Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 48 : Feynman Rules for G (p 1, p 2, . . . , p N) Continued

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A two loop example

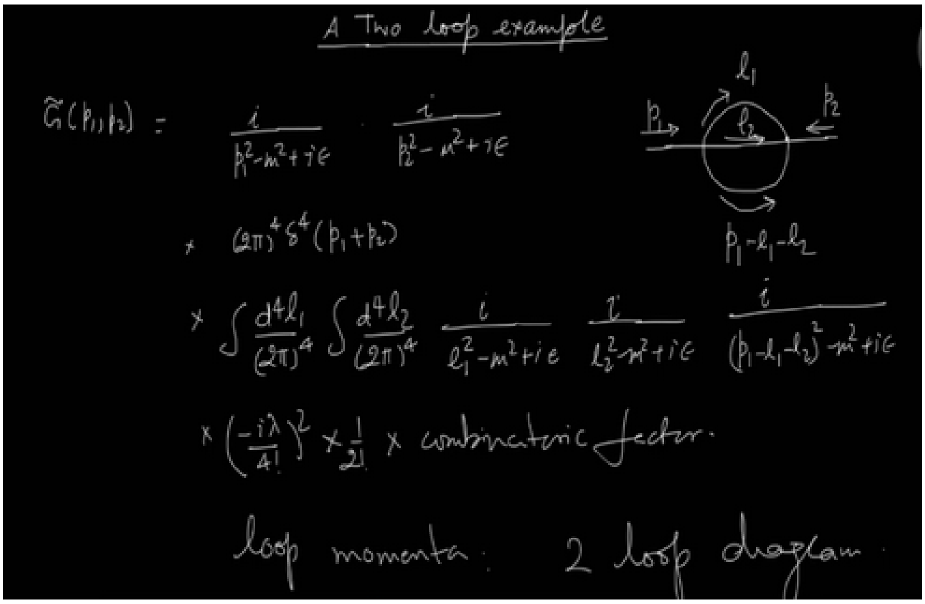


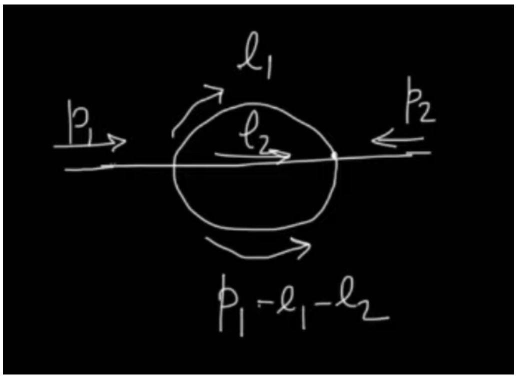
Figure 1: Refer Slide Time: 00:13

I will give you a 2 loop example now, arguing as an exercise but I think, it would be good to have one example at least. So, let us look at this Feynman diagram. This is again in phi-4 theory. So, what we are looking at is G tilde p 1 p 2. So, I have momenta p 1 and momenta p 2. So, everything is going into the diagram and I want to write down the expression for it.

So, if I use my Feynman rules that we have derived earlier in the previous, yeah here. So, I should include such a factor for each external propagator. So, let me do that. So, I should have I over p 1 square minus m square plus i epsilon times i over p 2 squared minus m square plus i epsilon then I should have an overall momentum conserving delta function. So, I have 2 phi to the 4 delta 4 p 1 + p 2 because both are ingoing so, I have p 1 + p 2.

This says that the sum of incoming momentum is 0. Good then, we also know that I will have a force other propagators but I will have some undetermined loop momenta which will be integrated over so, let us call this one l 1. So, p 1 comes in l 1 goes out but these 2 are still not fully determined by the momentum conservation that happens at this vertex. Remember before we arrived at these Feynman rules.

We had already seen here somewhere that at each vertex you should have yeah, at each vertex, we should include a delta function which conserves which imposes conservation of momentum added vertex at each vertex. So that is still there. But now, we do not write that explicitly but rather when we do the momentum assignment, we take care of it. So, here you see that I have 1 momentum incoming and 3 going out.



And clearly using just momentum conservation, I cannot fix all the 3. So, I will let us say I label this as l 1, still I cannot fix the remaining 2. So, let us have l 2 but now, this one is fixed. This one will be p 1 comes in and 1 1 an l 2 leave there so, this is what comes in here. And at this vertex everything is fine. L 1 comes in here l 2 comes in here, p 1 – l 1 and l 2 comes in here. So, everything is determined.

So eventually what you have is the following d 4 l 1 over 2 phi to the 4 d 4 l 2 over 2 phi to the 4. This is finally what we will get. And if you are not sure how I am getting this, you go back to the initial steps and repeat it and you will eventually end up here. Let me write down the propagators that you will get; so, these are these 3 propagators and what else you will have to include of course, factors of. And you can try to determine what that factor would be. So this is the expression for G tilde p 1 p 2. I am not sure I told last time these momenta are called loop momentum. So, your loop momenta there are 2 of them. And that is why this is a 2 loop diagram. It is a 2 loop diagram. So, all good here, there is 1 last piece that I should settle before we can say that the we have completed this part where we wanted to write Green's functions and they are Fourier transforms.

$$\begin{aligned}\tilde{G}(p_1,p_2) &= \frac{i}{p_1^2-m^2+i\epsilon}\frac{i}{p_2^2-m^2+i\epsilon}(2\pi)^4\delta^4(p_1+p_2)\\ &\times\int\frac{d^4l_1}{(2\pi)^4}\int\frac{d^4l_2}{(2\pi)^4}\left(\frac{i}{l_1^2-m^2+i\epsilon}\right)\left(\frac{i}{l_2^2-m^2+i\epsilon}\right)\\ &\times\frac{i}{(p_1-l_2-l_2)^2-m^2+i\epsilon}\times\left(\frac{-i\lambda}{4!}\right)^2\times\frac{1}{2!}\times\text{combinatoric factor}\end{aligned} \tag{1}$$

$$\lim\int_{-\Lambda(1-i\epsilon)}^{\Lambda(1-i\epsilon)} dz^0\, e^{-ik^0z^0} \tag{2}$$

not the usual expression for F.T. (e^{-ikz})

$$z^0=(1-i\epsilon)\tilde{z}^0 \quad ; \quad \tilde{z}^0 \text{ is real} \tag{3}$$

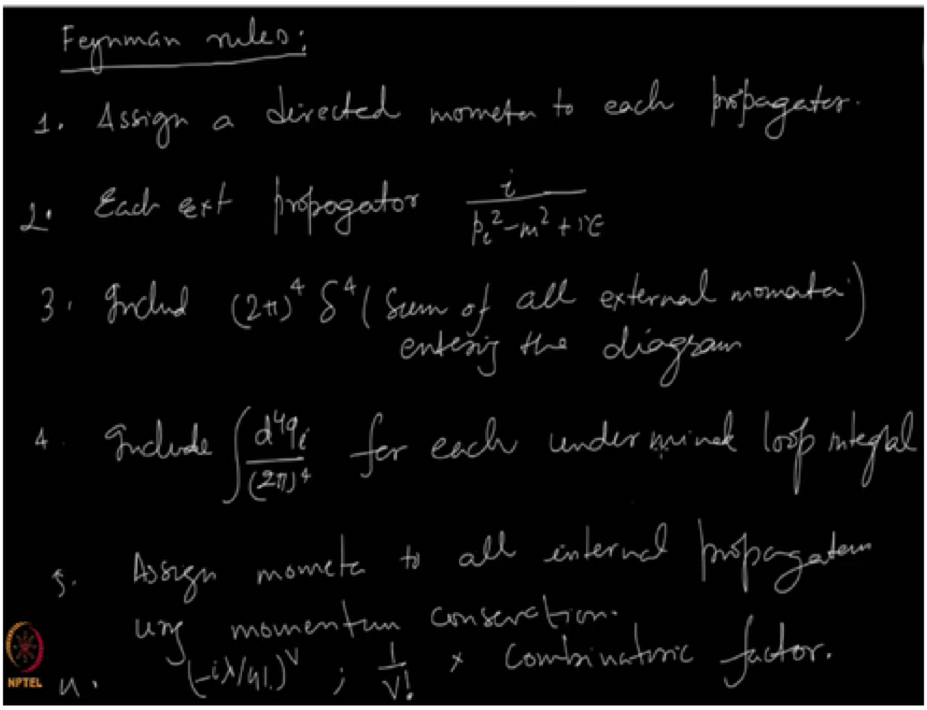


Figure 2: Refer Slide Time: 02:22

And that part is let me go back and show you. Here so, you see when we were writing the Green's functions, these objects, we had an integral over d tau where the time tau was running from minus lambda 1 – i epsilon to lambda 1 – i epsilon where we have to take lambda going to infinity and H is the Hamiltonian interacting part of the Hamiltonian which is further an integral over the Hamiltonian density.

So, there is a d q z here and then called tau is z 0. So, you have basically d 4 z that is what the notation that we have used but the time component z 0 runs from minus lambda 1 – i epsilon to this. These are imaginary numbers not real numbers but all the calculations that we have done later and we have been doing lots of Fourier transforms. In all those places, we have been pretending as if its usual Fourier transform where the integrals run from minus infinity to plus infinity.

Without paying attention to the fact that the limits were actually from minus lambda 1 – i epsilon 2 lambda 1 – i epsilon and then I should argue or should show that all my manipulations of Fourier transforms are justified even though I had such the integral over x or integral over z was running over all complex values. So that is what we want to settle now.

So, we had like this d z 0, what was tau I was showing, I am writing as d z 0 and it was running from minus lambda 1 – i epsilon to lambda 1 – i epsilon and then of course, we had to take lambda going to infinity and eventually at the end epsilon going to 0. This is what we had and we have been encountering integrals of this form because we have been having e to the minus i k dot z.

When you are taking Fourier transforms or when you are doing integrals over the internal vertices and then when you are doing the integrals which involve these pieces coming from the propagators. Maybe I can try to show you, if it is somewhere here, yeah for example, here you see we had these factors coming in maybe for this still, yeah this one. So, you had z 1 z 2 here and you have to integrate over these points and the time component of z 1 runs from minus lambda 1 – i epsilon to lambda 1 – i epsilon and you see that you have such factors here. So, where k

1 k 2 and all these are coming from and k 4 and k 5 k 6, they are all coming from here, these propagators. And here we have been using the formula like this.

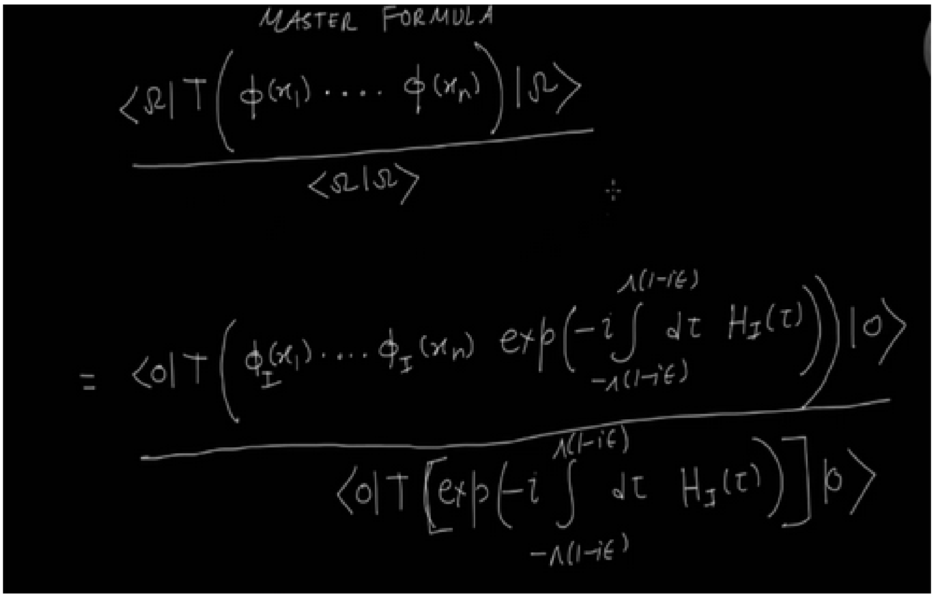


Figure 3: Refer Slide Time: 06:07

So, we wrote this as delta function as if we were just doing Fourier transform of the exponential. But now, we realize that is not something obvious that they should work out because the limits are not from minus infinity to plus infinity. So, let us see why this is justified. So, we have objects like this and of course, you have some other functions which are multiplying here and I will ignore them now, they do not disturb the argument.

Now, clearly this is not the usual expression for a Fourier transform, because usual expression will be involving this. This is what you will have, if you are doing usual Fourier transforms. So, let us try to arrange for this and that is not difficult because I can change the variables from z 0 to z tilde 0 where I defined z 0 to be 1 − i epsilon times z tilde 0. And I say z tilde 0 is real.

If I do so, then I get integral d z 0 becomes 1 − i epsilon, d z tilde z 0 and now, my integration limits run from minus infinity to plus infinity but the exponent changes. So, it becomes e to the minus i k 0 1 − i epsilon z tilde 0. So, again this is not looking like your Fourier transform but at least the integration limits are. But here these are not real objects.

$$(1 - i\epsilon) \int_{-\infty}^{\infty} d\tilde{z}^0 \, e^{-i\tilde{k}^0 (1-i\epsilon)\tilde{z}^0} \tag{4}$$

So, now what I can do is. I can define k 0 to be 1 − i epsilon times sorry 1 + i epsilon times k tilde 0. So, if I do so, then k 0 times 1 − i epsilon because when I take the inverse and when I multiply inverse of 1 + i epsilon on both sides, so, inverse of 1 + i epsilon is 1 − i epsilon and because 1 1 i epsilon times 1 + i epsilon gives you 1 plus order epsilon square term and we are working to a order epsilon. So, this can be dropped.

So, k 0 times 1 − i epsilon becomes k tilde 0. So, this becomes with this definition. So, I will not worry about this part because there you can easily take the epsilon going to 0 limit without any difficulty. So, I get this into the − i k tilde 0 z tilde 0, where both k tilde 0 and z tilde not are real. So, here I define k tilde 0 to be real. Now, this clearly looks like expression for Fourier transform with whatever function you have here will be Fourier transforming that this is good.

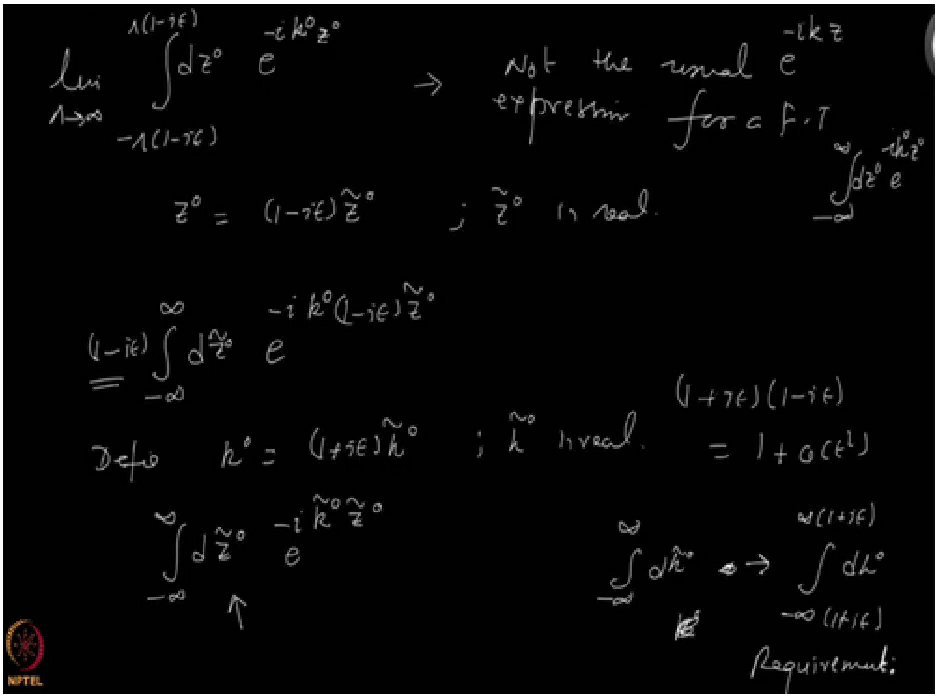


Figure 4: Refer Slide Time: 07:56

But then this requires that k 0 be imaginary. So, if I am doing a Fourier transform and that Fourier transform involves an integral over k tilde 0. So, there you will have integral k tilde 0 and this one will be running from minus infinity to plus infinity when you are doing some Fourier transforms. And which will mean that when this is running from minus infinity to plus infinity your k 0 will be running from minus infinity 1 + i epsilon.

Just a second, is that yeah, it will be k 0 will be running from minus infinity 1 + i epsilon to plus infinity 1 + i epsilon. Correct so, this is something we should arrange whenever we are doing I mean this is the i implication of making this change of variable. So, now, you have fixed the Fourier transform but this implies that wherever k 0 integral is involved the limit should run in this fashion.

Now let us see. So that is the requirement and I should ensure this and if I can ensure this all the time I am doing integrals over k 0 then all is good.

Define:

$$k^0 = (1 + i\epsilon)\tilde{k}^0 \quad ; \quad \tilde{k}^0 \text{ is real} \tag{5}$$

$$(1 + i\epsilon)(1 - i\epsilon) = 1 + \mathcal{O}(\epsilon^2) \tag{6}$$

$$\int_{-\infty}^{\infty} d\tilde{z}^0 \, e^{-i\tilde{k}^0 \cdot \tilde{z}^0} \tag{7}$$

Requirement:

$$\int_{-\infty}^{\infty} d\tilde{k}^0 \to \int_{-\infty(1+i\epsilon)}^{\infty(1-i\epsilon)} dk^0 \tag{8}$$

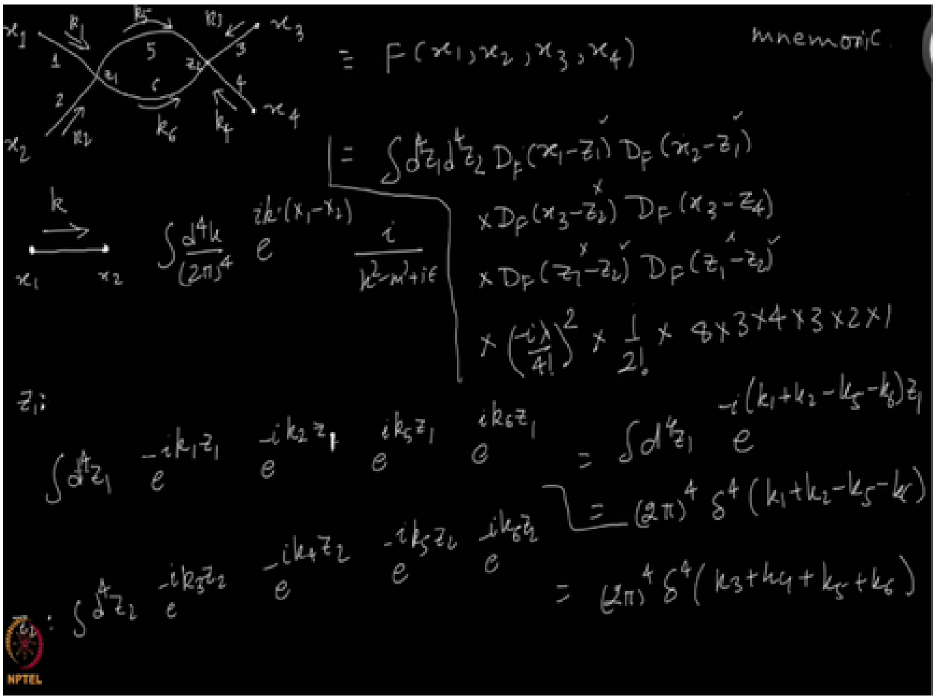


Figure 5: Refer Slide Time: 09:23

Now, we recall something that when we were doing integrals over k 0 so, let us go to k 0 space. So when we were doing integrals over of integrals coming from the Feynman propagators there we saw that the poles were like this. Remember the poles were on the because of the Feynman prescription the poles were slightly shifted off the axis and this is what I am indicating right now or if you keep the poles there, you are basically integrating over this line.

So, whichever way you see, either you shift the poles or you say that I am tilting the axis of integration. So, even initially the contour integral was the contour of integration was real axes but then we put the i epsilon prescription which is equivalent to saying that either you shift the poles or you tilt the contour. So, this contour so, let us say we are doing integral over this contour and you see that for real parts.

So, this is just each k 0 has been shifted by a positive imaginary part so, it is k 0 1 + i epsilon. So, what was k 0 here? Has been shifted to k 0 1 + i epsilon and similarly, what was some negative number here has been shifted to negative imaginary part. So, if k 0 is negative, so, let us look at this 1 + i epsilon, if this number is negative and this times epsilon is going to give you a negative imaginary part so, which is exactly what is happening.

So, you see that the Feynman prescription that we have been using has precisely done this that it has tilted the k 0 axis, the k 0 contours along this imaginary in the complex plane according to this. So, it has given each k 0 a imaginary piece And this is precisely what? We are asking here that the k 0 integrals should run from minus infinity 1 + i epsilon to infinity 1 + i epsilon and which is what Feynman integral has been doing, the Feynman prescription.

Which means that if I completely forget about these issues here and just replace the z 0 by real numbers and integrate over real axes, I would I do not have to bother because the requirement that contour integration should be like this is automatically arranged by the Feynman prescription. So, I am justified in doing all the manipulations which I did involving Fourier transforms precisely for this reason.

So that part is settled. Now, there is only 1 piece which is left and that I should discuss about is maybe I will show you, yeah here. So, till now, I have always been talking about the

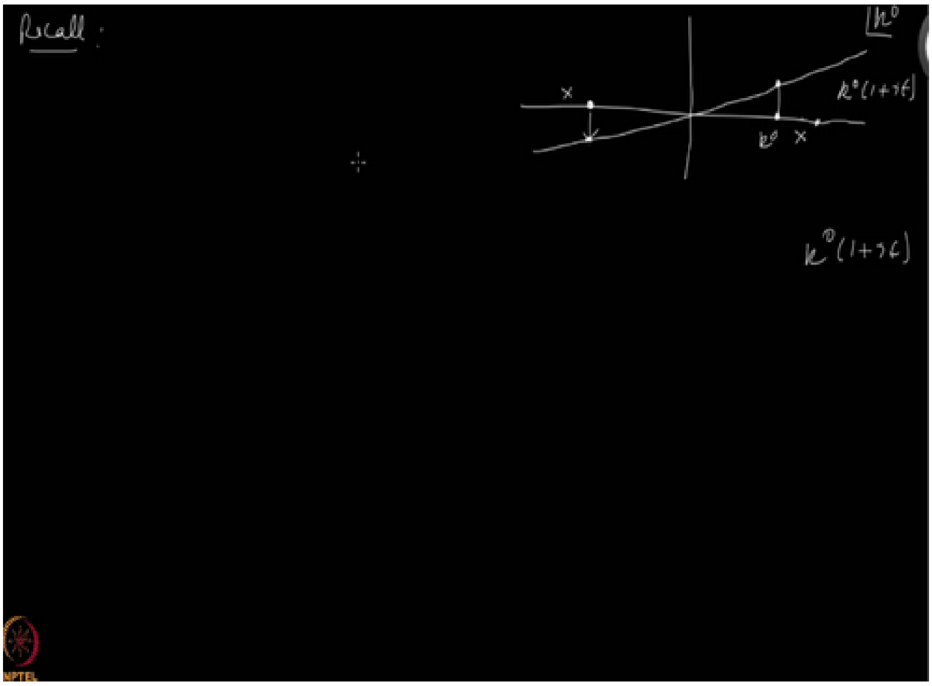


Figure 6: Refer Slide Time: 15:32

numerator, I have never talked about the denominator. So, next thing which we will do is look at the denominators and then this will complete our discussion on the Green's functions, at least whatever I plan to talk in this course. So, in the next video, let us start looking at the denominator.