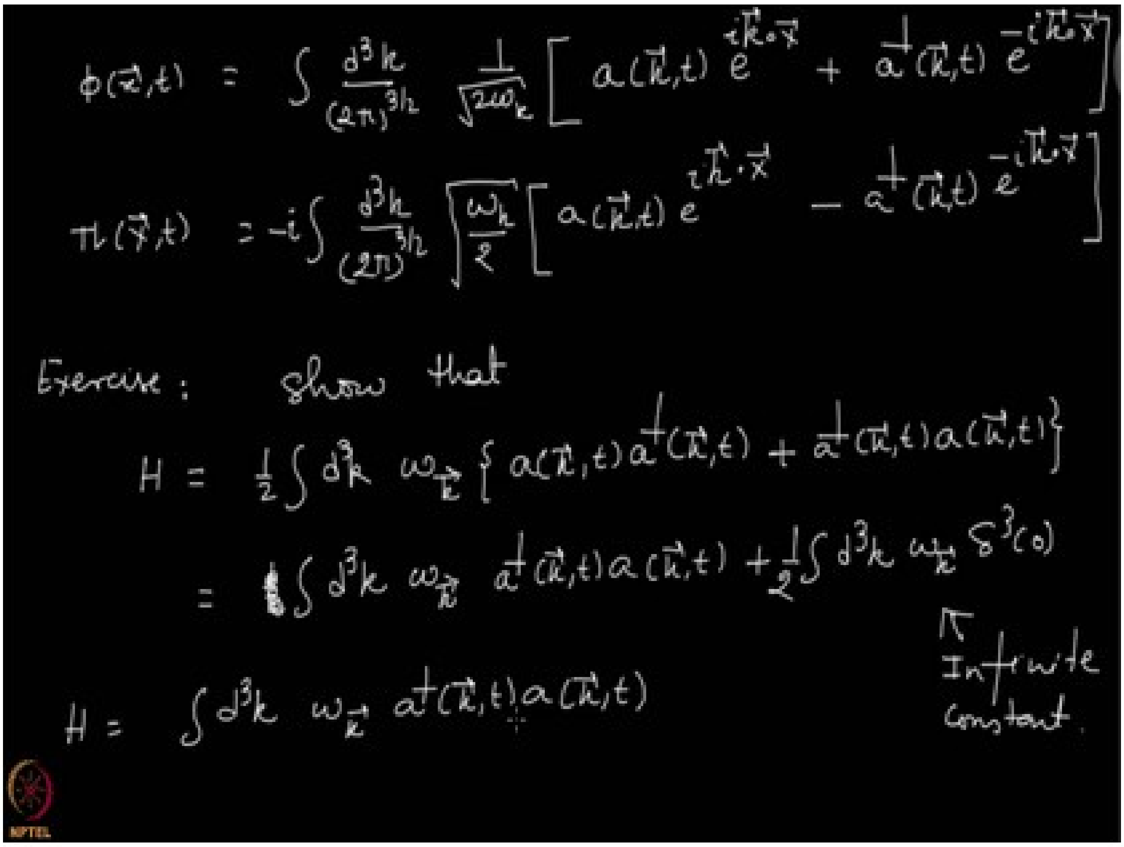
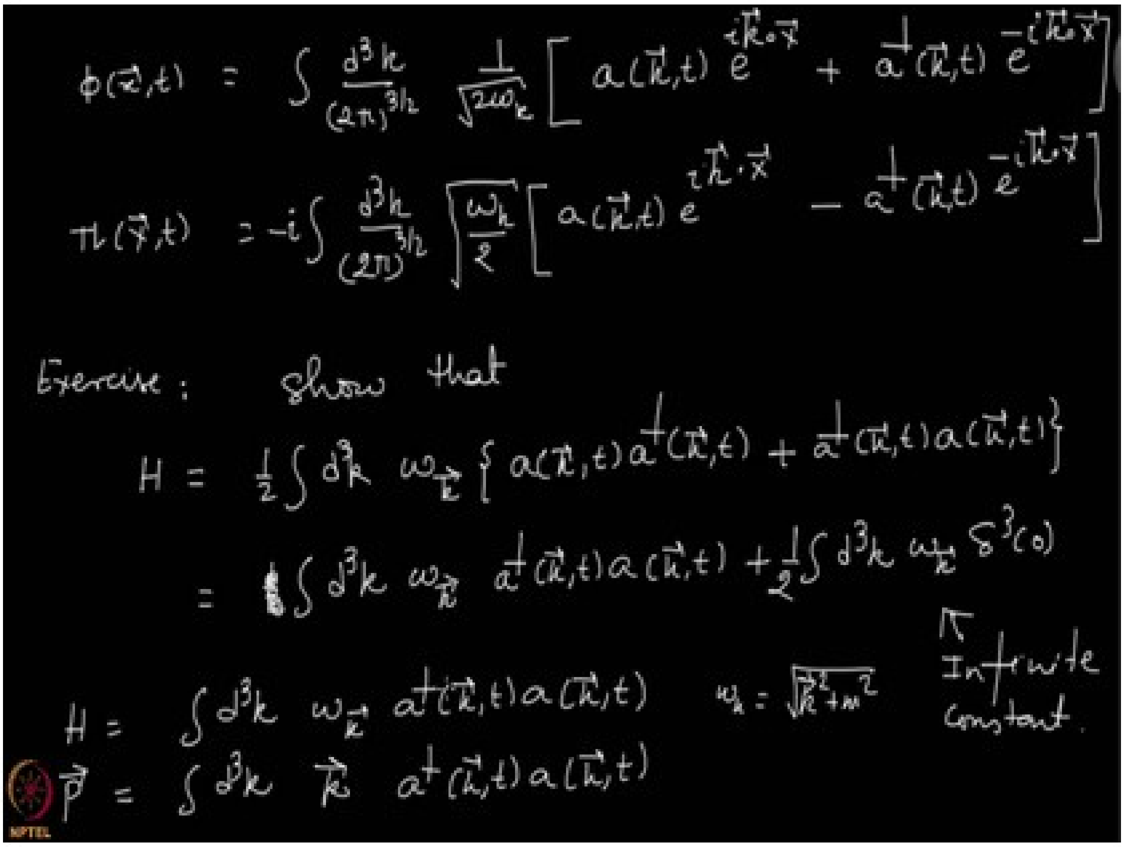


Figure 5: Refer Slide Time: 20:37

quantity neither momentum but they together form a 4 vector. So neither energy is invariant it does not have a proper it does not have a good Lorentz transformation property not the momentum has but the 2 together have very nice property under Lorentz transformation they transform exactly as a 4 vector transforms and that is why these four quantities E p 1, p 2 and p 3 they are written as p mu.

So, that is a four vector. Now the energy you are going to obtain from the Hamiltonian some operator will give you the momentum. So, you will have states which will have some energy and they will have some momentum in your theory and if the Hamiltonian operator is going to give the energy then there will be some operator which let us denote by capital P which we will call momentum operator and put a arrow.



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$$H = \int d^3k \, \omega_{\vec{k}} \, a^\dagger(\vec{k},t) a(\vec{k},t) \tag{26}$$

$$\vec{P} = \int d^3k \, \vec{K} \, a^\dagger(\vec{k},t) a(\vec{k},t) \tag{27}$$

So, that is 3 dimensional which will give the momentum of that state. So, you expect on the basis of relativity that these four operators h p 1 p 2 and p 3 whatever that p is that they will form they will together form a 4 vector. So, you expect a new operator P mu which will be this one. Now we have already derived an expression for the Hamiltonian which is this and you see this piece is k square plus m square omega k.

Now if you think of a free relativistic particle then if it has a momentum small k then this will be the energy of it now with this in mind it is easy to guess that the momentum operator p would be the following. So, all this will remain the same and here instead of omega k which is this I would expect to have k. Now I will do it late at a later point of time properly I will show you how to arrive at this in fact we will find the whole all the operators start but here I just want to make this note.

So, that I can still make progress without taking a digression; to Noether theorem and related topics, so, this is nice. Now we are almost going to begin to write down the states in this free field theory using and we' will look at the energies and moment of those states. So, I will stop here this lecture and we will continue in the next video.