Introduction to Quantum Field Theory

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Lecture 49: Cancellation of Bubble diagrams

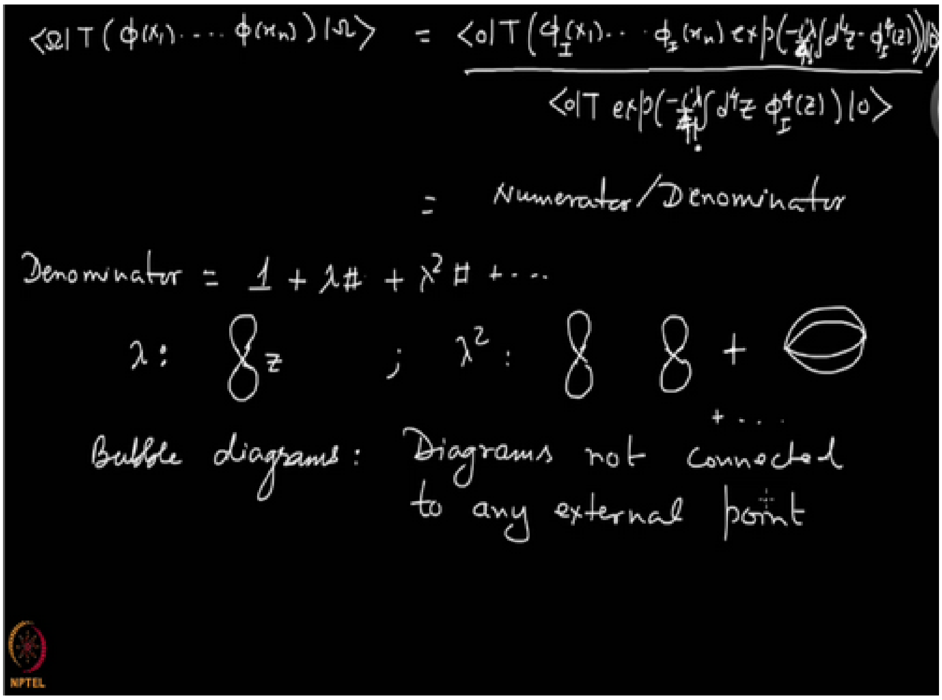

Figure 1: Refer Slide Time: 00:14

Let us recall the formula for Green's functions that we wrote which was this, if you have n external points x 1 to x n and you are looking at this correlation function in the interacting field theory we wrote down the master formula as. Then you had the exponential factor, this expression is in field theory. And then we have to divide this by the same thing as above except for the external fields.

And this all is an interacting picture and phi i's evolve according to free field equations of motion, sorry yeah, I have forgotten a factor of –i lambda over instead of h bar was 1; I should put a 4 factorial. So, this is what we had and then we wrote down diagrammatic expressions for these objects in perturbation theory. Look at the let us look at the kind of diagrams that you have in the denominator.

So, here there are no external points, there is no x 1, x 2 or x n. So, here you have only internal vertices. So, I will first write it as numerator divided by the denominator. By the numerator I mean what you have in here and by denominator I mean what you have in here. So, we have been writing Feynman diagrams for the numerator, let us look at denominator.

So, denominator has this, if you expand the exponential the first term would be 1 then you will have an order lambda term and then you will have order lambda square term and so forth.

And at order lambda you will have the kind of diagrams that will appear will be the following. So, let us say you draw this vertex with level z, internal point is called z and then the only diagram that you can have is this one.

If you are at order lambda square then the kinds of diagrams that you can have are of these forms. So that will order lambda square you get 1 diagram which is this, you also get a diagram which is, I will just add them up which will be this one and so forth. So, these kinds of diagrams which are not connected to any external point, these are called bubble diagrams. Diagrams not connected to any external point.

So, this diagrams are called bubble diagrams. Let us look at the numerator again and I want to this time look at 4 point functions or 4 point Green's function, I am looking at the numerator. So, at order lambda 0 will have so, let me label x 1, x 2, x 3, x 4 shall have these kinds of diagrams. They are not, their net order lambda will have this vertex lambda here. Then this is an order maybe I will write it in the next line. So, this lambda, these are lambda 0 terms and it means the right I will write, let us say G 4. So, I am writing the expansion of this. So, at order lambda you have lambda to the 0, this, these terms. Then at order lambda, you have terms like for example, this term is there, you also have terms like. So, this does not contribute any factor of lambda but this one contributes.

So here for example, the other diagrams and then at order lambda square, you will have among many other diagrams, the following. So, this contributes order lambda, this also contributes order lambda. So, these 2 vertices they make it order lambda square and you will also have these and this is at higher orders. This is a order lambda cube, this one is order again lambda cube 1 2 3.

So, many such diagrams maybe 1 more; I will draw to make it clear. So, you can also have a diagram of this out and a diagram of this out, you can count how many factors of lambda you have to multiply. So, you have all such kinds of diagrams appearing different orders. Now, I want to look at one particular diagram which is this one, I choose it at random there is no special reason for choosing this.

But the point I want to make can be made through this or any other diagrams. So, I choose one. So, let us look at the diagram, this. So, these are the external points x 1, x 2, x 3 and x 4. Let me label the internal points z 1, z 2, z 3 and z 4. So, what is the expression of this diagram? It is I write the Feynman propagator for each of these lines and include all the other factors. So, I will do it. So, the expression is the following. Let me give it a name. Let us call it, F, F of x 1 x 2 x 3 x 4.

So, this diagram I have given a name which is F and this is equal to D F x 1 - z 1 so that takes this propagator D F of x 2 - z 2, this propagator and all the other ones times D F of so, this one z 1 - z 2 and the second one also D F of z 1 - z 2, this one. And then I should write down all the, not yet, I have these ones. So, this is D F of z 3 - z 3, D F again z 3 - z 3 for this one and similarly the one or the other bubble.

Now, let us put all the factors of –i –i lambda over 4 factorial, you have 1 2 3 4 so, you get –i lambda over 4 factorial raised to the power 4. This is order lambda 4 so, this, when you are expanding the exponential to 4 th power to the 4 th order that is what gives you lambda 4. It gives you –i lambda over 4 factorial raised to the power 4 and 1 over 4 factorial because of this.

So, this is the 4 factor I am talking about. So, you get 1 over 4 factorial times the combinatoric factors. Let me write it. So, the factor, I will write on the next line. So, let us count how many ways. So, let us see x 1 x 2 x 3 x 4 here 1 vertex here, your 1 vertex, second vertex, third vertex and 4th vertex. So, you have 1 2 3 and 4. So, let us count. What is the factor here? So, x 1, this point is x 1 x 2 x 3 x 4.

So, I will urge you to do the calculation yourself first and then try to match with what I have got. And see whether you agree. So, the first one x 1 could go to any of the 8 or some other

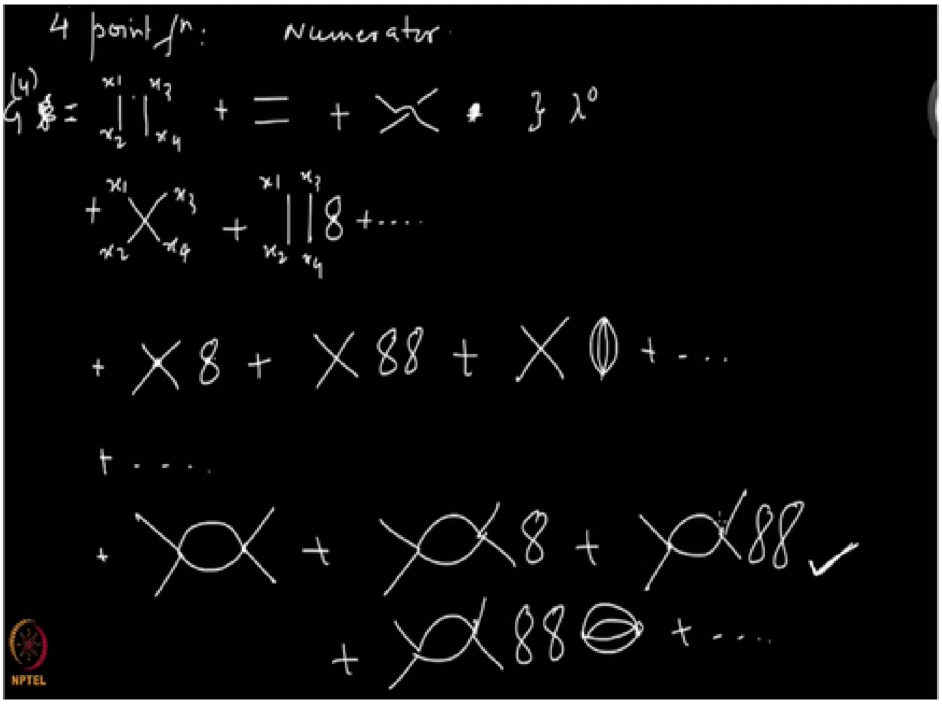

Figure 2: Refer Slide Time: 05:01

number, it could go here or it could go to this vertex or it could go to that vertex or it could go to this vertex. So, you have 16 possibilities because there are 4 sides with each having 4 little legs.

$$\langle\Omega|\phi_I(x_1)\cdots\phi_I(x_n)|\Omega\rangle = \frac{\langle 0|\,T\left(\phi(x_1)\phi(x_2)\cdots\phi(x_n)\text{exp}\left[\frac{-i\lambda}{4!}\int d^4z\,\phi_I^4(z)\right]\right)|0\rangle}{\langle 0|\,T\left(\text{exp}\left[\frac{-i\lambda}{4!}\int d^4z\,\phi_I^4(z)\right]\right)|0\rangle} = \frac{\text{Numerator}}{\text{Denominator}} \tag{1}$$

Denominator$= 1 + \lambda\# + \lambda^2\# + \cdots$

Bubble diagram: Diagram not connected to any external point
4-point function: Numerator
let us look at the diagram

So you have 16 possibilities, x 1 could go to any of the 16. So, let us connect it to one of them and whichever we connect, we call it z 1. So, if I had connected this one here, I would have called this one is z 1. It does not matter which one I am calling z 1 so, I will just connect here for ease. So, you have 16 possibilities then, this one has to go to the same vertex to which x 1 is connected, it cannot go to here. And because that will give a diagram, it does not look like this. So, this one has only 3 possibilities so, it goes to one of them. Then x 3 can go to any of these 3 vertices so, you have 12 options. So, it can go to any of the 12 points so, let us say it goes to one of them and whichever it connects to I will call it z 2. So, if I had connected this one here instead, I would have called this one is z 2 and everything will still look the same.

So, this is z 2. Now, z 4 has to go to the vertex to which x 3 is connected. So, z 4 has only 3 options. So, it goes there, it has 3 options. Now, this one can connect only to this because z 1 and z 2 are connected. So, this has only 2 possibilities, it goes either here are there. So, it takes 2

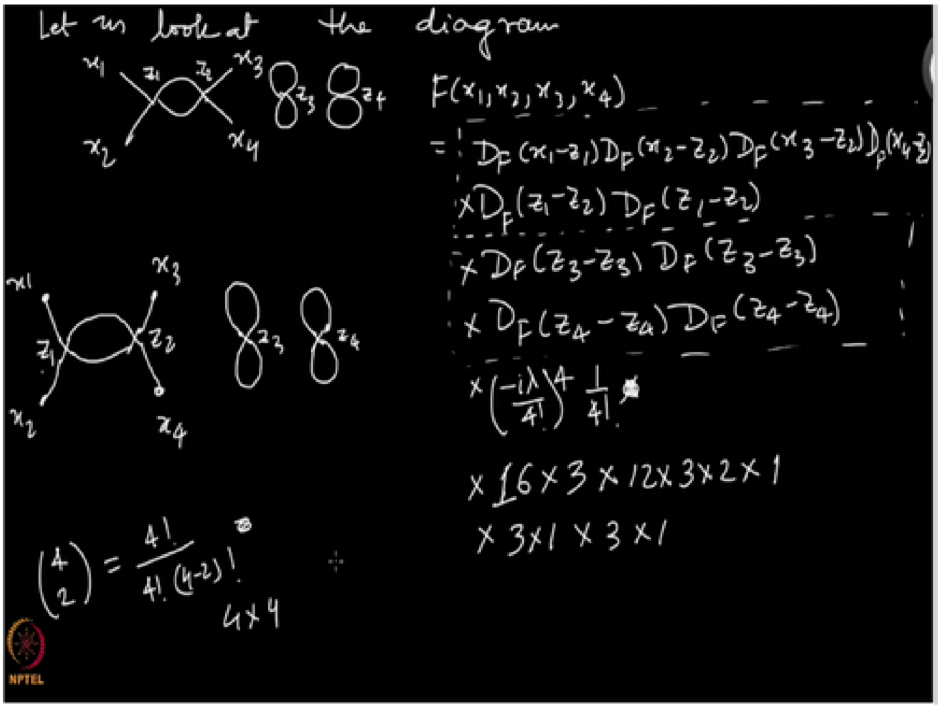

Figure 3: Refer Slide Time: 08:21

and there is only 1 possibility here. Then here, this can only connect to itself, it cannot connect here because that will be very different looking diagram.

So, we have to connect this to itself and there are only 3 possibilities. So, it could go here, here or here. So, it goes to one of them. So, there are 3 possibilities. Then there is only 1 and the same for this which is again 3 times 1. I think I got the factor right and if I made a mistake, we will discover it. So, anyway this is what we get and this is the full expression. Now, you see that this is looking like this object of course has all these propagators which make up this one and this one.

So, if you see z 3 and z 4 up to here, this one and then. So, this one has propagators of exactly this one and this piece has the one here below, the remaining these 2 lines have propagators only have these 2. So, we can think of this entire diagram as product of this piece times this piece. But not quite because the factors may not work out correctly. And let me show you what I mean by that.

$$\begin{aligned}
= \quad & D_F(x_1 - z_1) D_F(x_2 - z_2) D_F(x_3 - z_2) D_F(x_4 - z_2) \\
& \times D_F(z_1 - z_2) D_F(z_1 - z_2) D_F(z_3 - z_3) \\
& \times D_F(z_3 - z_3) D_F(z_4 - z_4) D_F(z_4 - z_4) \\
& \times \left(\frac{-i\lambda}{4!} \right)^4 \times \frac{1}{4} 16 \times 3 \times 12 \times 3 \times 2 \times 1 \\
& \times 3 \times 1 \times 3 \times 1
\end{aligned} \tag{2}$$

So yeah, let me do there. So, this I have done. Now, I will look at a different kind of a problem. So, let us look at F prime x 1, x 2, x 3, x 4 which is not the same as F. At least to begin with, it might turn out to be the same as F but at this moment, we do not have a reason to believe it is the same. Which is this? So, I am saying that this guy is this Feynman diagram. So, I am

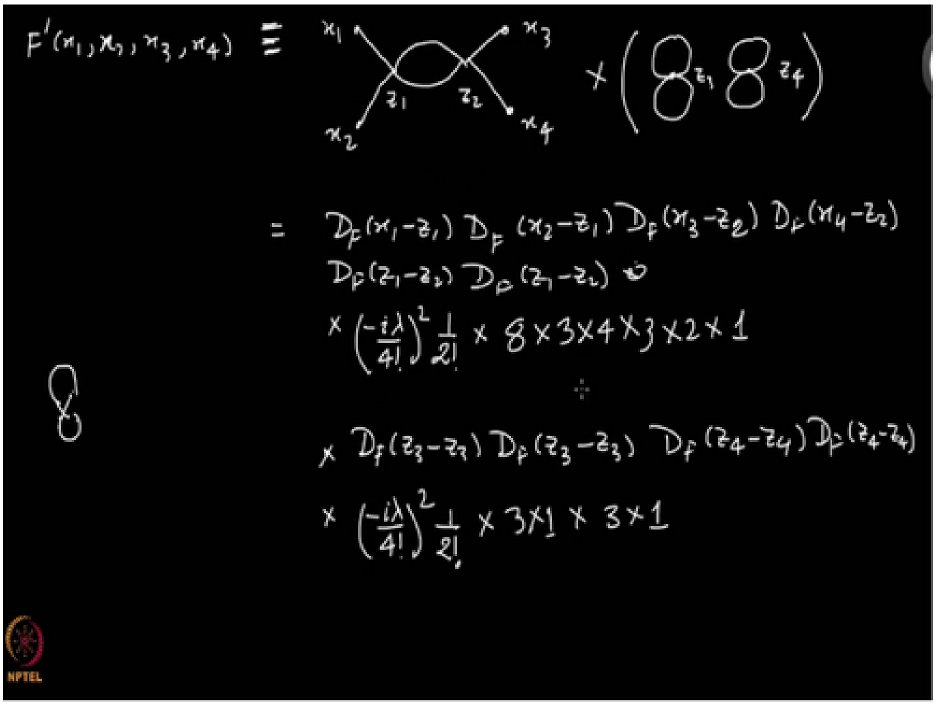

Figure 4: Refer Slide Time: 17:08

defining F prime to be this. So, F prime is defined to be a product of these 2. So, let us write down the expression of this one and then we will compare it with the expression of F.

So here, you get all the same propagators. Let me write it. It is boring to write but I will write nevertheless. So, this diagram is. See, the way I have written is you are given 2 diagrams and you are just multiplying them. This is different from the previous case, where I was given 1 diagram, this is 1 Feynman diagram and we wrote it this way. Now, I am given 2 Feynman diagrams. So, I should calculate this separately and calculate this separately.

So for this one, what are the other factors? So, this is order lambda square diagram. So, it has a - i lambda our 4 factorial raised to the power 2. This comes by expanding the exponential to second power. So, you have 1 over 2 factorial and let us write down the combinatoric factor for this. So, point x 1 has 8 places to go, 8. And this one has 3 places to go, 3 and then this one has 4 places to go, 4, this one has 3 places to go, this one has 2 places to go and finally one.

I did not, do it the way I used to because I believe that I have done it earlier many times. So, I am writing the expression. So, this is this one times, let us write this question for this. So, D F of z 3 - z 3, D F of z 3 - z 3 then D F of z 4 - z 4, D F z 4 - z 4. So, I have taken care of all the propagators. So, you have 1 2 3 4 propagators and that is here. Now, again this is an order lambda square diagram because there are 2 vertices here and there.

$$
\begin{aligned}
F'(x_1, x_2, x_3, x_4) &= D_F(x_1 - z_1) D_F(x_3 - z_2) D_F(x_4 - z_2) && (3)\\
&\quad \times D_F(z_1 - z_2) D_F(z_1 - z_1) \times \left(\frac{-i\lambda}{4!}\right)^2 && (4)\\
&\quad \times \frac{1}{2!} \times 8 \times 3 \times 4 \times 3 \times 2 \times 1 && (5)\\
&\quad \times D_F(z_3 - z_3) \times D_F(z_3 - z_3) D_F(z_4 - z_4) D_F(z_4 - z_4) && (6)\\
&\quad \times \left(\frac{-i\lambda}{4!}\right)^2 \times \frac{1}{2!} \times 3 \times 1 \times 3 \times 1 && (7)
\end{aligned}
$$

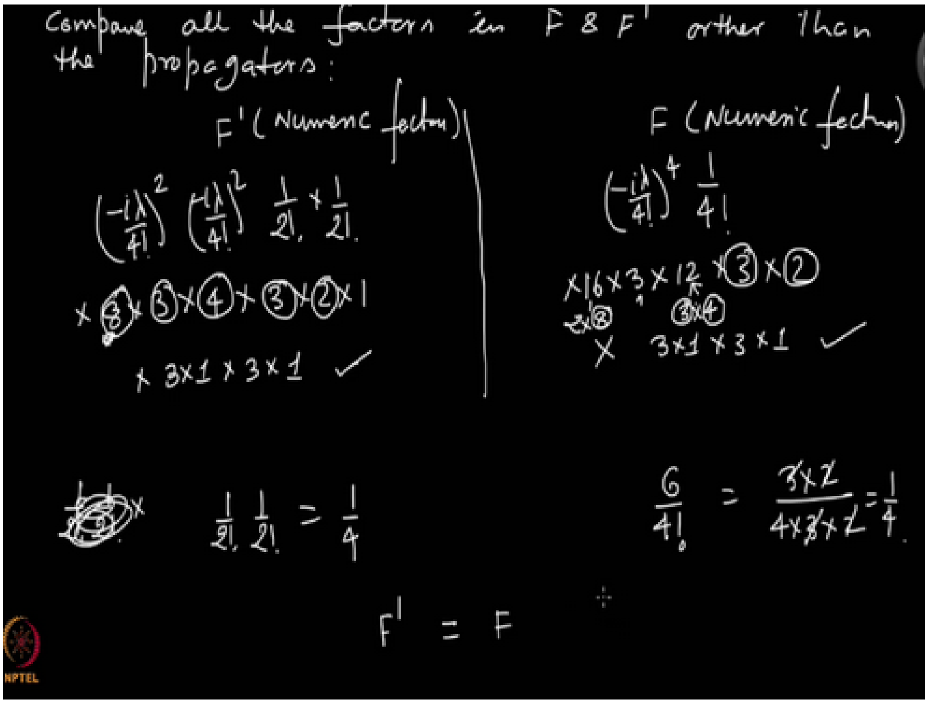

Figure 5: Refer Slide Time: 21:48

So, it also gives you –i lambda over 4 factorial squared, again because of the exponential expansion you get 1 over 2 factorial and then we have to count the combinatoric factor which is simple. Again, this one can go to any of the 3. So, it is 3. And then there is only 1 possibility, 3 times 1 into again same. So, this is what I meant by the product. So, you see, it is clearly this diagram, all its combinatoric factor and other factors times this one.

And here you should think of this as 1 diagram, not product of 2 diagrams. So, you see the difference, the factors in this and that are different. So, let us compare the combinatoric factors, not just all the numerical factors rather than combinatoric factors. So, let us compare, let me write down here.

So, let us compare all the factors in F and F prime, other than the propagators. Because propagators are identical in both there is no difference. Whatever you have in F, exactly the same ones appear in F prime also. So, they are identical, the only possible differences is, I mean the differences are only due to the other factors. So, let us write them here. So, let us write F prime and F. Let us see, so, I remember F prime has from the bubbles, it has 3 times 1 times 3 times 1 and this also had 3 times 1 times 3 times 1 that was coming from the bubbles. And then there was this factor, 6 let me look at my notebook so that I can write easily. So, this had 16 times 3 times 12 times 3 times 2 times 1. And then it had sorry which one I am writing. It is other way round, this was for F. So let me write here. This is wrong. I will erase it, maybe I should first erase it. So, it was 16 times 3 times 12 times 3 times 2. And F prime had the space yeah, F prime had 8 times 3 times 4 times 3 times 2 times 1. And then this is F prime. This have taken care of, this one comes with the 1 over 2 factorial. And this one comes also with a 1 over 2 factorial. And they both also have –i lambda over 4 factorial square. So, I have this was F prime.

So, you have –i lambda over 4 factorial squared then –i lambda over 4 factorial squared then you have 1 over 2 factorial times 1 over 2 factorial. And let us go back to the discussion of F, you had –i lambda 4 factorial to the power 4 and 1 over 4 factorial and let us compare these 2 now. So, this line is, so, these are all multiplied of course. So, this factor is same as this factor.

The factors of –i lambda 4 factorial, they are identical. So, let us look at this one. This one is 1 over 2 factorial times 1 over 2 factorials time the space so, this 8 I can write as sorry, this 16 I can write is 2 times 8, this 12 I can write as 4 times 3 and now, let us compare. So, this same the 4 is here, I will look at this and this 3 is there and then this 8 is here. So, what is the extra thing? The one which is extra is this 3 and these 2.

So, which is 6. So, you have 6 over 4 factorial and other factors are identical and here you have 1 over 2 factorial times 1 over 2 factorial which is 1 over 4 and this is again 1 over 4. So, you see that F prime is equal to F. Here I was looking at only the numeric factors but once you multiply it to these numeric factors the propagators which are identical in both the cases, we realize that the F prime and F are identical expressions. Which means that I have the following relation that this Feynman diagram is same as product of this times this. Even though they do not appear to be same but we have seen explicitly that the combinatoric factors and other factors combine in such a way that all the factors are same on both sides. So, this is what we have proved. And let us see why this has come out to be true. And we will see that this is not something that has happened accidentally for this particular diagram but rather it works for every diagram. Meaning, if you have a diagram which has some bubbles in it then that diagram can be written as a product of diagram with the bubbles times the diagram the part that appears in here without bubbles that appears as a product here.

So, this is going to be true in general and let us see why it has worked out this way. So, let us look at any diagram of this variety where you have some piece which does not have any bubbles and may have any number of legs. So, this one has no bubbles, any number of legs it may have and then there is this piece which has some bubbles. So, this diagram could be for example, this one, it could represent let us say this, whatever you prefer. This could be diagram which is this.

So, as I said, this is not 2 diagrams, not a product of 2 diagrams, 1 single diagram. So, I am still drawing this so, this could be something of this out. For example, this diagram could denote this and so, the claim we want to make is that this can be written as this diagram, standing on its own times the same bubble diagram which you have here, and the combinatorics factors will work out so that this equality holds.

And now, let me argue that this is indeed true. So, here, suppose you have n vertices in this and in this diagram which means in this case, you have 1 2 3 4 5. So, in this for this example, n will be 5. Let us look at this one, it is 1 2 3 4 5 that is looking too much of. So, we can have this and we can also have this. So, let us say this is actually this diagram then 1 2 3 4 5 6 7. So, this has 7 vertices.

So, I say that in general this will have some m number of vertices and m is 7 in this example. Now, when you are calculating the propagators, in this writing down the propagators, it will be identical on this case, so, I do not worry about it. The difference was coming from these pieces because instead of having a 1 over 4 factorial in this F, you had a product of 1 over 2 factorials in

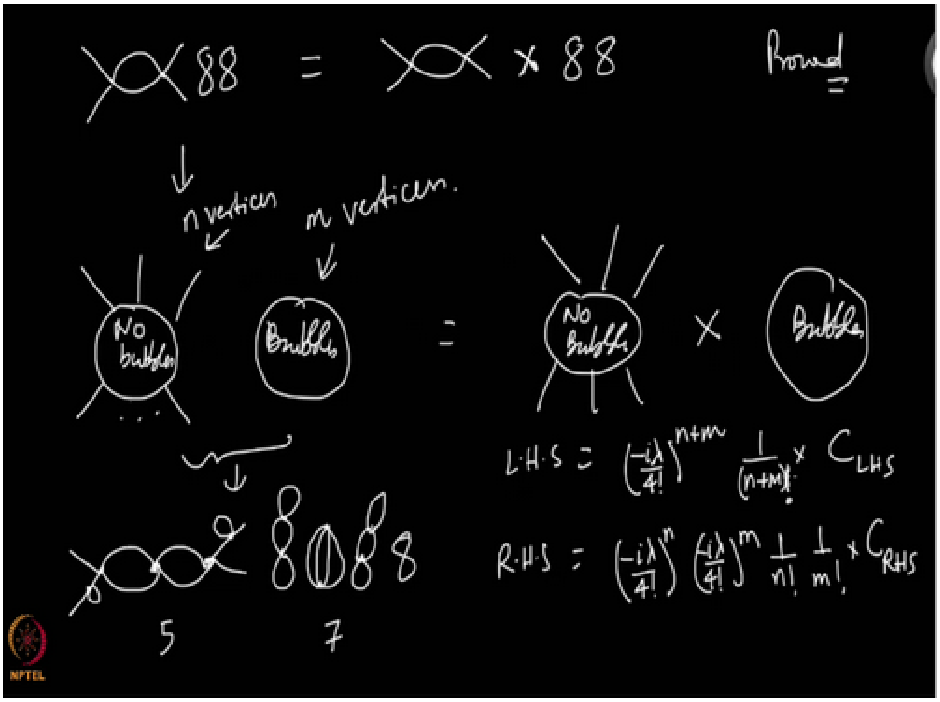

Figure 6: Refer Slide Time: 27:19

F prime.

So that will be the 1 source of difference. So, let us first calculate the factors of these, this vector is coming from the expansion of exponent and I do not worry about –i lambda over 4 factor because they are identical in both. So, this one left hand side will have so, this diagram becomes of order n + m. This will have n + m vertices. So, it is an order lambda n + m diagram.

And then you will have maybe I will write this piece just to help reading the order. And then this comes with 1 over n + m factorial times the combinatorics factor of this entire thing. So, let us call it C left hand side. C for combinatoric and for the left hand side. Now, right hand side will be a –i lambda 4 factorial power n because there are n vertices in here power m because m vertices here so that makes n + m again.

Or maybe I can write in 2 steps. And then this one will come with 1 over n factorial, this one will come with 1 over m factorial times the combinatoric factor on the right hand side. So, now, let us look at the combinatoric factors on the 2 sides and then compare. So, here when you are writing down the combinatoric factor for this diagram, you do the following. Let us go back to how we did that one.

Compare all the factors in F and F' other than the propagators

$$\mathbf{F'} = \left(\frac{-i\lambda}{4!}\right)^2 \times \left(\frac{-i\lambda}{4!}\right)^2 \frac{1}{2!} \times \frac{1}{2!} \times 8 \times 3 \times 4 \times 3 \times 2 \times 1 \quad (8)$$

$$\times 3 \times 1 \times 3 \times 1 \quad (9)$$

$$\mathbf{F} = \left(\frac{-i\lambda}{4!}\right)^4 \times \frac{1}{4!} \times 16 \times 3 \times 12 \times 3 \times 2 \quad (10)$$

$$\times 3 \times 1 \times 3 \times 1 \quad (11)$$

$$\mathbf{F'=F}$$

$$LHS = \left(\frac{-i\lambda}{4!}\right)^{n+m} \times \frac{1}{(n+m)!} \times C_{LHS} \tag{12}$$

$$RHS = \left(\frac{-i\lambda}{4!}\right)^{n} \times \left(\frac{-i\lambda}{4!}\right)^{m} \times \frac{1}{n!} \times \frac{1}{m!} \times C_{RHS} \tag{13}$$

So, here, this piece. When we were writing here, the x 1 had the possibility of going to any of these 4 vertices that I could have connected here or there or there or there. And that is why you got a factor of 16. The 16 was basically 4 times 4 because it is each one is 4 possibilities but they are 4 vertices. So, what I can do is I can think of it as this diagram going this x 1 going to so, I just count the combinatorics factor of this one.

And then I allow for the possibility that it could have gone to different vertices, different ones. So, when x 1 is going to connect to any of these 4, it had the following number of choices. I mean the vertices to which it could go towards the following. So, you had to begin with total of 4 vertices and you have selected 2 of them. So, now, I am looking at this piece actually and this is made up of 2 vertices.

So out of these 4 vertices, I have selected 2 of them. And that is this number. So, how many number, in how many ways you can select out of 4 vertices, 2 vertices it is this which is. And then on the top of it, you just have to calculate the combinatorics factors of these ones and these ones separately and because there is nothing else, let me show you. So, this is, what is this number? Let us go here, to be 0 I think there.

So, you see when I am drawing this diagram, the vertices I have so, you think of having certain vertices placed here and then you draw this diagram, but you could have chosen the vertices in this one also to draw and the total number of possibilities you have is the following. So, you have total of so, I am writing C LHS, you have total of n + m vertices and out of which you have chosen to use n of them to make your, one will come with whatever vertices are left, they make up the bubble and it comes with whatever combinatoric factor it has. And I forgot to say that this also comes with whatever other combinatoric factors it has.

So, the combinatoric factor of this times, the combinatoric factor of this one times the different choices that you were that you had for choosing the vertices from the remaining ones. So, these are the total number of these are the different combinatoric factors that you have. If it is not sounding very clear which it could be, you fall back to this example that I gave and then it is clear.

Let us look at the, what happened, it is frozen, it is not writing. Let us look at the right hand side. Right hand side had just C corresponding to no bubbles times C bubbles. So, because you just multiply this, the factor you get here and then what you get from here. And then now, I should see whether, if I include these factors they match.

So, let us multiply with 1 over n + m factorial on this one. Now, this is not just C but it is numeric factor now. Maybe I should not disturb that. So, numeric factor on the left hand side is 1 over n + m factorial times, I am writing this expression, this is n + m factorial. So, this is basically nCr and it is n factorial divided by r factorial times n - r factorial. So that is what I am writing. So, it is n + m factorial over n factorial times n + m - n which is m.

$$C_{LHS} = C_{(\text{no bubbles})} \times \binom{n+m}{n} \times C_{(\text{bubbles})} \tag{14}$$

$$C_{RHS} = C_{(\text{no bubbles})} \times C_{(\text{bubbles})} \tag{15}$$

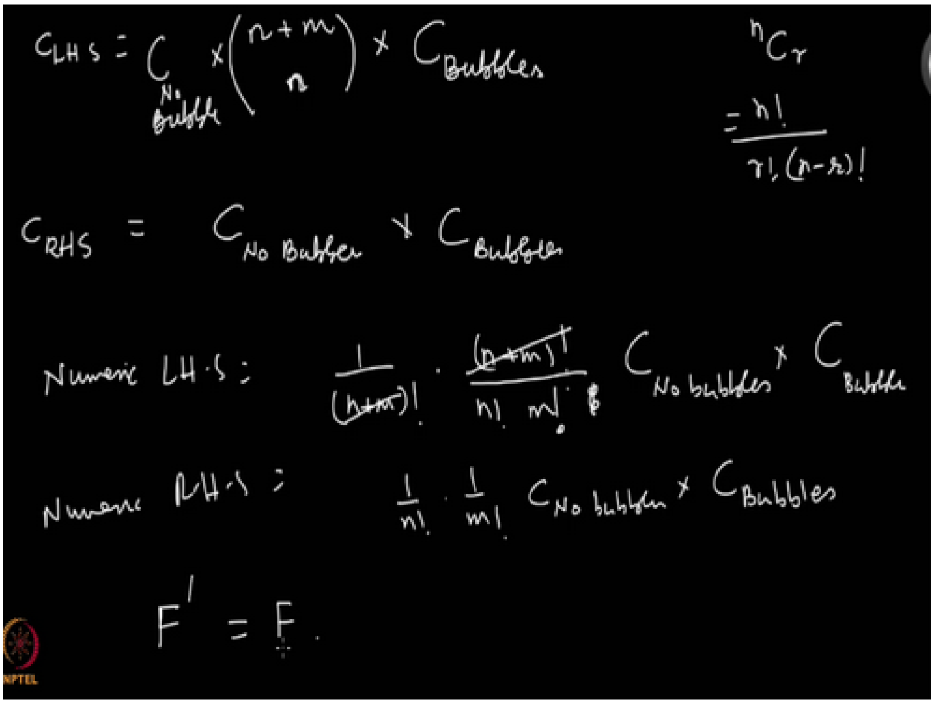

Figure 7: Refer Slide Time: 36:42

Numenic,

$$LHS = \frac{1}{(n+m)!} \times \frac{(n+m)!}{n!m!} C_{(\text{no bubbles})} C_{(\text{bubbles})} \quad (16)$$

$$RHS = \frac{1}{n!m!} C_{(\text{no bubbles})} C_{(\text{bubbles})} \quad (17)$$

$$(18)$$

F'=F

So, how it is, this piece time C with no bubbles times C with bubbles. And what is the numeric factor for right hand side? It is just 1 over n factorial times 1 over m factorial because that is what we had here. And the C RHS is just C times C of no bubbles times C of bubbles. And you see here that this cancels and this piece is exactly as this base. So, you see that all the numeric factors on the left hand side in the right hand side are same.

And that is why in general F prime is equal to F. So that is what we have proved, good. So, with this we can make a nice statement.

You see that the numerator of our master formula is basically you can write it as sum over diagrams without bubbles times 1 plus diagrams, sum over diagrams with bubbles. Let me tell you why I am saying that? So, if you look at again the 4 point function maybe we go here, we have gone many diagrams already here. So, you see any diagram, if you look at this one now, you can think of it as this times this that is what we have shown.

So, each diagram is some diagrams times some diagram without a bubble times a diagram with bubbles. And if you have a diagram here in the numerator which has no bubble then you can think of it this as multiplying one and that is the one here. So, you can write this and diagram denominator is any way sum of all diagrams with 1 +, sorry denominator is 1 plus sum over all bubble diagrams.

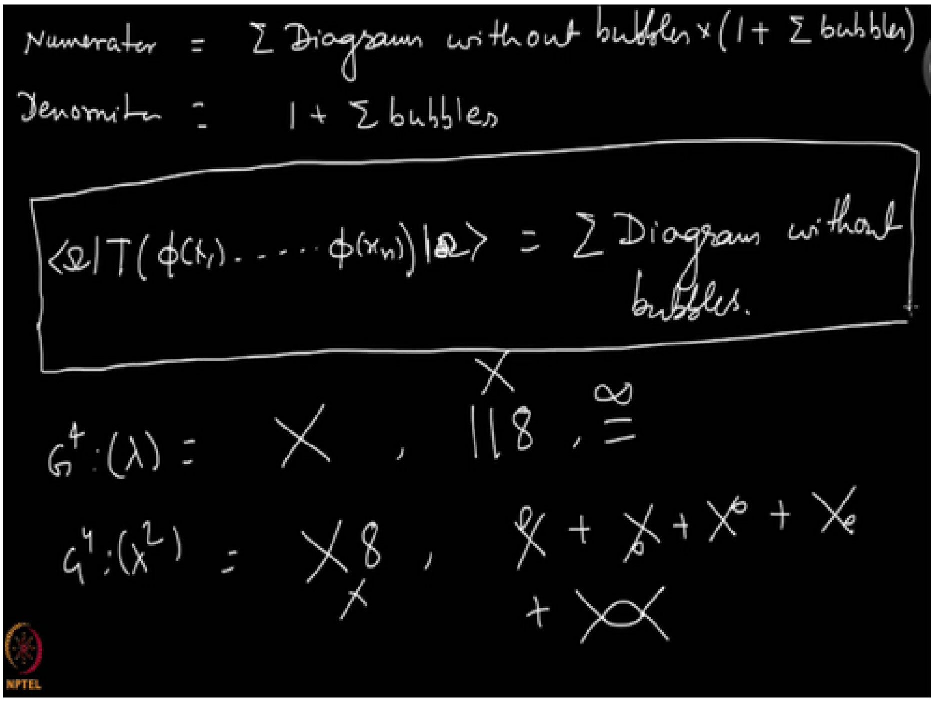

Figure 8: Refer Slide Time: 41:09

Numerator:

$$\sum \text{of diagram without bubbles } \times (1 + \sum \text{bubbles}) \tag{19}$$

Denominator

$$1 + \sum \text{bubbles} \tag{20}$$

$$\langle \Omega | T \Big(\phi(x_1) \cdots \phi(x_n) \Big) | \Omega \rangle = \sum \text{Diagram without bubbles} \tag{21}$$

So, you see now that this 1 plus bubbles in the numerator and 1 plus bubbles in the denominator they cancel. So, what you get is eventually this. The Green's function n point Green's function can be written as only sum of diagrams without bubbles. So, when you are drawing when you are writing an expression of a 4 point function.

Let us say a G 4 at, yeah let us say order lambda then you have only this diagram. You do not have to include this one because this is also there but this you can discard because this is a product of a diagram which is without bubble and with bubbles. So, this one you can discard or this one also you can discard. All you have to keep is this one at ordered lambda and if you were at order lambda square.

Let us see you have 2 to begin with you had this one for example but this you can discard because it is a product, we just have to write this sorry lambda square so, this, this. So, only these ones and all the diagrams and bubbles will be dropped. So, we have a good nice result there. We do not have to worry about the bubbles at all because they cancel between the numerator and the denominator. So, we will stop here and we will meet in the next video.