Introduction to Quantum Field Theory

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Lecture 6 : Klein Gordon Equation

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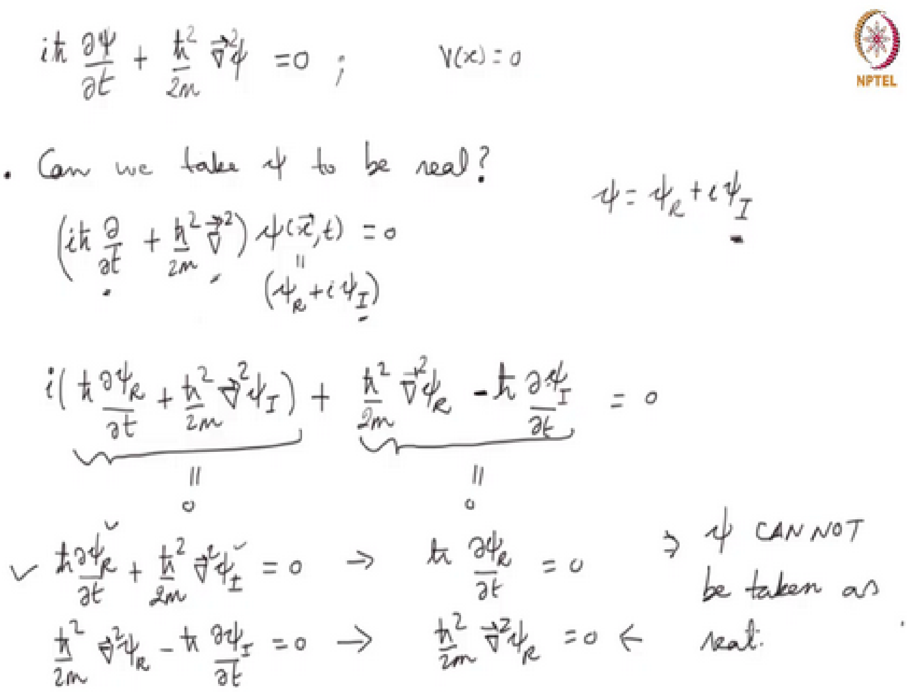
1 Revisiting the Schrodinger equation



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Till now we have been talking about Schrodinger equation and the corresponding Schrodinger theory. But that equation is a non-relativistic equation and we would like to make a theory which is relativistic. So I am going to introduce today Klein Gordon Fields. So I start with Klein Gordon equation. But before we do so let us quickly revisit and make some simple remarks about the Schrodinger equation and the corresponding theory. So here I have written down the Schrodinger equation I put the potential to be zero here. Can we ask one simple equation can we take Psi to be real? Do you have a freedom to take Psi to be real feel? Now note that you are the operator that you have here the Schrodinger operator It is ih bar del over del t + h bar square over 2m del square this operator action Psi put the potential to be zero here. Now if I can take Psi to be real then I should be able to put Psi imaginary to be 0 I write Psi as Psi real plus i times Psi imaginary.

Then I should have the freedom to put Psi i is equal to 0 that is what would mean by saying that I can choose Psi to be real. Let us see whether we have the freedom and you can probably see already that you do not have because of this factor of I the Psi real and Psi imaginary will get mixed up. In case of mixing between them and that is why you will not be able to do, will not be able to choose Psi to be real. Let us see slightly more explicitly.

So what I am going to do is substitute here Psi real + i times Psi imaginary that is what I do. I am just, operate with this operator on these two components. So first let us correct the imaginary parts. So when this derivative x and Psi are you get ih bar del Psi R over del t and when this operator here acts on Psi i you get a this iota here that is what I have collected outside. So you have here + h bar square over 2m gradients del square Psi i plus let us collect the real parts.

So when you have this operator acting on Psi i you get a real contribution. And when the first operator acts on this one the two i's give -1 here then you have h bar. Then you have del Psi i over delta T equal to 0, so that is what you get. Now if this equation has to be satisfied the real part and imaginary part must vanish separately because there is no way a real part can cancel against an imaginary part.

The 2 can cancel against -2 but you cannot cancel against two times i guess these two have to vanish separately means this is should be equal to zero and that should be equal to zero. So you get two equations of motion which are this let me write it down over delta t h Bar square over 2m gradient square Psi I this is equal to zero. And the other one is h bar square over 2m gradient del square Psi I h bar – delta Psi I over h bar delta t is equal to 0.

So you get these 2 equations now and of course they do not look like Schrodinger equation at all because you have Psi R here and Psi I here. So if you say ok I can take Psi to be real which we can put Psi I to be 0. If you do that then this time goes away. So look at this equation and this term goes away and it leaves behind only the space. And this one when I put Psi I to be 0 it leaves behind only h bar square over 2m. But this one does not even have a time derivative involved in this. So this is not even equation of motion. So you clearly will see that Psi I does not satisfy Schrodinger equation neither of this, this is a Schrodinger equation. So clearly Psi is truly a complex function. It cannot choose it to be real we can we will see when we once were written down the Klein Gordon equation what can be said about the fields which satisfy Klein Gordon equation this one point I want to make let me write it down.

So Psi cannot be taken as real. Another point to note that the reason this is not a relativistic equation the Schrodinger equation this one is because you have only one partial derivative of time. And 2, derivatives of space so del square is del over del 2 over del x square. Now if you want to have a relativistic theory your space and time should be an equal footing because they mix under Lorentz Transformation so you have to have them appearing equally. Now here you have one single derivative for time to derivatives of space.

So it is not a relativistic theory. So we should search for a relativistic equation and that is what brings us to Klein Gordon equation.

$$\iota\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}+\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x},t)=0 \tag{1}$$

Here we have taken

$$V(x)=0$$

Can we take $\psi(\vec{x},t)$ to be real?

$$\left(\iota\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\vec{\nabla}^2\right)\psi(\vec{x},t)=0 \tag{2}$$

If we take

$$\psi(\vec{x},t) = \psi_R(\vec{x},t) + \iota\psi_I(\vec{x},t) \tag{3}$$

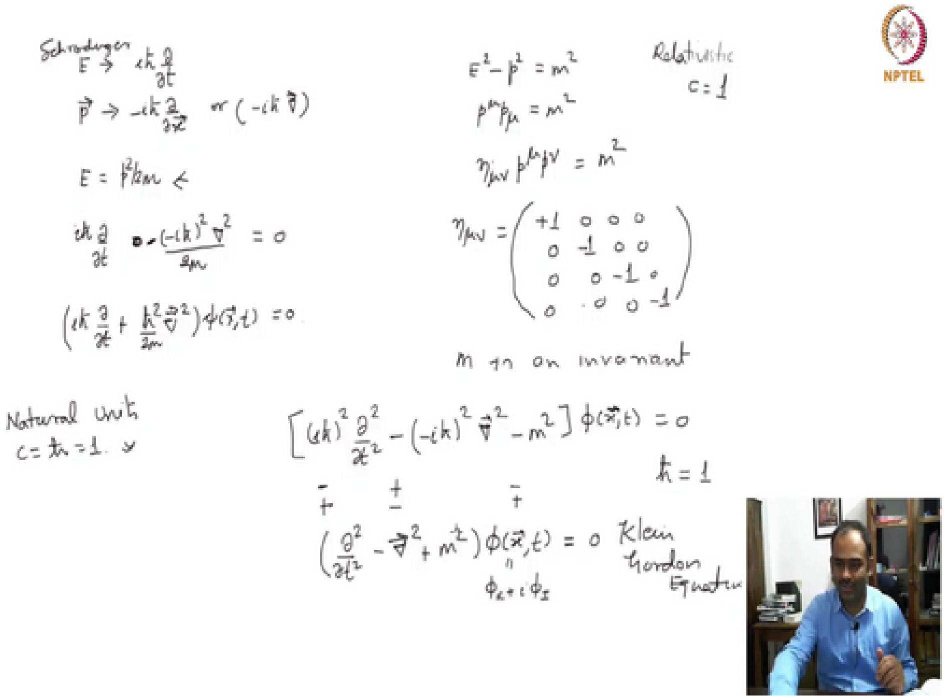
Substituting this in Eq.(1) will give us

$$\iota\left(\hbar\frac{\partial\psi_R(\vec{x},t)}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}^2\psi_I(\vec{x},t)\right) + \frac{\hbar^2}{2m}\vec{\nabla}^2\psi_R(\vec{x},t) - \hbar\frac{\partial\psi_I(\vec{x},t)}{\partial t} = 0 \tag{4}$$

And to do that, let us recall how we can write down Schrodinger equation. So, the prescription is this, you take energy and promote it to an operator but the following prescription. So you have energy is del over del t. Then you have momentum that also you promote to and operator and this is del over del x which is same as; these two are identical. And then you say the particle has to satisfy the dispersion relation with E is p square 2m I am looking at free particle there is no potential sites just free particle.

So if we substitutes these operators in here, you get ih bar del over del t is equal to p square so you have –ih bar square and you have del square. So that is the operator you get or you instead of equal to you can put a minus sign here. So this is p square and you put it 2m and you say this is equal to zero. So basically E - p square 2m is equal to zero that is what I am writing in the operator language which is ih bar del over del t.

So you have h bar square over 2m then your minus sign and then this also give another minus sign and that makes it plus and that is how you can see that you can arrive at Schrodinger equation. Now when you are doing relativistic theory and you are imagining a free particle the dispersion relation which is written here that gets changed. Now energy relation between energy and momentum is different and it is E square - p square is m square or you can also write it as p mu is equal to m square or still you can write it as eta mu nu p mu equal to m square.



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So these are 3 ways in which I can write the same equation eta mu nu I am taking to be +1, -1, that is the matrix we have. So I want to do the same thing which I did for the Schrodinger case for is relativistic case. So, what I do is I will substitute this operator which are here into this equation and arrive at a similar equation to this one. Before that let us have a small remark here. This is the same thing E square – p square so I will puts C equal to 1.

That thing is written as p mu is equal to 2m square. So the left hand side you see both the indices are contracted here if you see for example the same thing. The mu and nu are contracted with each other through eta mu nu meaning there is no free indices on the left hand side, which means

Real and imaginary part should vanish seperately, what we get is that the left hand side is a Lorentz's scalar which means that the right side should also be Lorentz's scalar. That because this two sides should have same transformation properties and the Lorentz transformations.

$$\hbar\frac{\partial\psi_R(\vec{x},t)}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}^2\psi_I(\vec{x},t) = 0 \quad (5)$$

$$\frac{\hbar^2}{2m}\vec{\nabla}^2\psi_R(\vec{x},t) - \hbar\frac{\partial\psi_I(\vec{x},t)}{\partial t} = 0 \quad (6)$$

If we put $\psi_I(\vec{x},t) = 0$, above equations will become

$$\hbar\frac{\partial\psi_R(\vec{x},t)}{\partial t} = 0 \quad (7)$$

$$\frac{\hbar^2}{2m}\vec{\nabla}^2\psi_R(\vec{x},t) = 0 \quad (8)$$

Seperately they do not follow Schrodinger equation, hence we can not put $\psi_I(\vec{x},t) = 0$
How we wrote Schrodinger equation,

- $E \to \iota\hbar\frac{\partial}{\partial t}$
- $P \to -\iota\hbar\frac{\partial}{\partial\vec{x}}$ or $(-\iota\hbar\vec{\nabla})$
- $E = \frac{P^2}{2m}$ for free particle

Substituting these in $E = \frac{P^2}{2m} = 0$, we get

$$\iota\hbar\frac{\partial}{\partial t} - \frac{(-\iota\hbar)^2}{2m}\vec{\nabla}^2 = 0 \quad (9)$$

$$\left(\iota\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}^2\right)\psi(\vec{x},t) = 0 \quad (10)$$

That is how we arrive at Schrodinger equation, which is a non-relativistic equation.

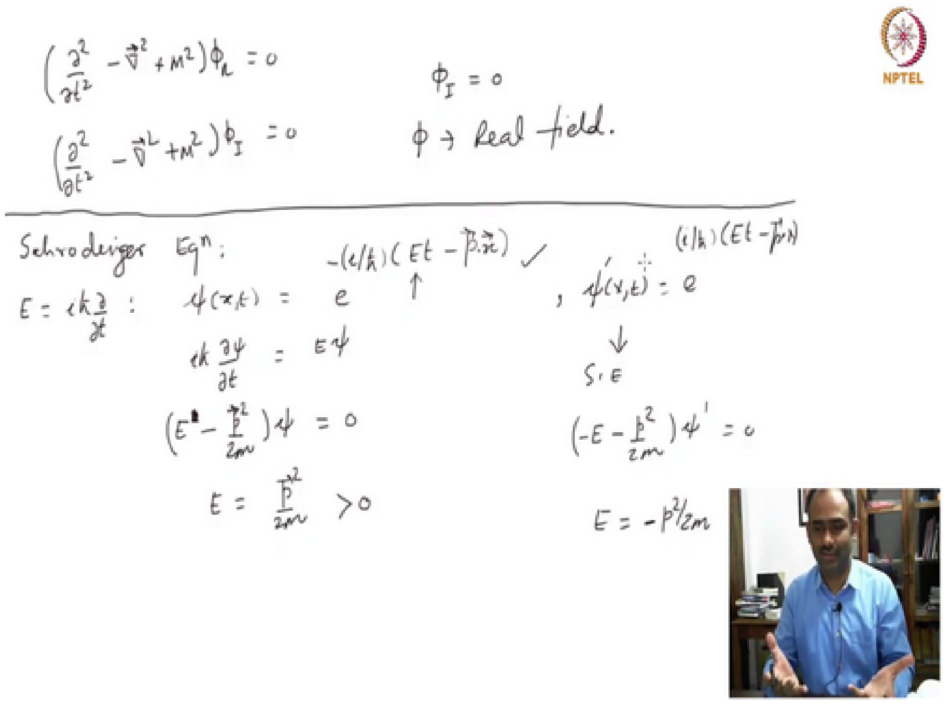
2 Klein-Gordon equation

Relativistic energy momentum relation is,

$$E^2 - p^2 = m^2 \quad (11)$$

$$p_\mu p^\mu = m^2 \quad (12)$$

$$\eta_{\mu\nu}p^\mu p^\nu = m^2 \quad (13)$$



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where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

No free indices, Lorentz scalar on both side, thus m is an invariant. Substituting E and p in above equation,

$$\left[(\iota\hbar)^2\frac{\partial^2}{\partial t^2} - (-\iota\hbar)^2\vec{\nabla}^2 - m^2\right]\phi(\vec{x},t) = 0 \tag{14}$$

So both sides are Lorentz scalar, so m square is a constant which does not depend on the frame of reference in which you are m square is a constant here or an invariant, whichever you say m square or m, m is an invariant. So, this m is called mass and it does not change under Lorentz Transformation this is a fixed quantity. Now, let us proceed with writing down the equation. So I put ih bar del by del t and square it.

So here we have 2 factors of ih bar f because I am squaring it and then your second derivative. I wanted the second derivative of p square is again –ih bar square and you have del square and this let me bring –m square to the left side and that should be equal to 0 so let us feel phi on which this operator and gives you 0 and that is how gives you write down the equation. Now, let us put h bar also to be 1 you choose units which are called natural units, in these units both C and h bar are chosen to be 1.

Now once you do that so you get i square gives you minus so you get a minus sign here this gives again a minus sign which makes it over all plus and then minus here and then act on phi is equal to 0. If I multiply entire equation with the minus sign the -1 you get this becomes a plus this becomes a minus and this becomes plus and then you get delta 2 over delta t square - del square + m square phi 0.

This is our Klein-Gordon equation. And now we can ask the same equation which we ask for the Schrodinger case Schrodinger equations, Schrodinger field can I choose phi to be real or as in that case it has its truly complex. Not that here the operator which you have the Klein Gordon operator that you have is a real operator there is no factor of i sitting here anywhere, which means if I write phi as phi real plus phi imaginary phi real plus phi imaginary you will get 2 equations one for phi R and one for phi I and they will be both the same equations. And this will be Klein Gordon equations which mean the real part of the field phi will evolve completely independently of phi i. So if it takes Phi to be complex you have two copies of the same thing because Phi I and Phi real they satisfy the same equation. Let me write it down.

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi(\vec{x},t) = 0 \tag{15}$$

Above equation is known as Klein-Gordon equation. Same question, can we take $\phi(\vec{x},t)$ as completely real or imaginary? and to check that we take

$$\phi(\vec{x},t) = \phi_R(\vec{x},t) + \iota\phi_I(\vec{x},t)$$

Upon substituting, $\phi_R(\vec{x},t)$ and $\phi_I(\vec{x},t)$ both satisfy same equation,

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi_R(\vec{x},t) = 0 \tag{16}$$

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right)\phi_I(\vec{x},t) = 0 \tag{17}$$

Thus $\phi(\vec{x},t)$ is a real field

3 Positive and negative energy states

3.1 Schrodinger equation

So if you do that substitution you are going to get this and the same equation for Phi imaginary. So if you take complex you get two copies of the same thing because they are not talking to each other. They are not coupled at all which also means that you can choose to put Phi i equal to zero and take phi to be real field. So this is to contrast with the situation in the case of Schrodinger. Now what we would like to do its proceed as before write down an action which will give you Klein Gordon as equation of motion when you take delta is equal to zero.

Before that I want to make one small observation again and let me do that. So let us return to Schrodinger equation first. There if you recall we just wrote down few minutes ago. I am concerned only with this operator right now. So let us take away function Psi which is –i over h bar Et - px or p bar x. Now when you add with this operator on Psi you will get the energy because this gets a full. So you get ih bar del Psi over del t E Psi.

And if you substitute the Psi in the Schrodinger equation you will obtain E square - p square over 2m Psi is equal to 0 which means that for this to be satisfied sorry, not E square the energy is p square over 2m and because p square is a square of vector it will be positive this is greater than 0. So you get energy to be positive here. Now if you instead of this Psi, you took Psi xt if you start taking this one it is called Psi prime to be e to the i over h bar taking a complex conjugate of Psi this item is complex conjugate of Psi.

So this Psi prime ET – p dot x then if you substitute this Schrodinger equation you will get -E - p square over 2m acting on the Psi prime is equal to 0 which would mean that E is negative. So you see that this wave function gives negative energy. So we do not work with this kind of wave functions. We work with this kind of wave functions that you can draw because it is giving you negative energy. Now here it was fine. Let us see what happens in the case of ah Klein Gordon theory.

See Klein Gordon theory we said that we can put Phi i to be 0 and we can work with real fields. So now real field you can construct as a sum of a complex field phi and its complex conjugate. So phi + phi star is what gives you two times Phi real. So you see that the same wave equation here are the same function here this one cannot serve as solution to the Klein Gordon theory. You have to also include this one because this is a complex conjugate and these two together form create a real valued field.

$$E \;=\; \iota\hbar\frac{\partial}{\partial t} \quad ; \quad \psi(\vec{x},t) \;=\; e^{-\frac{\iota}{\hbar}(Et-\vec{p}.\vec{x})} \tag{18}$$

$$\iota\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} \;=\; E\psi(\vec{x},t) \tag{19}$$

$$\left(E-\frac{p^2}{2m}\right)\psi(\vec{x},t) \;=\; 0 \tag{20}$$

That will give us positive energy

$$E=\frac{p^2}{2m}>0$$

$$\psi^*(\vec{x},t)=e^{\frac{\iota}{\hbar}(Et-\vec{p}.\vec{x})} \tag{21}$$

Schrodinger equation

$$\iota\hbar\frac{\partial\psi^*(\vec{x},t)}{\partial t} \;=\; E\psi^*(\vec{x},t) \tag{22}$$

$$\left(-E-\frac{p^2}{2m}\right)\psi^*(\vec{x},t) \;=\; 0 \tag{23}$$

Which will give us

$$E=-\frac{p^2}{2m}<0$$

Negative energy and hence we drop this wave-function.

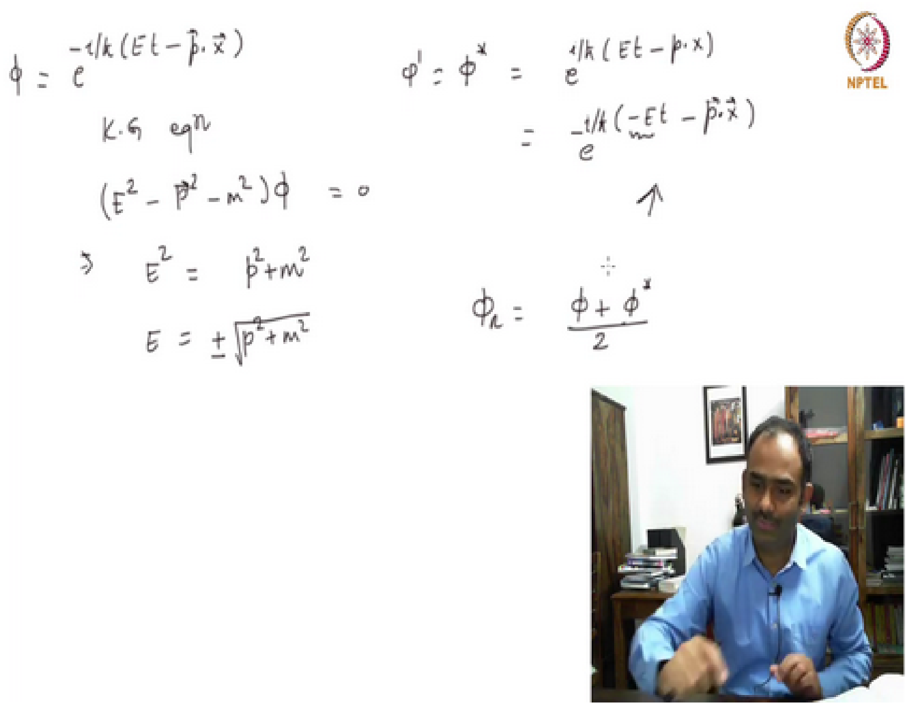
3.2 Klein-Gordon field

So let us see what happens because of this. So if you take e to the –i over h bar Et – p dot x and if you substitute it in the Klein Gordon equation then you will get the following. You will get E square something strange let us go back you will get this which would mean that E square is; but now you have two solutions possible, your E is both positive and negative that is one thing. But you cannot drop the negative energy once.

Look at this minus choice you cannot drop because you must include Phi star which is let us call this Phi prime which is Phi star which is E to the i over h bar ET – p dot x. And if you write it in this form with specific minus Psi outside it is basically -Et - p dot x. So, I am assuming that is E is positive right now. But then you that this quantity is the energy of the single of the state in the single particle Quantum theory because when you operate with delta over delta t it will give you energy -E and you need the solution.

You need the solution because you need both the parts. So it means that when you are looking at Klein Gordon theory, single particle Klein Gordon theory treating it as quantum mechanical systems then you have issue that you have both positive and negative energy states in your theory. But this does not bother as at all because for us Klein Gordon equation is not a quantum equation for us it is classical equation. We are going to take that as a classical equation the Phi is a classical fields and we are going to quantize it and to ask what is the energy of some particular state which we will have later you need to calculate Hamiltonian of the theory and operate on that state. And you can trust that the energy will come out to be positive when we start doing that. So we will leave the single-particle interpretation of this wave function and Klein Gordon equation in all these things here.

And now we move on and repeat the steps, which we did few lectures ago and start constructing the action of this relativistic Klein Gordon theory.



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$$\phi(\vec{x},t) = e^{-\frac{i}{\hbar}(Et-\vec{p}.\vec{x})} \tag{24}$$

$$(E^2 - p^2 - m^2)\phi(\vec{x},t) = 0 \tag{25}$$
$$E = \pm\sqrt{p^2+m^2} \tag{26}$$

$$\phi^\star(\vec{x},t) = e^{\frac{\iota}{\hbar}(Et-\vec{p}.\vec{x})} \tag{27}$$

$$= e^{-\frac{\iota}{\hbar}(-Et-\vec{p}.\vec{x})} \tag{28}$$

$$\phi_R(\vec{x},t) = \frac{\phi(\vec{x},t)+\phi^\star(\vec{x},t)}{2} \tag{29}$$

We cannot drop the negative energy wave-function