Introduction to Quantum Field Theory

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Lecture 5 : A Multiparticle System of Bosons

Recap

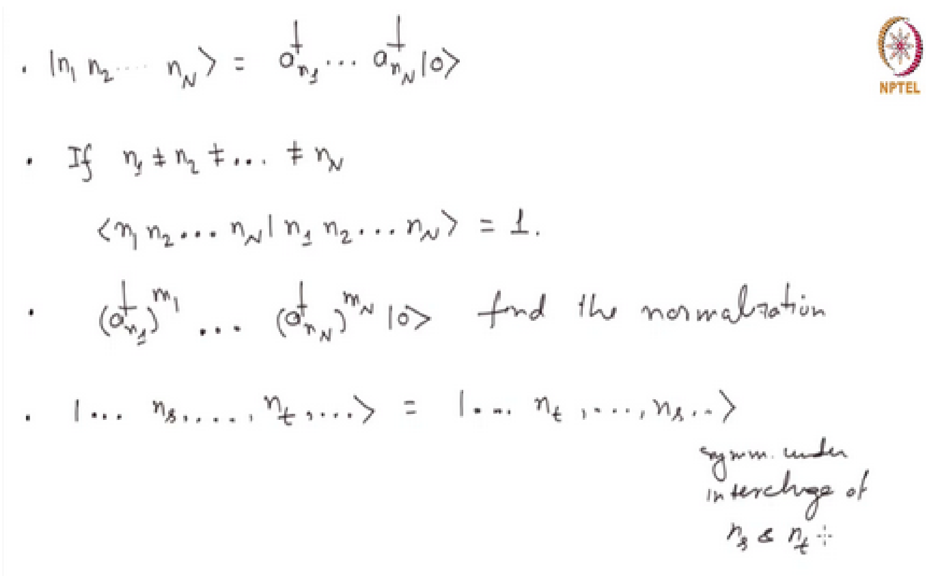


Figure 1: Refer Slide Time: 00:16

Let us start with a quick recap. So, last time we were looking at this action which is here. So, this when you take this action and you from here you find the Lagrange equations of motion you will get the Schrodinger equation and the operator small h here is this operator. And then what we did was we expanded psi like this. So, we expand it as the coefficients a n of t and times u n of x where u n are the eigen functions of this operator h.

And then we treated a n as the dynamical variable and impose the quantization condition by first calculating p n's. So, you have a n and then you calculate the corresponding conjugate momentum p n remember that p n r conjugate momentum they are not physical momentum of they are they are not physical momentum of anything in this theory these are canonical variables a n and p n. And this turned out to be i h bar a n star t and then we impose the commutation relation.

And if you see this let me suppress t you got delta n m and this theory upon quantization had a Hamiltonian which is sum of infinite number of harmonic oscillators. And then let me go to yeah next page and then we constructed the fork space and the states were the following. So, you can operate with a string of a daggers on the vacuum and that these states are eigenstates of the Hamiltonian and I also gave you an exercise to show that if n 1 is not equal to n 2 and so forth that none of these n i's are equal then in that case the state is properly normalized meaning if you take if you take this inner product then you get unity.

And I also give you an exercise to find the normalization of these states. So, if a n 1 dagger is repeated m 1 times a and capital subscript capital N dagger is subscript is repeated m sub capital N times then find the normalization. I hope you did that it is not difficult. And also we saw that the states are symmetric under interchange of any of these two. This is obvious because these are daggers and they commute. So, there is no problem.

So, symmetric under interchange of a n s and n t. So, that is the recap of what we did last time. Now let us consider a completely different system and that system will be not a second quantized theory that I am going to look at the first quantize theory. So, the system is the following. So, look at a system of capital N number of identical particles which are in the potential V of x. So, you have n number of particles they are all identical and they all are in the same potential V of x and also they are non-interacting. So, there is no interaction between these. Now I want to look at this system and I will show eventually that the system that we were discussing earlier here this one and this one are equivalent. So, these are equal; I mean at least as far as this system is concerned whatever you find here you would find in the previous description also. So, that is what I the plan. Now of course I am still assuming that these particles are non-relativistic and obey Schrodinger equation that is the first quantized version. So, we need to set up the Hamiltonian of the system and then we will find out the equations of write down the equations of motion. And then I want to construct the states in this system. So, that is the plan.

So, let us begin. So, let us label the ith particle in the system the coordinates of the ith particle by r i earlier i was using x but let me use r I, r subscript i and i because I am; so, this will be for particle number one it will be x, y. So, x 1, y 1, z 1 like this so, because these particles are not interacting with each other the Hamiltonian will be a simple sum it will be just the sum of the following terms you will have minus h bar square over 2m.

So, I am assuming all of them have the same mass because they all are identical and the partial derivatives here are in with respect to the ith coordinate the coordinate of that particle. So, V r i and then I should sum over all the particles occurring short let me write as h equal to h 1 + h 2 h N where N is fixed the when you're specifying the system you have to tell me how many particles you have in the system.

So, this is what it is or maybe probably I should write not capital H with but small h that will make it resemble what we have already done. So, h 1 and where the small h is just this. So, when I am writing gradient i I mean del over del r i which is just del over del x i del over del y i del over del z i. Sometimes I will use this notation which is nice. So, I hope you understand that when you have when you have for example this operator acting on psi wave function and the wave function will be you know a function of all these coordinates are all of them r 1 r 2 r 3 r n.

- $$S[\psi(\vec{x},t)] \;=\; \int dt d^3x \left[\psi^*(\vec{x},t)\left(i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t)\right)\right] \tag{1}$$

- $$h \;=\; \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(x) \tag{2}$$

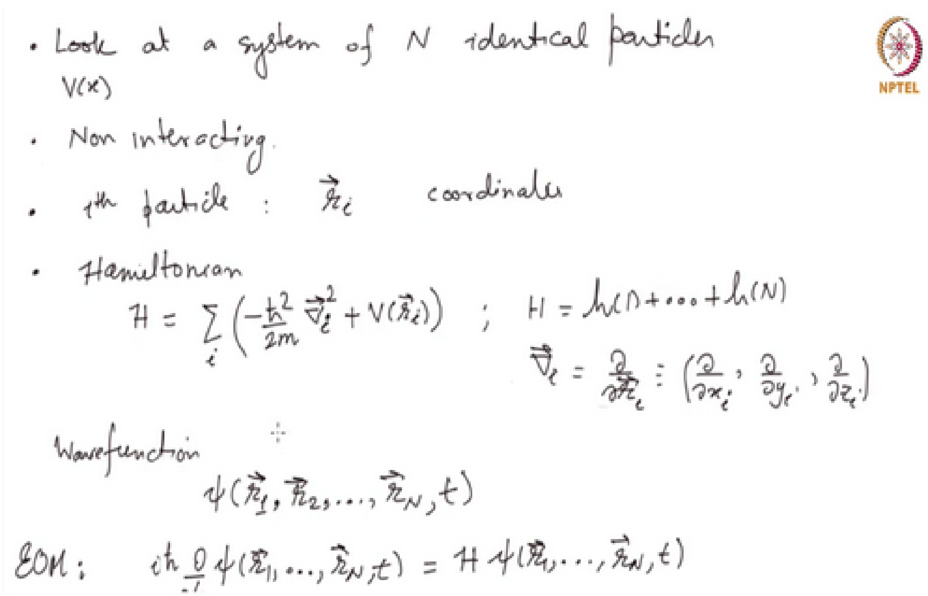
- $$\psi(\vec{x},t) \;=\; \sum_n a_n(t)u_n(\vec{x}) \tag{3}$$

- $$p_n = i\hbar a^\dagger(t) \tag{4}$$

- $$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \tag{5}$$

- $$[a_n, a_m^\dagger] = \delta_{mn} \tag{6}$$

- $$H = \sum e_n a_n^\dagger a_n \tag{7}$$



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And then, when you are taking such a partial derivative you have to differentiate with respect to the coordinates of the ith particle that is what this symbol means. So, now if I look at the wave function the wave function will be denoted by this. So, I should write psi it will be function of the coordinates of each of the particles. So r 1 r 2 and so forth r N and of course it will depend on time. And what will be the equation of motion for this system well equation of motion is the same Schrodinger equation.

So, you take i h let me write equation of motion i h bar the partial derivative of psi partial time derivative of psi and you have t and what should we have on the right hand side it should be x i right h of psi note that the Hamiltonian itself is symmetric under interchange of labels i. You see h is this. So, if you change interchange 1 with n for example here it becomes h n plus so on and so forth h 1.

So, it is symmetry and that is what you should expect because you are looking at a system of identical particles, so, it is symmetric and since the particles are identical. You know that there are two possibilities namely if you let me if you interchange particle i with j the wave function either remains the same or picks up a minus sign depending on whether these particles are bosons or fermions. So, here r j r i so, I have interchange ith and jth particles. So, this is one possibility and other possibility is the minus sign. So, plus is for bosons and minuses for fermions that is good.

So, let us look at now the eigenstates of the Hamiltonian of this theory and it is not difficult. So, earlier we were we have encountered this. So, if I take this small h which is the Hamiltonian for particle number one and I look at its eigenfunctions those we were denoting by u of n right. For a single particle the eigenfunctions we were denoting by u of n and with energy eigenvalue e n.

So, that is the eigenfunction that we can use to construct the the eigenfunctions of the full Hamiltonian and it is easy to see that u n 1 so I am putting one extra label here to distinguish between to distinguish it from the eigenstate of particle number 2. So, particle number 1 could be. So, this wave function u n one which corresponds to particle number 1. It could be the second you know second excited state this could be the third excited state and so, forth let me also indicate the see these depend on the coordinates.

So, let me in fact I can suppress r 1 r 2 because it is clear from this two that this is for the second particle but anyhow u n subscript N r N. This will be a eigenstates of the full theory will be product of the eigenfunctions of individual particles because that is easy to understand because your hamiltonian is just a sum of all the small h. So, when these this sum is acting on the product of wave functions that I have shown to you can easily see that you will again get a eigenfunction.

So, let me act with the Hamiltonian here and this is what we are looking for and let me write it down since double states are eigenstates you can easily see if you take for example h 1. So, this term here and act on this u n 1 u n 2 u n N then this will act on this one it does not do anything to these because the coordinate here is r 2 and this involves taking derivatives with respect to r 1. So, that that operator does not affect these ones at all.

- $$|n_1 n_2 \cdots n_N\rangle = a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_{N-1}} a^\dagger_{n_N} |0\rangle \tag{8}$$

- $$\text{if} \quad n_1 \neq n_2 \neq \cdots n_N \tag{9}$$
  $$\langle n_1 n_2 \cdots n_N | n_1 n_2 \cdots n_N\rangle = 1$$

- $$(a^\dagger_{n_1})^{m_1} \cdots (a^\dagger_{n_N})^{m_N} |0\rangle \quad , \quad \text{find the normalization} \tag{10}$$

- $$|\cdots n_s \cdots n_t \cdots\rangle = |\cdots n_t \cdots n_s \cdots\rangle \tag{11}$$
  symmteric under exchange of $n_s \leftrightarrow n_t$

- $$H\,|n_1 n_2 \cdots n_N\rangle = (e_{n_1} e_{n_2} \cdots e_{n_N})\,|n_1 n_2 \cdots n_N\rangle \tag{12}$$

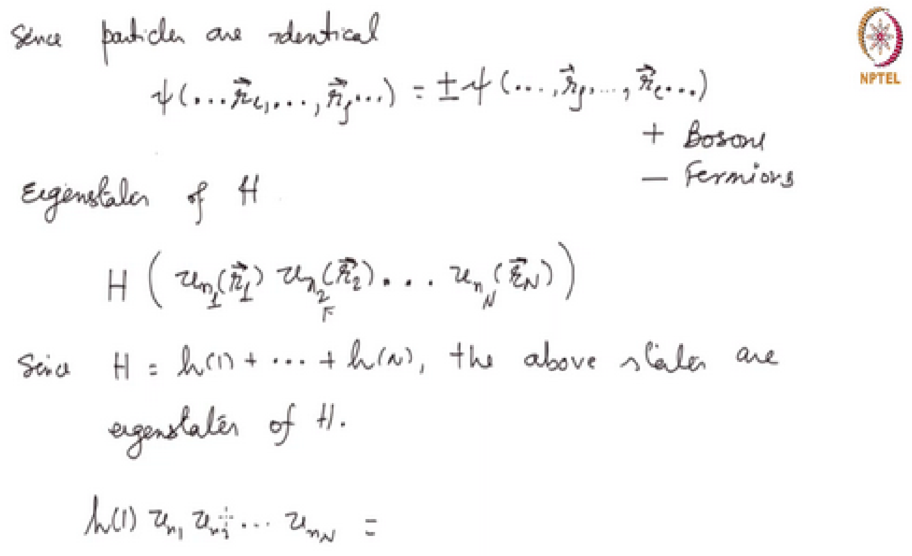

Figure 3: Refer Slide Time: 12:58

So, these all these pieces all these terms sit outside this operator and it is h 1 acting on u 1 is just e n 1 u n 1. So, you get the u n 1 back because that is an eigenstate and all these things are anyway there. So you see this is a this is how you are going to get e n 1 and then this other guy here h 2 will give you e n 2 and the same all the same factors back and similarly for each term and then you obtain the following.

So, h acting on u n 1 u n 2 u n sub N is e n 1 + e n 2 because you are going to generate another term from h 2 which will have the same form except for this vector and similarly this. So, these are the eigenstates. But now these states do not have the symmetry which I was talking about earlier that the states should be symmetric under interchange of these two with either picking up a plus and a minus sign.

So, for now I want to concentrate only on the system of particles that are bosons which means I want to pick up the plus sign here and clearly there is no such symmetry present here. So, we can symmetrize the states and then those new states which you construct they will carry the appropriate symmetry property.

- Look at system of N identical particles V(x)
- Non-interacting
- i'th particle: $\vec{r}$ coordinate
-
$$\text{Hamiltonian} \quad H = \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r}) \quad ; \quad H = h(1) + \cdots h(N) \tag{13}$$
-
$$\vec{\nabla} = \frac{\partial}{\partial \vec{r}_i} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right) \tag{14}$$

So, let me define a new defined state by the following. So, u here n 1 r 1 and you have all the factors let me u n 2 r 2 u n N r N then I want to have another term which in which these two

$$u_{(n_1\dots n_N)}(\vec{r}_1,\dots\vec{r}_N) = \frac{1}{\sqrt{N!}}\Big(u_{n_1}(\vec{r}_1)\,u_{n_2}(\vec{r}_2)\dots u_{n_N}(\vec{r}_N) + \text{permutation of } \vec{r}_1,\dots,\vec{r}_N\Big)$$

$N!$ permutation

$$u_{n_1}(\vec{r}_1)\,u_{n_2}(\vec{r}_2)\,u_{n_3}(\vec{r}_3)\dots u_{n_N}(\vec{r}_N)$$
$$+\,u_{n_1}(\vec{r}_2)\,u_{n_2}(\vec{r}_1)\,u_{n_3}(\vec{r}_3)\dots u_{n_N}(\vec{r}_N)$$

Figure 4: Refer Slide Time: 20:35

are interchanged the r 1 and r 2 are interchange and n 1 and n 2 are the same meaning particle number 1. So, here the particle number 1 is in the eigenstate u n 1 and this one is in state u n 2 I want to have another term in which you have u n 1 r 2 u n 1 r 1.

So, the particle number 1 is now in state sorry this is the same thing. So, that is what did I do u n 1 u n 2 here yeah so r 1 is the coordinate which tells about the particle right that this is for the particle number 1. So, this is saying that the particle number 1 is in the second eigenstate second excited state I mean f n 2 is 2 and and this one particle number 2o is in n1'th, eigenstate of the Hamiltonian the small h.

So, what we want to do here is we just want to make the state completely symmetric under any interchange . So, let me make it a little bit more clearer. So, if let us say I want to make this state symmetric under interchange of particle number 1 and 2 and I do not worry about other particles then I would do the following then in that case I will just do u n 1 r 1. So, I am just writing out writing down what was above r 2 u n 3 r 3 u n N r N.

And then I will add to this n 2 r 1 I have interchange r 2 and r 1 and these pieces these factors I keep the same if I do so, then I have arranged for a symmetry between r 1 r 2 interchange or particle number 1 and particle number 2 interchange but we want to symmetrize fully under the interchange of all the particles. So, I should do the following. So, I should write plus all the permutations all the permutations of r 1 r 2 r 3 so on and so forth up to r N.

Now if you do. So, just like here we permuted r 1 and r 2. So, this got permuted like this if you permute if you add all the possibilities all the permutations the state which you are going to get will be fully symmetric and there are such n factorial such permutations. So, I would divide this state by I mean this thing I will normalize by dividing by 1 over square root of n factorial and this is what I defined to be this. So, these are your eigenstates symmetrized eigenstates of the Hamiltonian good then. So, I will give you a few exercises now. Exercise: check that the states these eigenstates are normalized meaning if you calculate d cube r 1 d cube r N if you do this integral over u u star you get 1 . So, u I am suppressing r 1 r 2. Now you get 1 f and 1 is not equal to n 2 if none of these are equal. So, that is one thing you should show. And the second thing that you should show is or you find out is the following find the normalization when if you have m 1 particles in n 1 state sorry not n 1.

So, the first excited state right. So, the small h has excited has states u 1 u 2 u 3 and so forth. The n 1 n 2 n 3 they take values 1 2 3 like this. So, now we are specifying that you have m 1 particles which are in the first first state u 1 in the u 1 state and you have m 2 particles in u 2 state and so forth. And such that your m 1 + m 2 they all add up to capital N because you have total number of total number of particles is fixed to be capital N.

So, in that case you find the normalization of the states and once you do that you will realize that the following is true that if you compare the second quantized state system which we were discussing earlier before we started this discussion before we started this discussion of this new system the second quantized system that we were studying if you look at these states and compare with the states which we have constructed just.

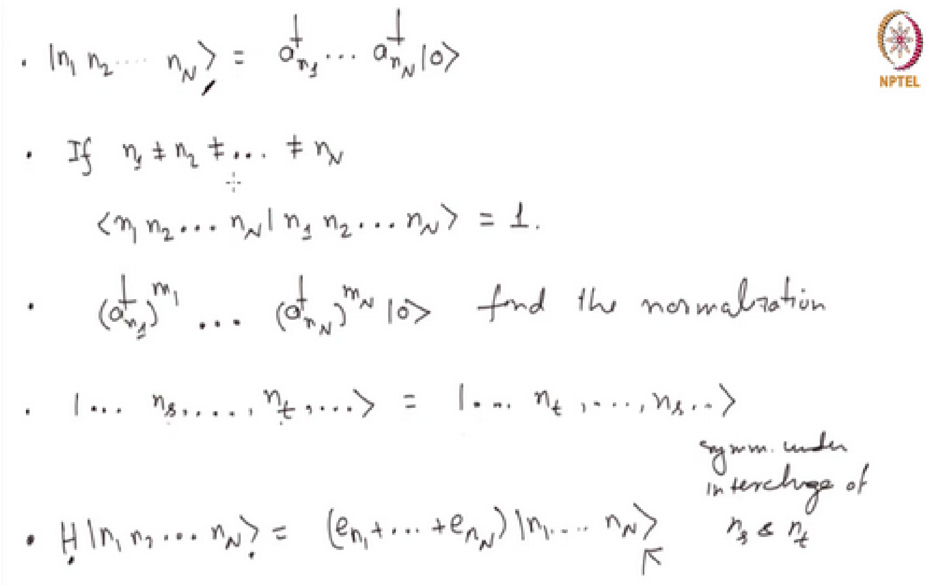

Figure 5: Refer Slide Time: 25:50

Wave function

$$\psi(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N, t) \tag{15}$$

EOM

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N, t) = H\,\psi(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N, t) \tag{16}$$

$$\psi(\cdots \vec{r}_i \cdots \vec{r}_j \cdots) = \pm\psi(\cdots \vec{r}_j \cdots \vec{r}_i \cdots) \tag{17}$$

$+ \rightarrow$ Boson
$- \rightarrow$ Fermion

Eigenstate of H

$$H\Bigg(u_{n_1}(\vec{r}_1)u_{n_2}(\vec{r}_2)\cdots u_{n_N}(\vec{r}_N)\Bigg) \tag{18}$$

Since

$$H = h(1) + h(2) \cdots h(N) \tag{19}$$

The above states are eigenstates of H.

$$h(1)\, u_{n_1}(\vec{r}_1)u_{n_2}\cdots u_{n_N}(\vec{r}_N) = e_{n_1}\, u_{n_1}(\vec{r}_1)u_{n_2}\cdots u_{n_N}(\vec{r}_N) \tag{20}$$

Now these ones here this one for example and similarly the ones which you know where several of them are in the same state u 1 or u 2 and similarly here if you consider these ones you will see that there is a one-to-one correspondence between the states of this system and the states of that system. Also I think I did not write here but I should have maybe I can do it. Now here I would like to add that.

Let us do here, if you take Hamiltonian and act on this state you get the sum of energies. So, you see that there is a one-to-one correspondence between these states and the states that we have here in this theory. So, if you take this state and this Hamiltonian you get exactly the same thing as you are getting there right under this Hamiltonian which is very different which is Hamiltonian of an infinite number of harmonic oscillators each with a different frequency.

But as far as if you say I will this Hamiltonian corresponds to the Hamiltonian of that theory and this state corresponds to the eigenstate the states of that theory then you see there is a one to one correspondence and the symmetric symmetry properties are same under interchange of any two labels. So, if you interchange any of the two it is symmetric and here also we have constructed the states which are let me go here are symmetric under interchange of any of the two these are normalized they are normalized.

So, you can conclude that this system which is a system of n capital n non-interacting bosons which are all under which are all in the same potential V of x, V of which are all in the same potential v this system is equivalent to a second quantized system which is given by a infinite number of harmonic oscillators. The advantage there is that though in this case when you want to add one more particle let us say instead of capital N you want to go to capital N + 1 your Hamiltonian changes and you know you have to construct states differently but there you do not have to change anything.

$$\begin{aligned} u_{(n_1\cdots n_N)}(\vec{r}_1\cdots\vec{r}_N) =& \frac{1}{\sqrt{N!}}\Big(u_{n_1}(\vec{r}_1)u_{n_2}(\vec{r}_2)\cdots u_{n_N}(\vec{r}_N)\Big) \\ &+\text{permutation of } \vec{r}_1\cdots\vec{r}_N \end{aligned} \tag{21}$$

N! permutations

$$\begin{aligned} & u_{n_1}(\vec{r}_1)u_{n_2}(\vec{r}_2)u_{n_3}(\vec{r}_3)\cdots u_{n_N}(\vec{r}_N) \\ + \quad & u_{n_1}(\vec{r}_2)u_{n_2}(\vec{r}_1)u_{n_3}(\vec{r}_3)\cdots u_{n_N}(\vec{r}_N) \end{aligned} \tag{22}$$

Exercise: Check that the states

$$u_{(n_1\cdots n_N)}(\vec{r}_1\cdots\vec{r}_N) \;\; \text{are normalized}$$

$$\int d^3\vec{r}_1\cdots d^3\vec{r}_N\, u^*_{(n_1\cdots n_N)}\, u_{(n_1\cdots n_N)} = 1 \tag{23}$$

$$\text{if}\, n_1 \neq n_2 \neq \cdots \neq n_N \tag{24}$$

Exercise: Find the normalization when

$$\begin{aligned}
&m_1 \text{ particles in } u_1 \text{ state}\\
&m_2 \text{ particles in } u_2 \text{ state}\\
&.\\
&.\\
&m_n \text{ particles in } u_n \text{ state}
\end{aligned} \tag{25}$$

Where,

$$m_1 + m_2 \cdots m_n = N \tag{26}$$

It is the same thing you are just exciting more number of oscillators. So, this is the equivalence between these two systems and I could go further and discuss how different operators are defined in this first quantized theory which we discussed just. Now and what are their corresponding operators in the second quantize theory that we discussed earlier but I would leave it as an exercise for you to do or read some book or other literature.

And what I wish to take up next is look at the start looking at relativistic theories. So, I will leave this study of multi-particle system which we undertook here the reason I undertook was because it makes the step of second quantization very easily understandable in the context of a very familiar equation Schrodinger equation and this multi-particle system I want to leave now and want to start looking at relativistic field theories and then we will quantize them.

So, the next thing I would do is look at Klein-Gordon theory at the classical level and then we will quantize it. So, that is the plan for the next video.