Introduction to Quantum Field Theory

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Lecture 50 : Examples of Feynman Diagrams

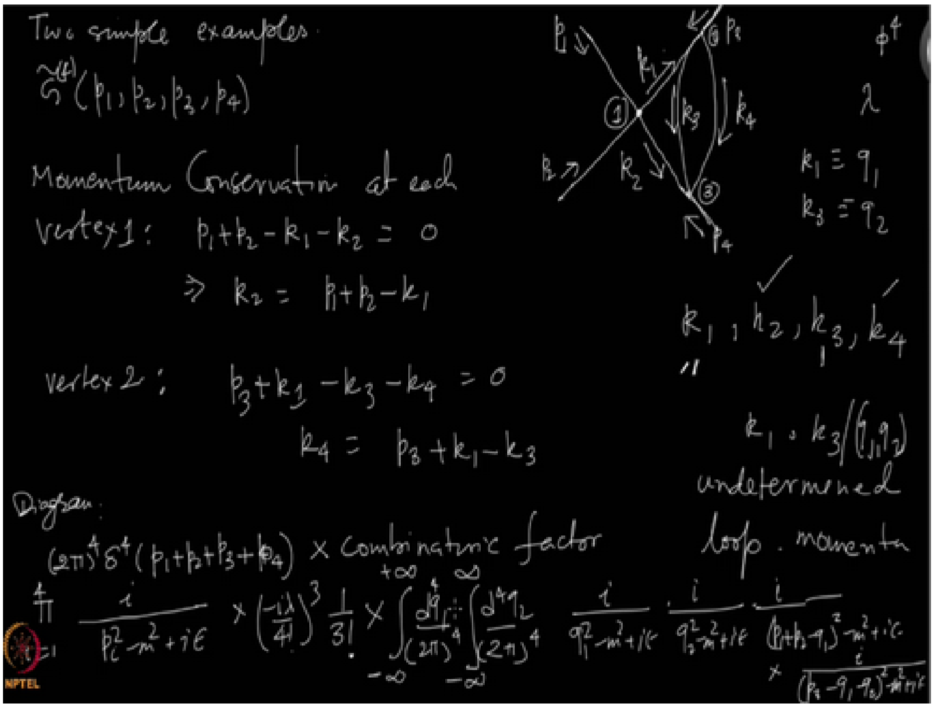

Figure 1: Refer Slide Time: 00:16

Two sample example

$$\tilde{G}^4(p_1, p_2, p_3, p_4)$$

I want to give 2 simple examples. So, first is this Feynman diagram. So, I have a 4 point vertex here and a 4 point vertex here. So, clearly this is from phi 4 theory. So, this is a 4 point function and I am interested in looking at G tilde 4. So, this is 4 point function and I am in Fourier space that is what I specify by writing G tilde and the external moment are p 1, p 2, p 3 and p 4. So, this is p 1, p 2, p 3, p 4.

And as you can see that this is an order lambda cube diagram, 1 vertex is here. So that gives you order factor of lambda and 2 vertices here so that is lambda cube. And I want to write down the expression for this Feynman diagram. So, this is easy. First let us use momentum conservation at each vertex and determine how many undetermined loop momenta we have. So, first momentum conservation at each vertex.

So, p 1 + p 2 that is coming in at this vertex and you cannot determine the momentum that will flow through both these lines. So, let us say k 1 flows through this and k 2 flows through that. So, we just say we do not know what it will come out to be and I will just give them name and then determined using momentum conservation. So, let me first label first assign a momenta to each propagator. That is what I am doing.

Let us say k 3 comes in here, it flows like that. And k 4 flows through this propagator like this. Let us call this vertex 1, this one as vertex 2, this one as vertex 3. So, at vertex 1 we have p 1 + p 2, these are external momenta. And k 1 and k 2 are external momenta which are leaving the vertex so, –k 1 and –k 2 are entering the vertex and that sum should be 0. That is what you get from momentum conservation.

Which means that k 2 I can write in terms of p 1, p 2 and k 1. So, k 2 is k 1 sorry, k 2 is p 1 + p 2 − k 1. So, this one is now, determined in terms of these remaining 3. That is good let us look at what happens at vertex 2? At vertex 2, you have p 3 coming in k 1 coming in, k 3 leaving out which means –k 3 entering the vertex, k 4 coming out so, again –k 4 entering in that vertex and that sums to 0.

Sum of all the momenta entering or exiting a vertex has to be 0. This is what momentum conservation is and from this as before you can write down k 4 as p 3 + k 1 − k 3. Good so, we have k 4. So, let us see what has happened? We have apart from the external momenta we have k 1, k 2, k 3 and k 4 of which we have been able to express k 2 in terms of k 1 and k 4 in terms of again k 1 and k 3.

So, you see you have 2 momenta, k 1 and k 3 which are undetermined which are not fixed by momentum conservation. So, good, so, there is nothing that you can do about this so, they will be there in the problem. So, now, what we can do is, we just write down the expression of this diagram and then discuss about it. We have run out of space here but let me squeeze it in here. So, expression of the diagram.

So, first factor let us put the overall energy momentum conserving delta function that you know you will get which is p 1 + p 2 + p 3 + p 4 = 0. So that is what this imposes then you have 1 2 3 this propagator and 4 propagators which do not involve momenta k 1 and k 3. They involve only the external momenta and those I can write down. So, I have product i = 1 to 4 p i square minus m square plus i epsilon. So, I have taken care of these 4 propagators. Then you have louder lambda cube expression. So, you will have –i lambda over 4 factorial cube and this is lambda cube. So, it comes to the 1 over 3, factorial and then I should multiply with the combinatoric factor. Let me write down that thing here and this is an exercise for you to do find the factor which you should multiply. And I should just point out here that in several books you will find usually that the vertex is chosen as –i lambda rather than minus lambda over 4 factorial and then the counting becomes different but we will follow this way. So, we just keep the vertices –i lambda over 4 factorial and then we just count the number of ways in which you get the same diagram without worrying about symmetry factors and other things.

Momentum conservation at each vertex

$$\begin{aligned} \text{Vertex 1} \quad : \quad & p_1 + p_2 - k_1 - k_2 = 0 \\ & k_2 = p_1 + p_2 - k_1 \end{aligned} \tag{1}$$

$$\begin{aligned} \text{Vertex 2} \quad : \quad & p_3 + k_1 - k_3 - k_4 = 0 \\ & k_4 = p_3 + k_1 - k_3 \end{aligned} \tag{2}$$

This will be a simple. So, anyhow then we have still these propagators 1, 2, 3 and 4 and for all of this we still have to include the momentum integrals. So, what I will do now is; I will rename

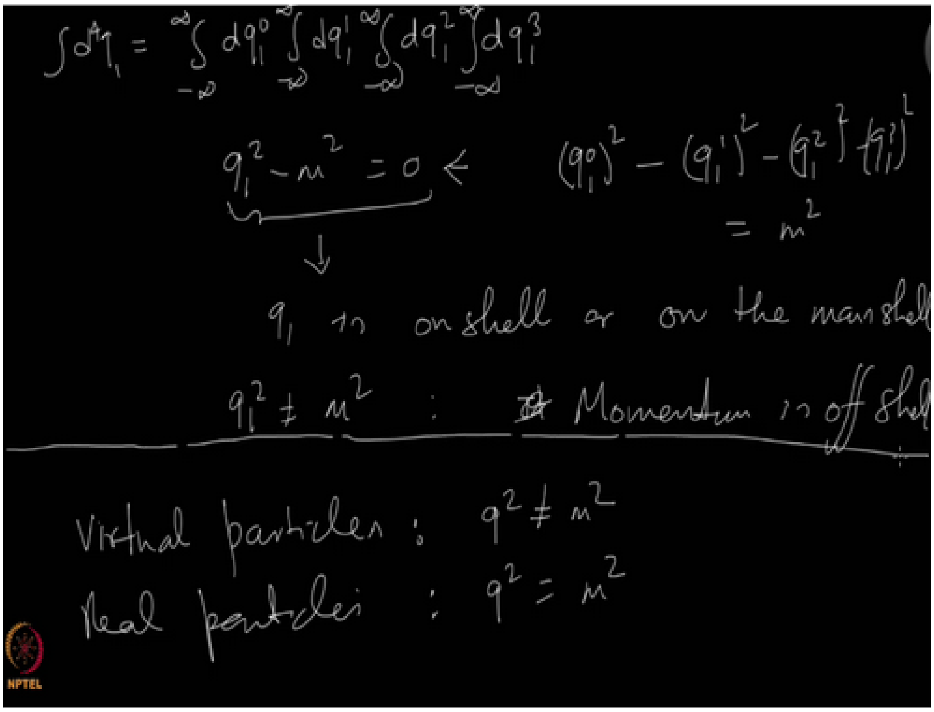

Figure 2: Refer Slide Time: 11:10

the momenta. So, instead of calling k 1, I will call it q 1 and k 3, I will call it q 2. So, this is q 1 now and it says q 2. So, d 4 q 1 over 2 phi to the 4, d 4 q 2 over 2 phi to the 4.

And then we have the propagators, i over this one will give you q square q 1 Square minus m square +i epsilon. This one will give you i over q 2 square minus m square + i epsilon and this one is p 1 + p 2 – q 1, k 1 is q 1 now, p 1 + p 2 – q 1 whole square minus m squared +i epsilon. And the last factor is i over, this is for k 4 so, you have p 3 – q 1 – q 2 whole squared – m squared +i epsilon.

So that is the expression of this and I hope you remember that all the other integrals over k 4, k 3, k 2, k 1 all the others have been already used up. So, at this stage the Feynman rule was you have to include only those integrals over only those momenta which are not which are left undetermined. All other ones have been used up when using and they have used up the delta functions at the vertices.

So, this is the simplified version of Feynman rules which we have used here. And so, you see that this is a 2 loop integral or 2 loop Feynman diagram because you have 2 undermined loop momenta. So, these moments k 1 k 3 or now, what we are calling us q 1 and q 2? They are called undetermined loop momenta and as you see that these run from minus infinity to plus infinity. And clearly because these integrals let me write them like this default q 1 is basically d q 1 0, d q 1 1, d q 1 2, d q 1 3. So, these are basically x y and z components and this is the temporal component of the loop momenta and they take all possible values which means that q 1 square – m square = 0 will not be in general satisfied by this and because this is a very specific demand that this says that q 1 zeros, the time component –q 1 1 should be equal to m squared.

But here these are completely unconstrained they take all possible values. So, there will be many configurations of these momenta for which this will not be satisfied and there will be of course, some configurations for which this will also be satisfied because they take all possible values. Now, when this is true, we say that q 1 is on shell or on the mass shell. This is what you gives you the mass shell condition.

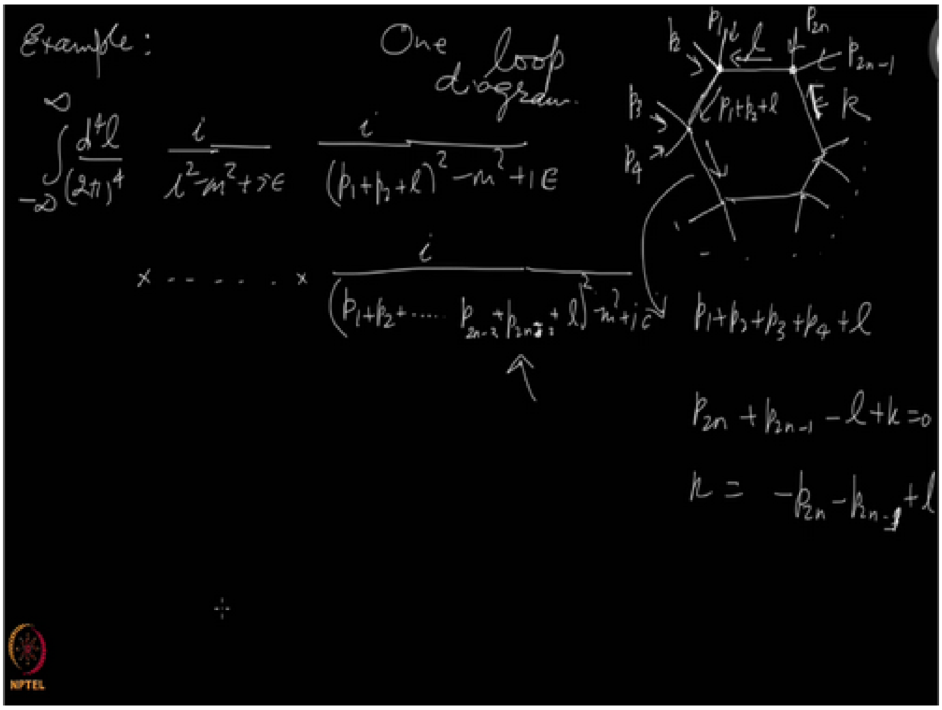

Figure 3: Refer Slide Time: 15:51

This is called as mass shell condition. So, if your momentum that is flowing through the propagator, if it the square of it gives you and by the square I mean this square of it gives you m square then it is on shell otherwise, if this is not true then we say it the momentum is off shell or off the mass shell. Let us just name for seeing whether this condition is satisfied or not.

Diagram

So, you see that the loop momenta when you are integrating y the loop momenta take off shell values and of course some of the configurations of the loop momenta will also be on shell but a majority of them or the largest chunk of it is basically off shell. So that is I think what I wanted to say about this diagram and I mean right now, we are not trying to interpret anything as particles here.

Because we have not we have just taken these as some Green's functions and or Fourier transform of these but eventually in the next course when we start talking about s matrix, you will see that these can be interpreted as particles which are moving in space and time and they are interacting at vertices. But that interpretation we will save for later and at this stage we just look at these as mathematical objects that we are trying to calculate.

But when we do that interpretation, there you will be able to see that these will correspond to real particles, real meaning on the mass shell. So, these are really relativistic particles which are satisfying the on shell condition. So, there are real particles or on shell particles. But these ones, these propagators, these will correspond to particles which are off the mass shell.

Which means; they are called that is why they are called virtual particles. So, this is for later but just to tell you this so, you say virtual particles and real particles. And virtual particle is the one whose momentum let us see, if it has a momentum q and if it is, supposed to correspond to a particle of mass m then q square is not equal to m square, so that is a virtual particle. And if q square is equal to m squared then that is a real particle.

But you do not have to worry about the particle interpretation at this stage. Let me give you 1 more very simple example of another Feynman diagram and we will write the expression for that one also.

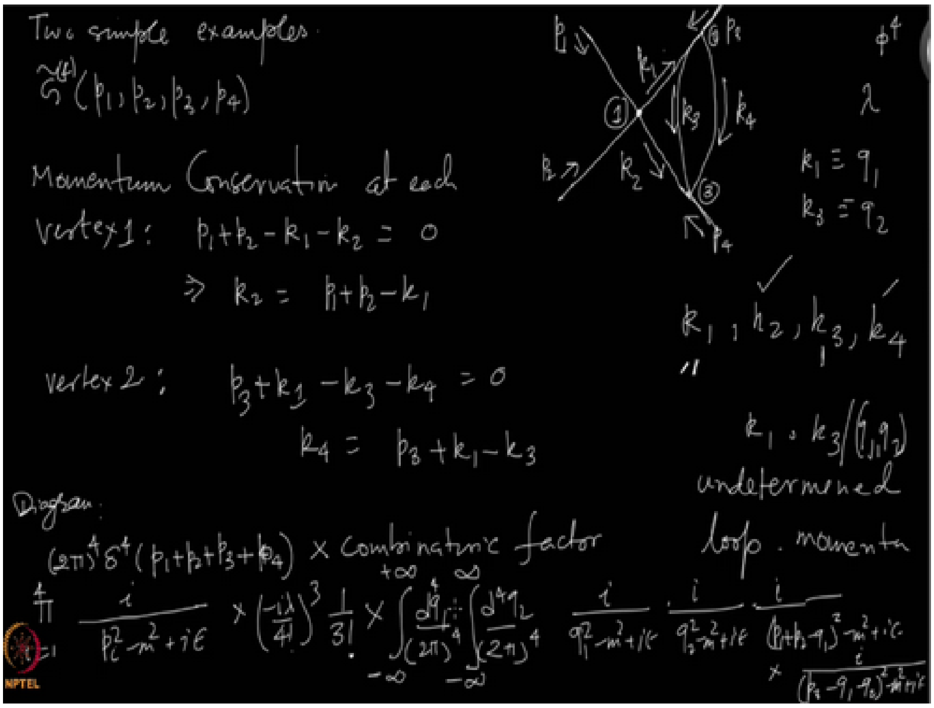

Figure 4: Refer Slide Time: 21:54

$$(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\times\text{combinatoric factor} \tag{3}$$

$$\prod_{i=1}^{4}\frac{i}{p_1^2-m^2+i\epsilon}\times\left(\frac{-i\lambda}{4!}\right)\times\frac{1}{3!}\times\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4} \tag{4}$$

$$\times\left(\frac{i}{q_1^2-m^2+i\epsilon}\right)\left(\frac{i}{q_2^2-m^2+i\epsilon}\right)\times\left(\frac{i}{(p_1+p_2-q_1)^2-m^2+i\epsilon}\right) \tag{5}$$

$$\times\left(\frac{i}{(p_3-q_q-q_2)^2-m^2+i\epsilon}\right) \tag{6}$$

The point that I want to make in this next example is that unlike here, where you have many propagators and you got a loop diagram which is 2 loop so, you might imagine that you know having lots of propagators will make it a higher loop thing but not necessarily so, and we can think of a simple diagram which is 1 loop but has many propagators and we can easily write down the expression.

So, because we are in phi 4 theory, we should have vertices which have 4 legs coming out from the vertex so, 4. So, at this at each such point you have these. So, let us say this is p 1 which enters in here; this is p 2 which enters in there and p 3, p 4 and so forth. And then you come to p 2 n, p 2 n – 1. And I could have put p n but I am putting p 2 n because we are in phi 4 theory.

And you can check that if you have odd number of ah phi's in the Green's function then that it vanishes, that you can, you will be able to check very easily. So, if you try to draw such diagrams, you will never be able to connect all the vertices and external lines, if you have odd number of external lines. Maybe I will show you one after I finish this diagram. So, let us see what it looks like. So, as before, we will assign momentum to each of the propagators.

So external ones are okay so, let us start from here, let us call it l and it flows like this. And then I should put something here but then I realize I do not have to because this one is completely fixed. So, p 1 enters at the vertex, p 2 interest at the vertex, l enters into the vertex and that is what it has to come out because of momentum conservation at that vertex. So, you have here p 1 + p 2 + l flowing in this line. Similarly, here you will have, maybe I will write here, p 1 + p 2 + p 3 + p 4 + l flowing in this one. So, you see, it will always be the undetermined moment will always be l which is in these. So, it is clearly only 1 loop diagram and I think I will leave it as an exercise to write the full expression of this. I will just like the loop integral part. So, you put all the other factors and external propagators everything.

Find the combinatoric factor but as far as loop integral is concerned, this is what we have. Again, this will be this propagator will be mostly off shell this one because you are integrating over l all possible configurations are allowed and then you will eventually have this one which you can write in different ways but so, I will just write this one minus and of course, you can use momentum conservation at this vertex.

To write this piece as involving momentum these 2, add because this plus, this plus, this plus, plus so, let us, say this is called this one, let me call it k for a moment. So, you have p 2 n + p 2 n – 1 – l that is what enters in this vertex, +k should be equal to 0. So, k is simply – p 2 n – p 2 n – 1 + l. So that is what you can put in here instead of this fact what I have written. So that is the loop momentum integral that will be here.

And clearly this is a 1 loop diagram. So, if you have a complicated diagram and you are unsure what is the loop order, this is how you can do you just assign momenta and use momentum conservation delta function at each vertex and then you can fix or figure out which ones are undetermined.

$$\int d^4q_1 = \int_{-\infty}^{\infty} d^4q_1^0 \int_{-\infty}^{\infty} d^4q_1^1 \int_{-\infty}^{\infty} d^4q_1^2 \int_{-\infty}^{\infty} d^4q_1^3 \tag{7}$$

So, there is another example and I just said let me say about the odd number diagrams with odd numbers of external lines or equivalently Green's functions which involve odd number of fields here. So, let us call phi 1 to phi 2 n +1. So, this will always be odd number. So, let us see why you will get 0 and the consequence. I mean, this is a consequence of phi going to minus phi symmetry.

So, if you change phi by minus phi, things should not change and that is what reflects into telling that if you have an odd number of fields in the Green's function that vanishes but let us see that at the Feynman diagram level. So, suppose you have you are looking at a 3 point function. So, at each external point, you remember you get 1 tick and suppose you are looking at 1 loop sorry, at order lambda.

So, when you have order lambda you have this vertex. So, you can contract this with that and 1 remains. And which means that because this is uncontracted, this vanishes. So, let me remind you why this vanishes. Remember that when we were looking at the time ordered product of the fields and this is what everything had boiled down to when we were trying to write down this expression.

And there, if we turn this into normal ordered product and that helped because once you have everything in normal ordered and you have vacuum on left and right, only those terms survive which are fully contracted. We were contracting, we had several fields here and we were contracting this. So, you see that if you have odd number of them, one will always remain uncontracted and that uncontracted one will kill the vacuum.

Sorry, yeah that is what is going to give you 0 because that will kill the vacuum. These are complex numbers, so, they were fine. But the one which we will be left, it will be an operator and it will kill the vacuum both on the left and right and it will give you 0. So that is the reason why, if you have odd number of external lines in phi 4 theory you will get 0. But if you are looking at phi cube theory where the interaction term had a phi cube then you will have a vertex.

Let us say g phi cube then you will have a vertex –i g phi cube oh sorry –i g. And then you can construct in that theory Green's functions with odd number of external lines or external points because this you can then connect, this is not vanishing, this is fine. And I will ask you to convince yourself that this is a consequence of this symmetry which you are seeing here.

So, I believe that you have now, some more experience of writing down expressions for Feynman diagrams and what loop momenta are and why as you will see later, these particles which are virtual particles. Why they are called virtual particles if they are running in the loops because they are of from the shell. So, let us meet in the next video.