Introduction to Quantum Field Theory

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Lecture 8 :Quantization of Klein-Gordon Theory continued (1)

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1 Free Real Scalar fields

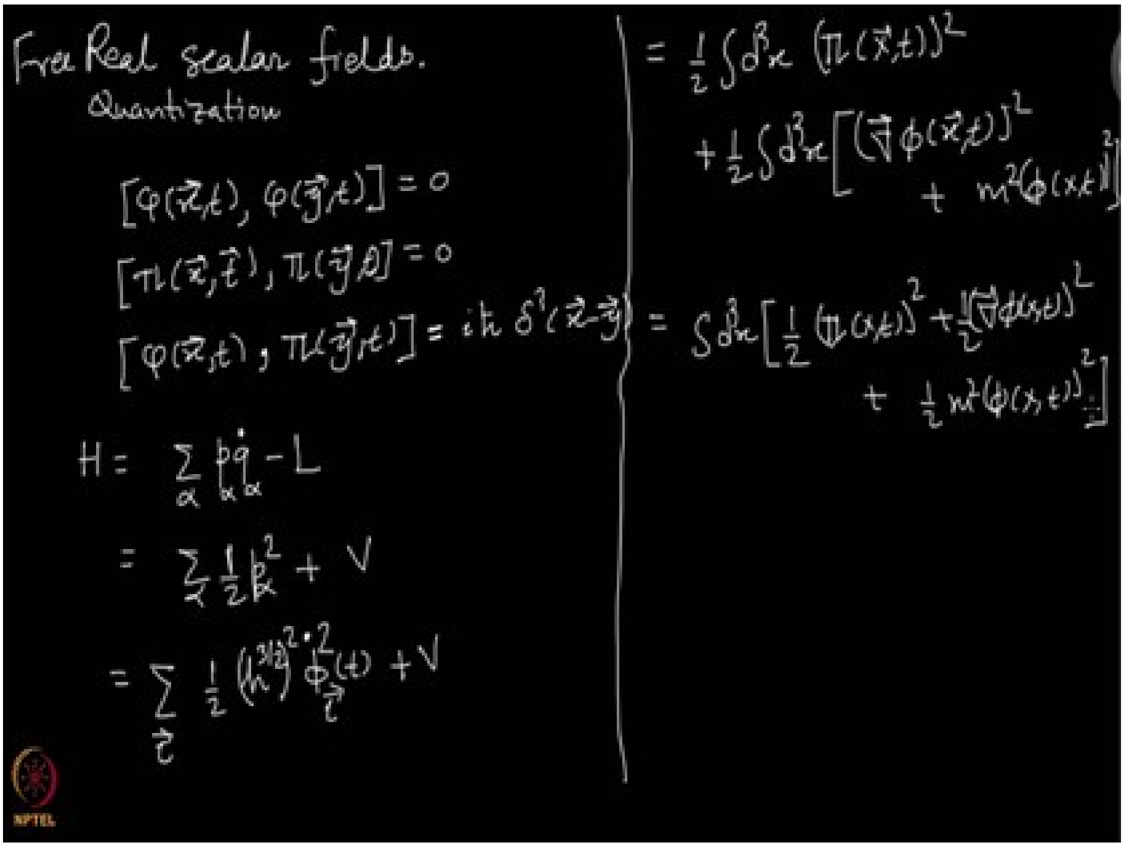


Figure 1: Refer Slide Time: 00:16

Let us continue our discussion of real scalar fields, and we started looking at quantizing the theory of real scalar fields. And let me remind you that we are looking at free field theory all the terms are quadratic in the fields. So, that the equations of motions are is linear. So, we are looking at free field theory and last time we had written down the commutation relations. So, if you promote the field phi to be an operator.

So, you put a hat here I will in this lecture I will not put a hat it is understood that it is an operator. So, we found that these are the commutation relations. So, these are called equal time commutation relations because you see that t here and the t there are same. And then of course the conjugate momenta pi they also commute and then you have the field with its. So, pi is the momentum density.

You have pi yt and these are canonically conjugate pairs and we saw that the Poisson bracket of phi and pi if they were I mean when you are treating them as classical fields that gives a Poisson bracket of which is equal to delta cube of x - y and when we quantize it we have to put an ih bar delta cube x - y that is good. Now I want to write down the Hamiltonian. Now that we have the momentum density with us we can write down the Hamiltonian.

The Hamiltonian is written in terms of the momentum density rather than the phi dot's or q dot's and that is easy we have already done the work in the last video. So you will recall that the Hamiltonian is basically pq dot where let me put alpha here I am thinking of any general system with several generalized coordinates which are labelled by alpha. So, q of alpha are the generalized coordinates for that system and this is what you have minus the Lagrangian.

Now if you look at what q alpha we had let us look at that where are we; yes there is something not good here somehow h 3 half is missing. So, we saw here that p i is q i dot here is good and then q i is defined to be h 3 halves of phi. So, here I had missed the h 3 half I am sorry for that no it is good, here I had used but there I forget to write here it was fine. So, now what I can do is construct the Hamiltonian I need the conjugate momentum I which is h 3 halves of phi dot and your phi dot is basically pi.

So here we can have when we are constructing the Hamiltonian we will get. So, this q dot for this case will become p alpha and in the Lagrangian also when you put it will also give you half p alpha squared. So, you will get half p alpha square plus all the potential terms coming from the Lagrangian. So, that is the generic structure. Now for our specific case we have half summation over let me first write p alpha.

So, p alpha is h 3 half of phi i correct phi i of t that is what we saw in the previous slide and then I should square it. So, this gets squared and becomes an h cube this gets squared, of course and you have to sum over all the lattice points and plus your potential term. Now if you look at the

Quantization

$$
\begin{aligned}
[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{y},t)] &= 0 \\
[\hat{\pi}(\vec{x},t),\hat{\Pi}(\vec{y},t)] &= 0 \\
[\hat{\phi}(\vec{x},t),\hat{\Pi}(\vec{y},t)] &= \iota\hbar\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{1}
$$

The hamiltonian of theory,

$$H = \sum_{\alpha} p_\alpha q_\alpha - L \tag{2}$$

$$H = \sum_{\alpha} \frac{1}{2} p_\alpha^2 + V \tag{3}$$

$$H = \sum_{\alpha} \frac{1}{2}(h^{3/2})^2 \int \frac{d^3x}{(2\pi)^{3/2}} \dot{\phi}_{\vec{i}}^2(t) + V \tag{4}$$

$$H = \frac{1}{2}\int d^3x (\Pi(\vec{x},t))^2 + \frac{1}{2}\int d^3x [(\vec{\nabla}\phi(\vec{x},t))^2 + m^2(\phi(\vec{x},t))^2] \tag{5}$$

$$H = \int d^3x \left[\frac{1}{2}(\Pi(\vec{x},t))^2 + \frac{1}{2}(\vec{\nabla}\phi(\vec{x},t))^2 + \frac{1}{2}m^2(\phi(\vec{x},t))^2\right] \tag{6}$$

first piece this is h cube and you have sum over all the cells that makes integral d cube x when you go to the continuum limit and phi i dot is what is basically pi.

So, this goes over to the following it becomes half integral d cube x squared ah] and the potential terms they are easy they come from the Lagrangian and they are simply this what I can put here itself plus m squared phi xt square. So, that is the Hamiltonian we have let me write it down a little bit quickly it is d cube x half pi xt whole squared plus gradient of phi xt whole squared and there is a half of course and half m squared phi squared that is the Hamiltonian for this system very good.

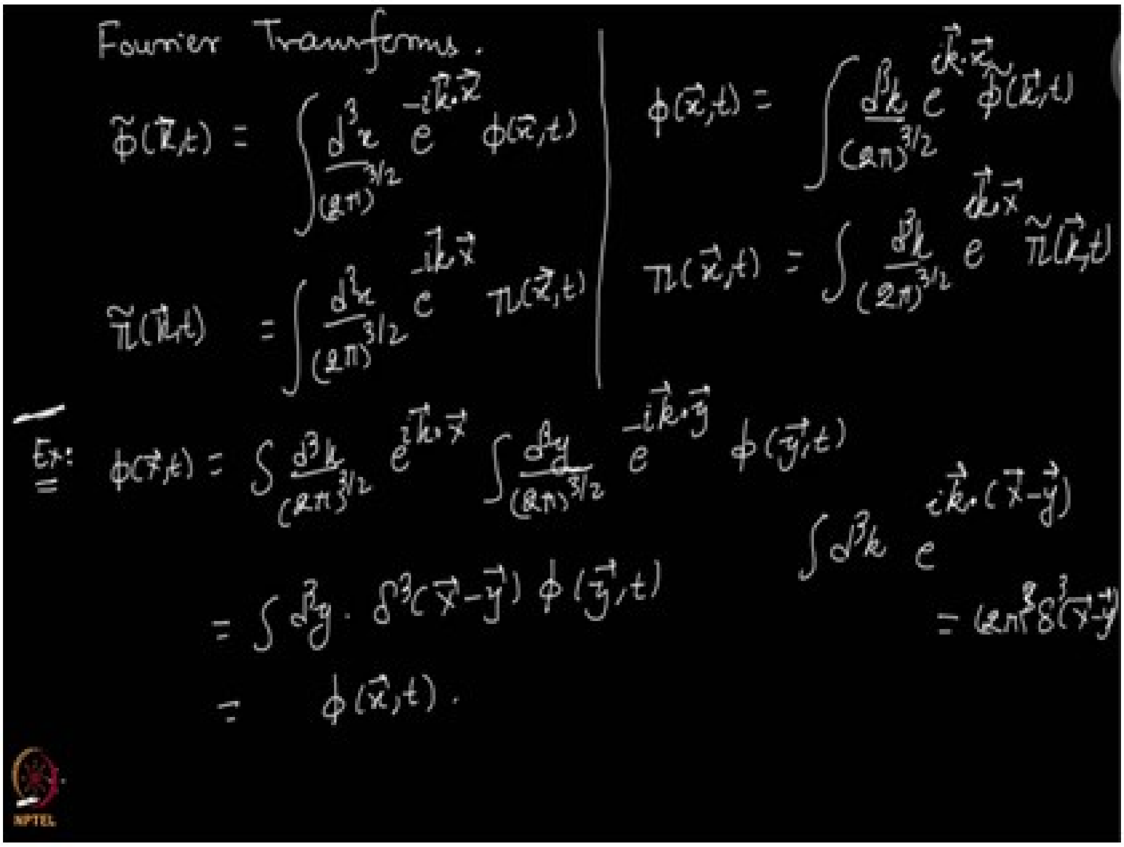


Figure 2: Refer Slide Time: 08:36

$$\tilde{\phi}(\vec{k},t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-\iota\vec{k}\cdot\vec{x}}\phi(\vec{x},t) \tag{7}$$

$$\tilde{\Pi}(\vec{k},t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-\iota\vec{k}\cdot\vec{x}}\Pi(\vec{x},t) \tag{8}$$

Now what we will do is we will instead of working in the coordinate space or the x space we will go to Fourier space. So, we want to do certain Fourier transforms and want to cast everything using these Fourier transforms. So, that is what I will do now. And you will see that it is going to be very helpful. So, let me define phi tilde kt to be a Fourier transform of the field phi and the transform is with respect to the variable x. So, phi xt e to the - i k dot x let me not write dot all right again there is no problem and d cube x over two pi 3 halves. So, I am going to keep the factors of two pi symmetrically between the Fourier transforms of phi and phi x and phi k

The inverse fourier transforms will be

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{\iota\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k},t) \tag{9}$$

$$\Pi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{\iota\vec{k}\cdot\vec{x}} \tilde{\Pi}(\vec{k},t) \tag{10}$$

Checking our newly defined fourier transforms

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{\iota\vec{k}\cdot\vec{x}} \int \frac{d^3y}{(2\pi)^{3/2}} e^{-\iota\vec{k}\cdot\vec{y}} \phi(\vec{y},t) \tag{11}$$

As we know

$$\int d^3k e^{\iota\vec{k}\cdot(\vec{x}-\vec{y})} = (2\pi)^{3/2}\delta^3(\vec{x}-\vec{y}) \tag{12}$$

We get

$$\phi(\vec{x},t) = \int d^3y \delta^3(\vec{x}-\vec{y})\phi(\vec{y},t) \tag{13}$$

$$\phi(\vec{x},t) = \phi(\vec{x},t) \tag{14}$$

Similarly I define a Fourier transform of the momentum density pi d cube x or 2 pi 3 x over 2 i tilde kt these are just the definitions of Fourier transforms. And we can also write down how phi and phi tilde will be related. I mean if I want to express phi in terms of phi tilde they are related to the inverse Fourier transforms let me write it down here. So, if you have phi xt then you can write it like this phi kt e to the ik dot x integral d cube k over 2 by 3 halves and the integration limits go from minus infinity to plus infinity.

So you have pi xt and again the same thing d cube k over 2 pi 3 halves e to the i k dot x pi k t i should put a tilde here and tilde that. It will be a good nice exercise to just check whether everything is consistent here. So, what you do is start with the phi substitute for phi tilde this expression and your right hand side should again evaluate to phi. This should be easy see when you are substituting here. So, let us start with this right hand side let us do this it is a minor exercise but let us do for a warm up.

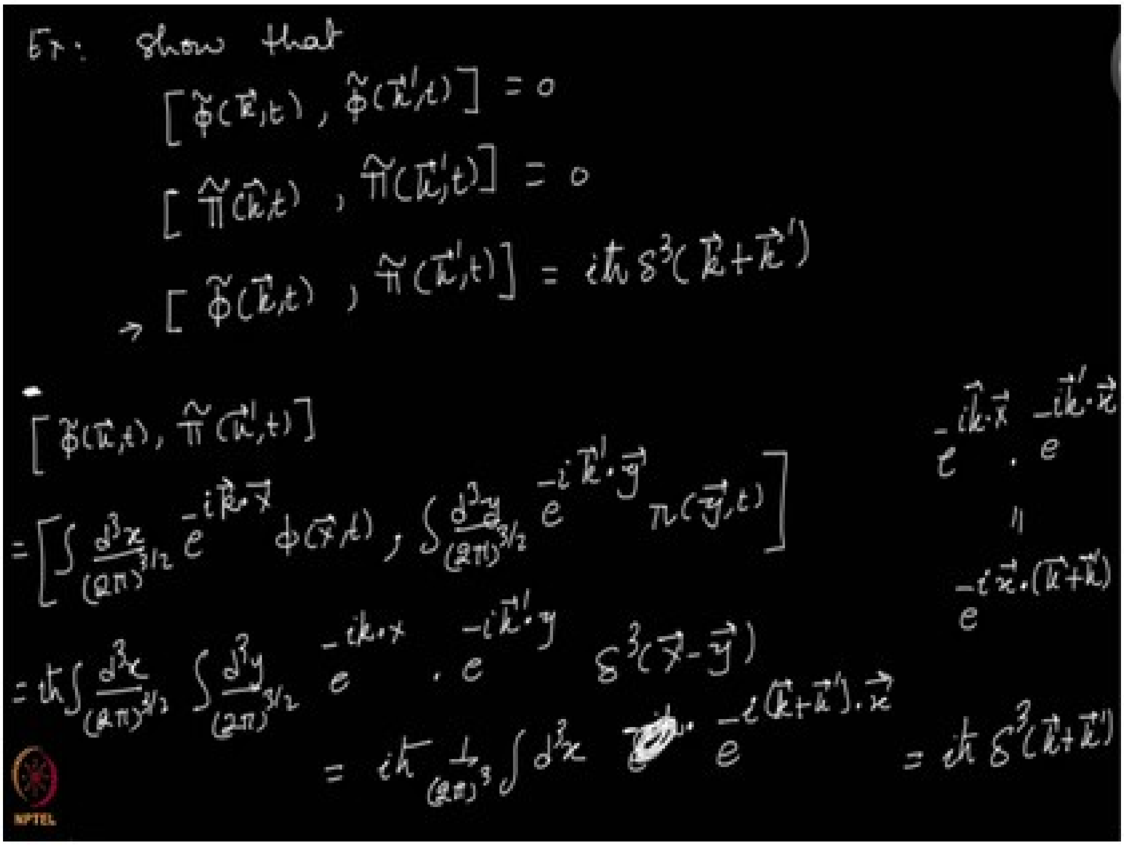
So, I am looking at this pi xt is equal to d cube k over 2 pi 3 over 2 e to the i k dot x and then i have phi tilde kt. So, I substitute phi tilde kt which is here integral I should not use x as the variable let us use y because if you use x that is a problem because you already have an x here and anyway the x in this expression is dummy. So, you should be using something else. So, let us use d cube y 2 pi 3 over 2.

This is a simple thing but should be kept in mind that you cannot use the dummy variable to vx here that that is a disaster. Then you have e to the - i k dot y and phi yt that is good. Now you can combine the two exponentials it will give you e to the i k dot x - y. So, I have integral d cube k 2 by 3 half then integral d cube y over 2 pi let me do it in two steps it is too much of writing. So, I can combine these two it will give me e to the i k dot x - y.

So, these two then I can do an integral over k. So, when I do an integral over k I will use this I will use d cube k this thing of the exponential is just a delta function but there is a factor of 2 pi cube delta cube x - phi. So, I get a factor of 2 pi cube and that factor of 2 pi cube cancels with these two 3 halves. So, there is no factor of 2 pi left I have a delta function and the integral over k is gone. So, you are left with integral d cube y delta cube x – y times phi y of t.

And now this delta function will just force the y to become x pi x t. So, we have also verified that everything is good with the Fourier transforms that we have defined it was meant to be a minor exercise here good then I will give you again some simple exercises to do which you can do yourself.

Thus with confidence we can say that everything is good with our fourier transform formulae



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Show that

$$\begin{aligned} [\tilde{\phi}(\vec{k},t),\tilde{\phi}(\vec{k}',t)] &= 0 \\ [\tilde{\Pi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] &= 0 \\ [\tilde{\phi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] &= \imath\hbar\delta^3(\vec{k}+\vec{k}') \end{aligned} \tag{15}$$

So, that if you take phi tilde kt and phi tilde k prime t and take the commutator of it you get zero that is that will be very easy to show because phi tildes are basically coming from the phi's and apart from the integral and exponential factors it is basically phi and the two phi's commute. So, correspondingly the phi tilde, will also commute and the same story for the pi. So, this will also be obviously true.

And similarly it is also easy to verify that if you take phi tilde kt and pi tilde k prime t we should get the following you should get ih bar delta cube k + k prime note that it's not -k prime when you had in terms of phi you had a phi x t pi y t you had delta cube x - y but here you have plus and this should be easy to check. Maybe even though it is easy let me just show how to do this. So I am going to show you this one.

So, phi tilde k t phi tilde k prime t this I can write as now phi tilde is the Fourier transform of phi. So, let me write this with x 2 by 3 halves and then you have e to the i - i k dot x. So, you see x is dummy. So, this result does not depend on x and similarly you will get something similar expressions from phi pi tilde let me write down that also. And I should use some other dummy variable because it is going to be multiplied with this one.

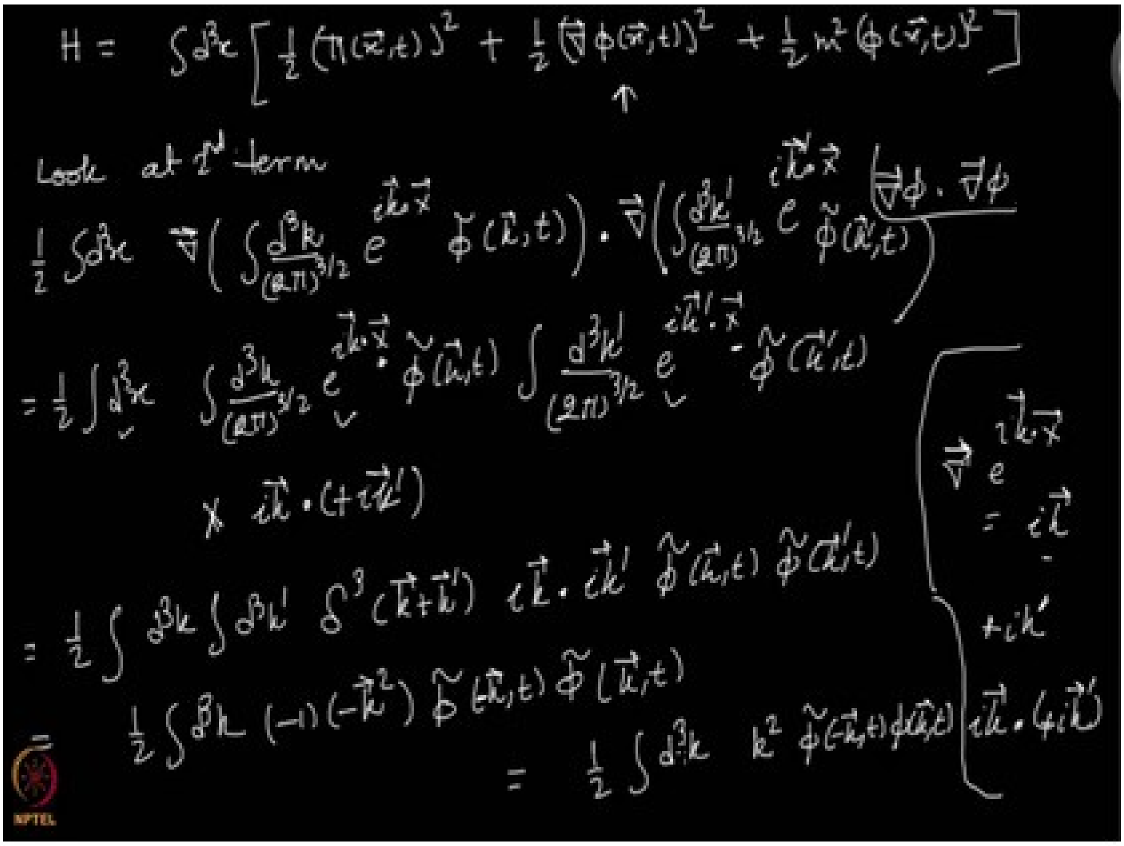

Figure 4: Refer Slide Time: 22:04

So, it should be using something else -i k prime dot y and then you have your pi y t other than the phi and the pi which are operators here everything else is just numbers and summations. So, that that can come out and you get integral d cube x 3 halves integral d cube y 3 half 2 pi 3 halves and then you have e to the -i k dot x times e to the -i k prime dot y and then the commutation relation gives you i h bar delta cube x - y right.

Solving third commutation relation,

$$[\tilde{\phi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] = \left[\int \frac{d^3x}{(2\pi)^{3/2}} e^{-\iota\vec{k}\cdot\vec{x}}\phi(\vec{x},t), \int \frac{d^3y}{(2\pi)^{3/2}} e^{-\iota\vec{k}'\cdot\vec{y}}\Pi(\vec{y},t)\right] \tag{16}$$

$$[\tilde{\phi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] = \iota\hbar\int \frac{d^3x}{(2\pi)^{3/2}}\int \frac{d^3y}{(2\pi)^{3/2}} e^{-\iota\vec{k}\cdot\vec{x}}\cdot e^{-\iota\vec{k}'\cdot\vec{y}}\delta^3(\vec{x}-\vec{y}) \tag{17}$$

$$[\tilde{\phi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] = \iota\hbar\frac{1}{(2\pi)^3}\int d^3x e^{-\iota(\vec{k}+\vec{k}')\cdot\vec{x}} \tag{18}$$

$$[\tilde{\phi}(\vec{k},t),\tilde{\Pi}(\vec{k}',t)] = \iota\hbar\delta^3(\vec{k}+\vec{k}') \tag{19}$$

So, we start with our Hamiltonian which is square plus half gradient of phi plus half m squared that is our Hamiltonian. Note that this does not have the structure of a large and variant quantity there is a plus sign here and this even if you express in terms of the dots phi dots it will not have the structure of Lorentz's invariant quantity and that is good I mean that is fine there is no problem because Hamiltonian is not a Lorentz's invariant quantity.

Hamiltonian gives you energy and energy depends on the frame of reference you are in. So, it would be actually if it turned out to be something Lorentz invariant here then you should be alarmed that you have made a mistake. So, this is fine. Now which one I want to do. So, I will not do all the parts of it because they are fairly straightforward. Let me just look at only this piece and show what it will look like when expressed in the phi tilde's. And these will be then very obvious to see from what I do that what the result will be. So, let us look at the term this one second term and express it in Fourier variable. So, it is d cube x is a half gradient. So, great this is gradient phi dot gradient phi that is what is greater than phi squared. Now I write for phi i will write integral d cube k over 2 pi 3 by 2 e to the i k dot x and then you have phi tilde k dot x k phi tilde kt.

Because phi tilde was defined with the e to the -i k dot x and phi is defined with e to the i k dot x. So, which is good then you have a dot product and you have again a gradient here d cube k prime over 2 pi 3 halves e to the i k prime dot x and you have a phi tilde k prime t here is this bracket is closed and that is all. Now this is equal to half integral d cube x perfect let me yeah let me keep it this way. So, integral dq k over 2 by 3 halves then when this gradient acts on this one it will give you i k.

So, this is a scalar object as far as rotations is concerned I am not talking about Fourier transformation. So, k dot x is a scalar under rotations this because there is an arrow that arrow is to signify that this transforms as a vector, vector under rotations. So, this entire object should be a 3 vector and that is why you have a; the result has a k vector that is correct. So, this one will give you ik. Then you have a dot product and then this, will similarly when this gradient acts on here it will give you -ik and these two will be dotted.

So, you will get ik dot - ik not k prime this one gives you this thing and then because you are taking derivatives over exponentials they will give you back the same things. So, here it will become e to the ik dot x phi tilde k t and again the same thing d cube k prime over 2 pi cube e to the ik prime dot x phi tilde k prime t and then you have this vector ik dot - y i have a minus here no there should be no minus sorry ik prime that is that is fine.

Now you see you have e to the ik dot x. So, there is an x here e to the ik dot k prime dot x. So, you have an x here and I can do an integral over x because other than these two exponentials there is no other place where you have an x. So, this integral is very easy because that gives you

a delta function. So, if I take this and this I will get a 2 pi i will get a 2 pi cube delta cube of k + k prime because they will combine together to give you k + k prime.

So, my 2 pi cube will cancel these two factors the integral will be gone over x and these two factors will gone will be gone and leave behind a delta q. So, let me write that down. So, you have half integral d cube k over as I said that is gone d cube k integral d cube k prime then you have a delta cube sorry delta cube of k + k prime and then that is all gone. Now and then you have the two factors of phi tilde and this one the k and k prime.

So, I have phi tilde maybe I will put the phi tilde's and then so, I have i k dot i k prime and then phi tilde kt phi tilde k prime t and this becomes half integral d cube k. Now let me do the integral over k prime. So, this doing this integral because of this delta function we will set wherever k prime appears as -k delta cube of k + k prime hits when k prime is equal to -k. So, this k prime gets turned into -k and this k prime turns into -k again the integral over k prime will be gone.

So, I will have I square is -1. So, I have -1 then k dot k prime times into -k. So, I have -k square that is fine and then I have phi tilde kt and finally -kt. So, let me put the -kt first and phi tilde of it is does not matter you can write in whatever order you want because right now I am treating the Hamiltonian as a classical the phi is to be classical fields. Anyway even if they were even if you treat them as operators you have seen that they commute.

Shifting our focus back to the hamiltonian of the theory

$$H = \int d^3x \left[\frac{1}{2}(\Pi(\vec{x},t))^2 + \frac{1}{2}(\vec{\nabla}\phi(\vec{x},t))^2 + \frac{1}{2}m^2(\phi(\vec{x},t))^2\right] \tag{20}$$

Looking at 2^{nd} term, $\vec{\nabla}\phi \cdot \vec{\nabla}\phi$

$$= \frac{1}{2}\int d^3x \vec{\nabla}\bigg(\int \frac{d^3k}{(2\pi)^{3/2}} e^{\iota\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k},t)\cdot\vec{\nabla}\bigg(\int \frac{d^3k'}{(2\pi)^{3/2}} e^{\iota\vec{k}'\cdot\vec{x}}\tilde{\phi}(\vec{k}',t)\bigg) \tag{21}$$

$$= \frac{1}{2}\int d^3x \vec{\nabla}\bigg(\int \frac{d^3k}{(2\pi)^{3/2}} e^{\iota\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k},t)\int \frac{d^3k'}{(2\pi)^{3/2}} e^{\iota\vec{k}'\cdot\vec{x}}\tilde{\phi}(\vec{k}',t)\bigg)\iota\vec{k}\cdot(+\iota\vec{k}') \tag{22}$$

$$= \frac{1}{2}\int d^3k \int d^3k' \delta^3(\vec{k}+\vec{k}')\iota\vec{k}\cdot\iota\vec{k}'\tilde{\phi}(\vec{k},t)\tilde{\phi}(\vec{k}',t) \tag{23}$$

$$= \frac{1}{2}\int d^3k(-1)(-\vec{k}^2)\tilde{\phi}(-\vec{k},t)\tilde{\phi}(\vec{k},t) \tag{24}$$

$$= \frac{1}{2}\int d^3k k^2\tilde{\phi}(-\vec{k},t)\tilde{\phi}(\vec{k},t) \tag{25}$$

So, the minus goes away and you have finally half integral d cube k k square phi tilde of -k t and phi of kt. So, this was easy and the other two are even easier because they do not even have a derivative. So, whatever the effect of derivative was there in the calculation that is gone. So, you will simply have instead of k square you will have 1 here because the k square was pulled out from the exponentials because of the gradient. So, the first term here will give you half d cube k pi tilde -kt pi k t this we have done anyway and this will give you half d cube sorry half d cube k m square phi tilde - kt phi kt. So, I can now easily write down the final result. My Hamiltonian would look like so, we have pi tilde -k t pi k + half it is nicer to put half here that looks that way you do not forget the half you have half and then you had a k square coming from the gradient term and the m square from that other term and they you can both take both of them and combine here and you will get phi tilde of – kt phi of kt and this we will define to be omega k square.

So, your omega k square is k square plus m square that is the Hamiltonian in terms of this Fourier transform variables. We will continue the development of the subject of quantization of real scalar fields in the next video.

The hamiltonian now,

$$H = \int d^3k \left[\frac{1}{2}\tilde{\Pi}(-\vec{k},t)\tilde{\Pi}(\vec{k},t) + (k^2+m^2)\tilde{\phi}(-\vec{k},t)\tilde{\phi}(\vec{k},t)\right] \tag{26}$$

We will write

$$\omega_k^2 = \vec{k}^2 + m^2 \tag{27}$$

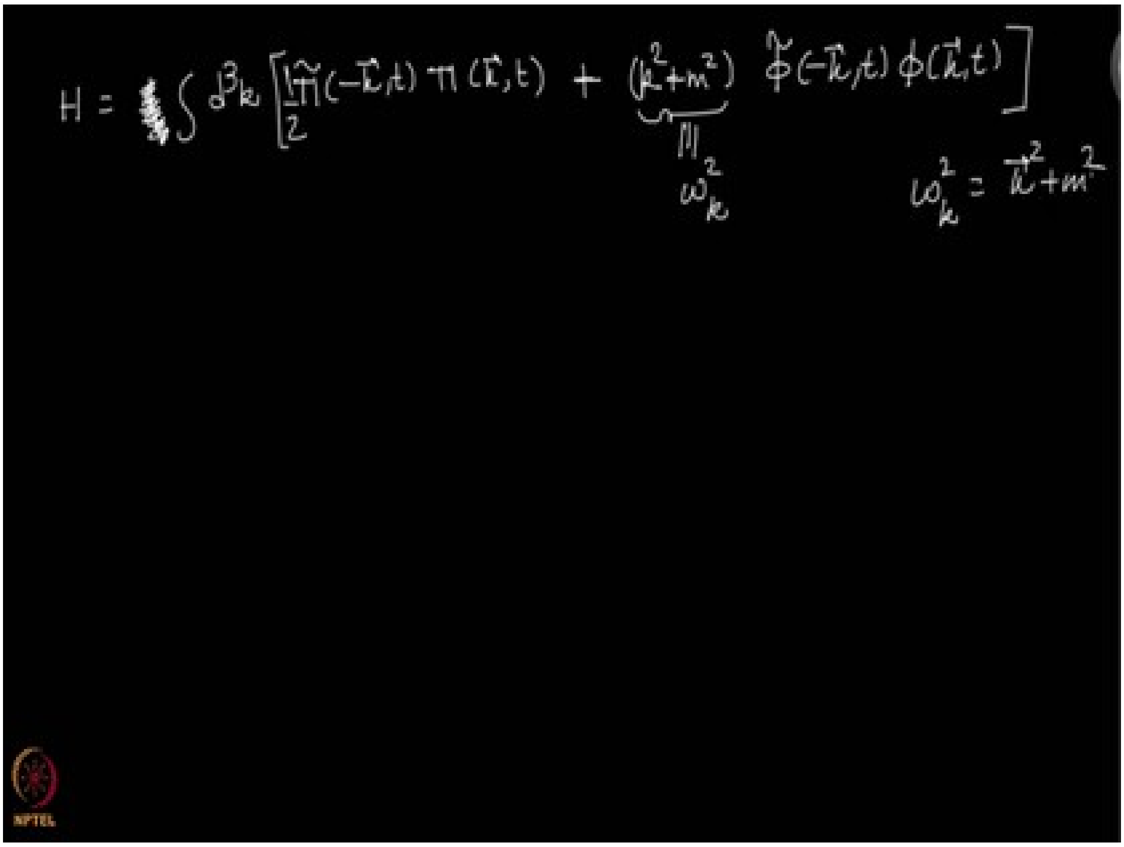

Figure 5: Refer Slide Time: 32:38