Introduction to Quantum Field Theory

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Lecture 47 : Feynman Rules for G (p 1, p 2, . . . , p N)

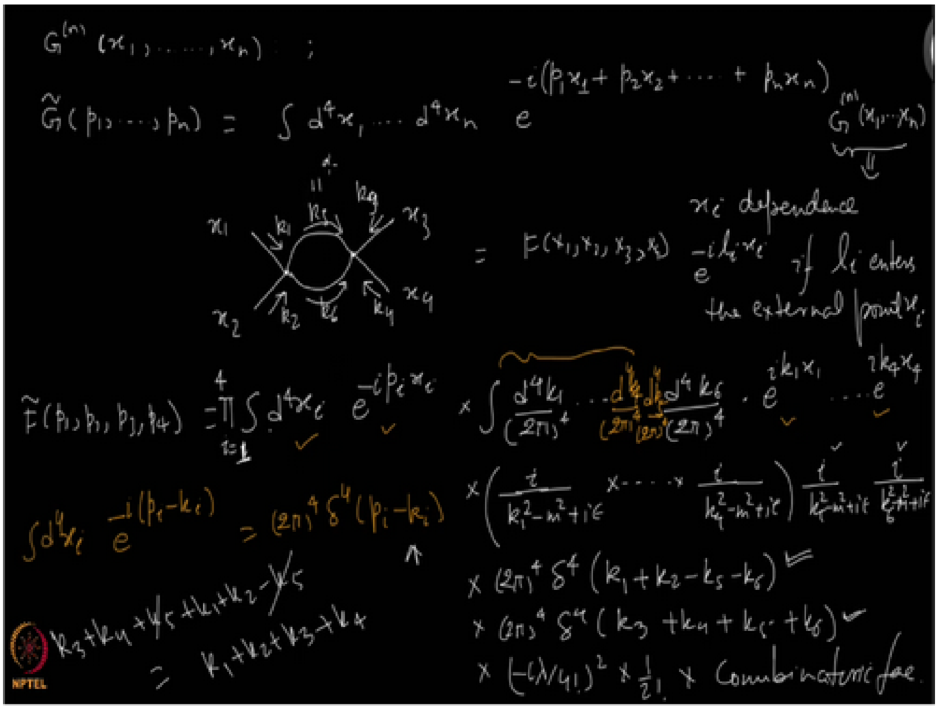


Figure 1: Refer Slide Time: 00:13

In the last lecture we wrote down momentum space Feynman rules for Green's functions. These ones and which you can use to write down expressions for Feynman diagrams using those rules. Another object that will be of interest to us will be the Fourier transform of this object and today we want to write down the Feynman rules corresponding to the Fourier transform of this. So, let me first define the Fourier transform of this object.

So, this is function of n variables x 1 to x n. So, I defined Fourier transform g tilde of p 1 to p n by Fourier transform each of the variables. So, here you have d 4 x i let me d 4 x 1 to d 4 x n and then e to the minus i p 1 x 1 and then for the x 2 variable I will have e to the minus i p 2 x 2 which I can put here and so, forth for all the terms up to x n. And then of course, we should have here G that is the Fourier transform.

And we will see later that G tilde is an object of interest and we want to that is why write down the Feynman rules for this. So, this will be simple because we already know how to write down an expression for G n. And all we have introduced is this exponential factor and there is an integral over the x x i's. Now recall that the dependence on x i of G n was quite simple.

The dependence of this on x i was like this. So, x I dependence was merely e to the minus i l i x I, if l i enters the external point. So, l i is the momentum that enters into the external point x i then you put e to the minus i l i x i and if l i exists then you put e to the plus i l i x i that is what we had. And now, this is why it is very simple to handle the x 1 because G depends on x i only through this factor.

This is the only place where x i appears. Now, if this is the only place let us look at x 1. So, x 1 is the x 1 appears only through this exponential factor in G. And this is the other place where you have x 1. So, you can combine these 2 exponentials and you can do the integral over x 1. So, these 2 explanations will combine and when you integrate over x 1 you will get a delta function.

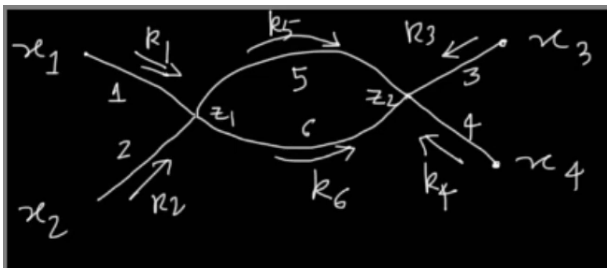
So, handling x 1 will be simple so, going to the Fourier space from x to p that will be quite easy. So, that is what we are going to do, it will be better to have an explicit example where I can show these calculations rather than talking in general. So, let us go back to our same diagram which we have been looking so far, which is this one. This diagram and look at it in the Fourier space so, let us see. So, here I am right now I am drawing G n this diagram which I was calling F. So, we have this object and we had assign the momenta like this last time. So, let us do it again the same way. But you can assign it differently, if you wish. And this was k 5 which was assigned this way and this was k 6 which was going this way.

And we have already done the integrals over the vertices. So, they are not there anymore. They left behind the delta functions, momentum and momentum conserving delta functions at the vertices. Anyhow, so, let us look at this. So, this we had called as F of x 1, x 2, x 3, x 4. And now we want to have F tilde of p 1, p 2, p 3 and p 4. And this will be exactly this factor. So, I should write it down d 4 x i and e to the minus i p i x i.

$G^n(x_1 \cdots x_n)$:

$$\tilde{G}(p_1 \cdots p_n) = \int d^4x_1 \cdots d^4x_n e^{-i(p_1x_1+\cdots+p_nx_n)} G^n(x_1 \cdots x_n) \tag{1}$$

$G^n(x_1 \cdots x_n)$ x_i depends on $e^{-l_ix_i}$, if l_i enters the external point x_i



$$\begin{aligned}
\tilde{F}(p_1, p_2, p_3, p_4) = & \prod_{i=1}^{4} \int d^4x_i \; e^{-ip_ix_i} \times \int \frac{d^4k_1}{(2\pi)^4} \cdots \int \frac{d^4k_6}{(2\pi)^4} \\
& \times e^{ik_1x_1} \cdots e^{ik_4x_4} \times \frac{i}{k_1^2 - m^2 + i\epsilon} \cdots \frac{i}{k_4^2 - m^2 + i\epsilon} \\
& \times (2\pi)^4 \delta^4(k_1 + k_2 - k_5 - k_6) \, (2\pi)^4 \delta^4(k_3 + k_4 + k_5 + k_6) \\
& \times \left(\frac{-i\lambda}{4!} \right)^2 \times \frac{1}{2!} \times \text{combinatoric factor}
\end{aligned} \tag{2}$$

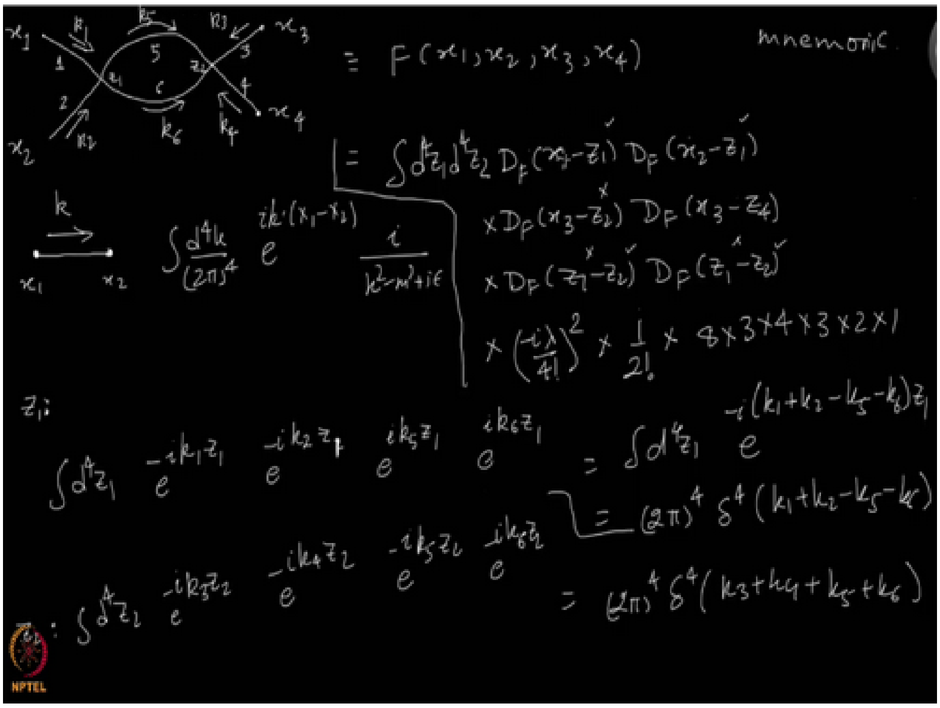


Figure 2: Refer Slide Time: 04:50

So, I have written for ith one, this piece and this piece and of course, I have to multiply this 4 times, I mean, multiply these 4 factors. So, now, I have taken care of these parts. Now, I should only write G n or rather F of this which you recall was the following. So, for each propagator, I had an integral corresponding to the momentum that runs through their propagator. So, we had d 4 k 1 over 2 phi to the 4, up to the d 4 k 6.

We have 6 propagators and for each one we have to have an integral. And recall what the piece was corresponding to the external points, it was simply this. So, we include all the factors corresponding to the external points and you get e to the i k 1 x 1 up to e to the i k 4 x 4, only 4 external points. So, you have this and then I should have all the 6 propagators which I will split into 2 when I am writing 2 parts.

So, you have k 1 square minus m square plus i epsilon i over k 4 square minus m square plus i epsilon. So, that is one. And the remaining 2 are i over k 5 square minus m square plus i epsilon and i over k 6 square minus m square plus i epsilon. So, I have taken care of all the propagators. And then we remember that we had delta functions to phi 4 times the delta functions at each vertex.

And we have to see that the momentum conservation holds there. So, k 1 + k 2 enters and k 5 + k 6 exists. So, the sum of all the momenta entering should be 0 which is k 1 + k 2 − k 5 − k 6. So, the sum of all momenta entering is 0. And you had another one k 3, so, this was k 3 + k 4 + k 5 + k 6. And then the other factors of minus lambda over 4 factorial squared times 1 over 2 factorial times the combinatoric factor.

So, let us do the integral over x 1, x 2, x 3 and x 4 for the reasons I was talking above. So, let us say when I am doing the integral over x 1, I take this piece and I integrate over x 1, I get a 2 phi 4 delta 4 k 1 − p 1. So, let me write here integral d 4 x let me try to change the color. So, if I do the integral d 4, this might not be good, d 4 x i, e to the minus i p i − k i.

This will give 2 phi to the 4 delta 4 p i − k i which will when you integrate over the k i because you have these k integrals. In fact, I should have split this also into 2 factors like this which is fine does not matter. So, anyway when you do the integral over k 1, k 2, k 3 and k 4, the k 1 will get the k 1 will be forced to become p 1 because of this delta function everywhere in this expression.

k 2 will be forced to become p 2 everywhere in the expression and so forth. So, let us do that. So, this will be gone, this will be gone and these facts 4 factors will be gone. And they will leave behind these 2 phi 4 Delta 4s. And I will also do one more integral now we will do 4 integrals so, d 4 k 1 up to d 4 k 4 but over 2 phi to the 4 and these 2 will remain k 5 over 2 phi to the 4.

So we will do up to these 4 and when I do these 4 integrals, I will be just setting k 1 = p 1, k 2 = p 2 and so forth and these 2 phi 4s will cancel against these denominators. So, 2 phi 4s will also go away. So, the answer will be this I should start with writing d 4 k 5 and d 4 k 6. So, what we get is d 4 k 5 over 2 phi to the 4 d 4 k 6 over 2 phi to the 4, let us go back, we have these 4 sitting here and these 4 there is no integral over k 1 sorry. So, when I am doing this integral, this will be set, k 1 will be set to p 1. Let me go back again. So, I have 4 factors and they will be i over p 1 square minus m square plus i epsilon up to p 4 square minus m square plus i epsilon. Now, these do not depend on these integrals, these variables k 5 and k 6 in fact, they do not depend on any other variables because these are the externally specified momenta. So, I could have just written it like this, I could have written it outside the integral.

So, maybe I should do that I think I cannot so, let us leave it. So, I have i over p i squared minus m square plus i epsilon and we have all the 4 so, that is the product. Then you have integral d 4 k 5 over 2 phi to the 4, integral d 4 k 6 over 2 phi to the 4th and then what else we have, we have these 2 propagators. So, I write i over k 5 square minus m square plus i epsilon, i over k 6 squared minus m squared plus i epsilon, it is fine, these 2 are gone.

And then we have 2 these Dirac delta functions and all these factors. Now, what I will do is, I will do a k 5 integral or rather a k 6 integral. Let us see which one I was doing in my note. I will do a k 6 integral now. And that is easy because I can use up this delta function. So, the 2 phi 4 in the denominator here that will cancel against this 2 phi 4 so, that goes away.

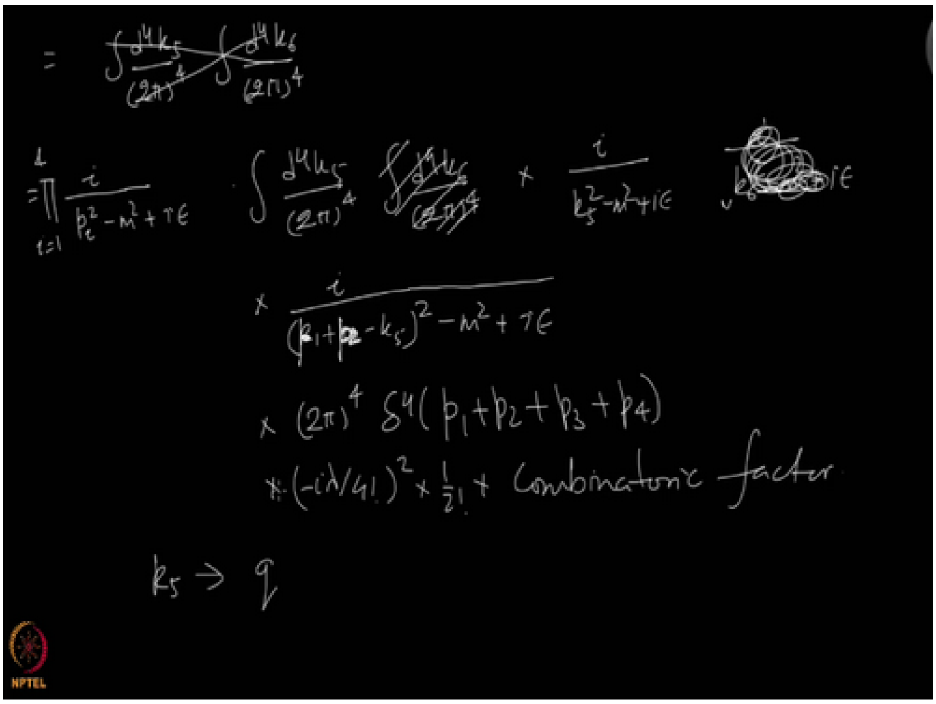


Figure 3: Refer Slide Time: 13:34

And when I do integral d 4 k 6 of this delta, it will set k 6 equal to k 1 + k 2 − k 5 at all the places where it appears. So, let me do that. So, let me I will change this one now, because I am doing that integral that does not work. So, this one will be changed to i over what was that k 1 + k 2 – k 5 whole squared minus m square plus i epsilon. I have just done the case 6 integral.

And here you had a case x squared which I have changed to this. Let us see what more. So, this I have taken care of, this one is taken care of, this integral I have done, this factor of 2 phi 4 is taken care of because it cancels against this one. So, now what I have is 2 phi to the 4 times delta for k 3 + k 4. And here for K 6, I should write k 1 + k 2 – k 5. So, let us see, k 3 I am writing the argument of Delta 4 here k 3 + k 4 + k 5 and for k 6.

I should write k 1 + k 2 – k 5 and this cancels which is just, there is something going wrong and what is going wrong? Yes I know that is fine. Let me just write it, I have made a slight mistake, this is what you will get. What I forgot is that when I did the integral over k 1 up to k 4 these delta functions, the their arguments will also change right? So, this k 1 will become p 1 and k 2 will become p 2.

$$\int d^4x_i\; e^{-i(p_i-k_i)x_i} = (2\pi)^4\delta^4(p_i-k_i) \tag{3}$$

$$\begin{aligned}=&\prod_{i=1}^{4}\frac{i}{p_i^2-m^2+i\epsilon}\int\frac{d^4k_5}{(2\pi)^4}\times\frac{i}{k_5^2-m^2+i\epsilon}\times\frac{i}{(p_1+p_2-k_5)^2-m^2+i\epsilon}\\ &\times(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\times\left(\frac{-i\lambda}{4!}\right)^2\times\frac{1}{2!}\times\text{combinatoric factor}\end{aligned} \tag{4}$$

Because we that is what we are doing here. So, instead of this actually, I will get p 1 + p 2 + p 3 + p 4. So, here it is fine. The way it is written is fine. But the next step, actually, everything is fine. So, since I am writing just now, it is all. There is nothing wrong. So, I get 2 phi to the 4 delta 4 p 1 + p 2 + p 3 + p 4. So, this is also taken care of and then you have these last factors.

Yeah, here also there is a mistake because now k 1 should become p 1 and because we had an integral delta function integral which had forced the p 1, p 2, p sorry k 1, k 2, k 3, k 4 to p 1, p 2, p 3, p 4. Now, this is fine. So, you see there is only 1 momentum over which we still need to do an integral and that is k 5. All other are completely fixed because they are fixed by the specification of G. See G tilde has this p 1 up to p 4 in our expression. So, they are given externally, they are specified to you. And what is left is only 1 integral over the dummy variable k 5 which we are going to call as q now. So, what we have here is now, the following so, let me go back to this diagram and k 5 now I am calling as q. So, 1 momentum here and you see that the final result is the following. So, I will just in the next slide, I will write k 5 as q. So, the final result is this, for Fourier transform of this which was integral. So, we have 1 q here which is going like this which was being called k 5. And all the external momentum they have been fixed, the moment which was flowing in here has been fixed to p 1 and p 2 and p 3 and p 4 let me go back. So, when we did this integral, we set k 1 equal to p 1.

So, I can draw the same diagram with p 1 here because p 1 has k 1 which was flowing like this here has been set to p 2 and the k 2 which was flowing like this has been set to p 2 and k 3 and k 4 have been set to p 3 and p 4. And remember what happened to this one, this because you have momentum conservation everything else is fixed. So, this one has momentum. What momentum is running here? It is p 1 + p 2 is entering into the vertex.

But q leaves out. So, whatever left is here, so, you have p 1 + p 2 coming in minus q leaving out. So, this is what goes in here and that is why you have this thing p 1 + p 2 – q, q, it is k 5 has become q. So, you see that all the momenta in this diagram, all the momentum on the propagators have been completely fixed except for 1 the q or which you still have an integral and the result is the following.

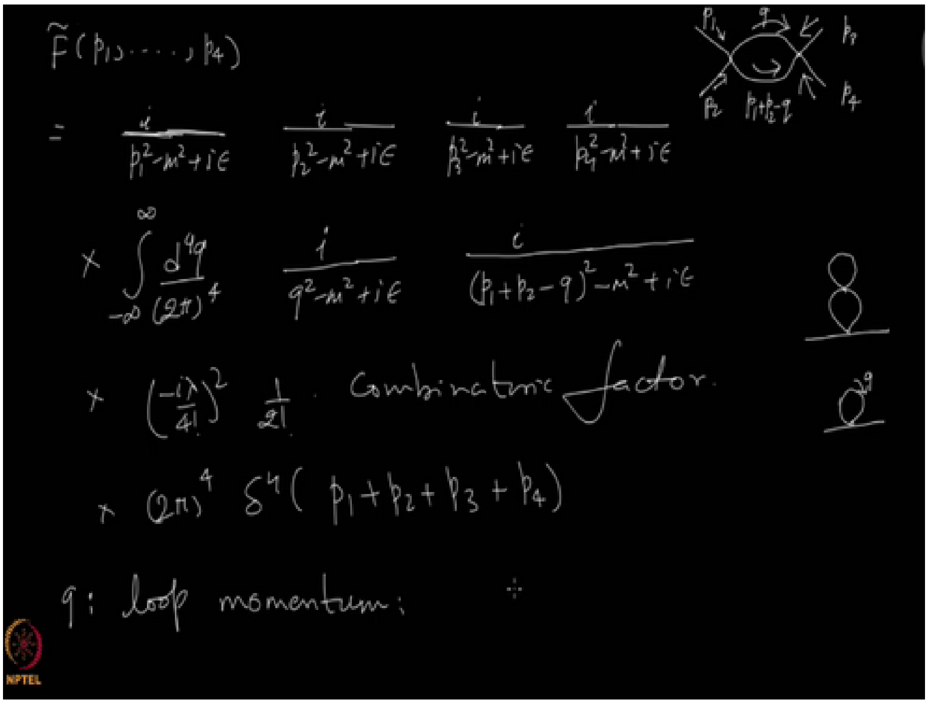


Figure 4: Refer Slide Time: 21:24

$$k_5 \to q$$

$$\tilde{F}(p_1 \cdots p_4) = \frac{i}{p_1^2 - m^2 + i\epsilon} \times \cdots \times \frac{i}{p_4^2 - m^2 + i\epsilon} \tag{5}$$

$$\times \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \times \frac{i}{(p_1 + p_2 - q)^2 - m^2 + i\epsilon} \tag{6}$$

$$\times \left(\frac{-i\lambda}{4!} \right)^2 \times \frac{1}{2!} \times \text{combinatoric factor} \tag{7}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \tag{8}$$

$q \to$ loop momenta

So, let me write this as F tilde p 1 to p 4. So, this is not now, G x F of x 1 to x 4. This is F tilde p 1 to p 4. So, I am specifying the external momenta here now, based on what the calculation that we did and the result is this. First factor was for each external propagator, I have such a piece and by external propagator, I mean the propagator that connects to the external points. So, you have i over p 1 square minus m square plus i epsilon.

For this 1 you have p 2 square minus m square plus i epsilon, p 3 square minus m square plus i epsilon. So, you see that these external lines have nicely factored out of all other things. Then what do we have? Then we have d 4 q over 2 phi to the 4 that is here. And then I have these 2 propagators, this one and that one. Let me write that down. So, this is the one which earlier we called k 5.

And this has become now, i over q square minus m square plus i epsilon and then this one has become i over p 1 + p 2 − q square minus m square plus i epsilon. Then what else? You had a minus lambda over 4 factorial from each vertex 1 over 2 factorial from the exponential and then the combinatorics factor. This is all correct, not yet. This 1 piece which is missing.

Which is this piece times 2 phi to the 4 delta 4 p 1 + p 2 + p 3 + p 4. We say this one is overall energy momentum conserving delta function. So, it says that the sum of all momenta for momenta p 1, p 2, p 3 and p 4 entering into the diagram is 0 which just means, whatever enters should exit. So, because right now, in the diagram we have everything entering the sum should be 0.

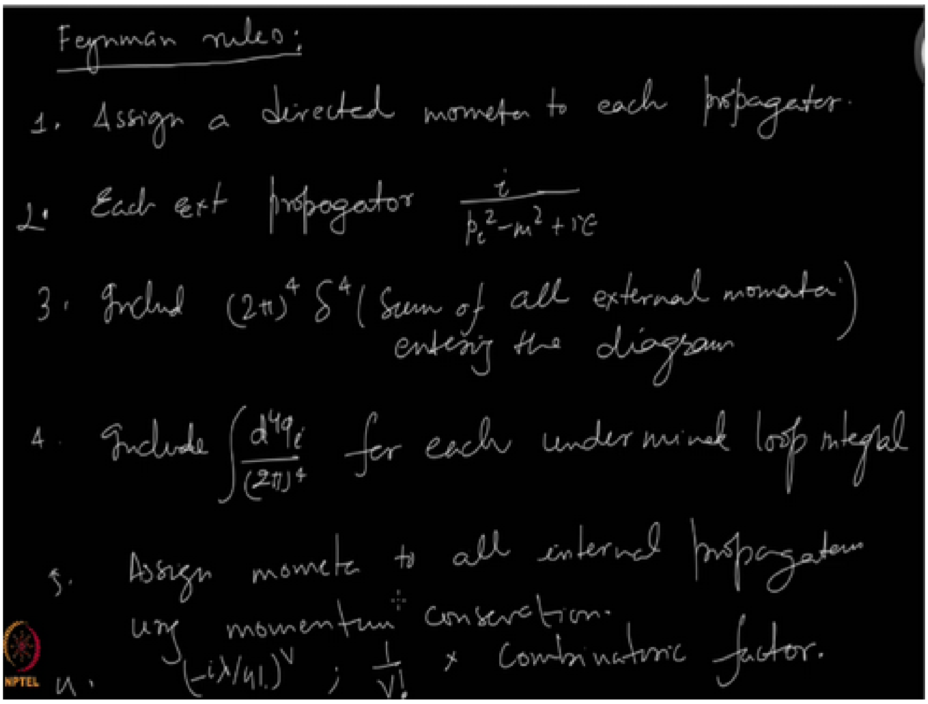


Figure 5: Refer Slide Time: 28:47

if I had drawn the p 3 and p 4 lines going out, if I had arranged it that way then you would have gotten p 1 + p 2 − p 3 − p 4. That would mean p 1 + p 2 = p 3 + p 4. But because everything is in going right now, in this diagram you have the sum of all momenta entering to be 0. So, that is the expression for the diagram which we have and this is really the diagram in the Fourier space. So, what we see that is that in this case you have only 1 momentum left unintegrated and that is called loop momentum. So, q is called the loop momentum and q runs of course from all the components run from minus infinity to plus infinity and this diagram has only 1 unintegrated I mean only 1 loop momentum because this is a 1 loop diagram. So, there is only 1 loop here.

And that is why when we did this computation we got only 1 variable q which was still remaining and which needs integration. But if you are looking at a diagram which was having more than one loops. So, let us say a 2 loop. For example, if you look at a 2 point function like this. So, this one, if you had only this, if you do the same exercise as I have done here for this 4 point function.

If you were to repeat the exercise for this which I would encourage you to do and do the same steps you will see you are still again going to be left with 1 loop momentum q. So, you will be left with one integral but if you start doing this one, you will be left with 2 q 1 and q 2 because that is a 2 loop. So, for each loop, you will have 1 loop momentum which you will need to integrate over. Good then so, what we can do is make the Feynman rules for calculating F tilde or G tilde.

Feynman rules

1. Assign a directed momenta to each propagator.

2. Each external propagator
$$\frac{i}{p_i^2 - m^2 + i\epsilon}$$

3. Include $(2\pi)^4\delta^4$(sum of all external momenta entering the diagram)

4. Include $\int \frac{d^4 q_i}{(2\pi)^4}$ for each undetermined loop integral.

5. Assign momenta to all external propagator using momentum conservation.

6. $\left(\frac{-i\lambda}{4!}\right)^V \times \frac{1}{V!} \times$ combinatoric factor

Exercise:

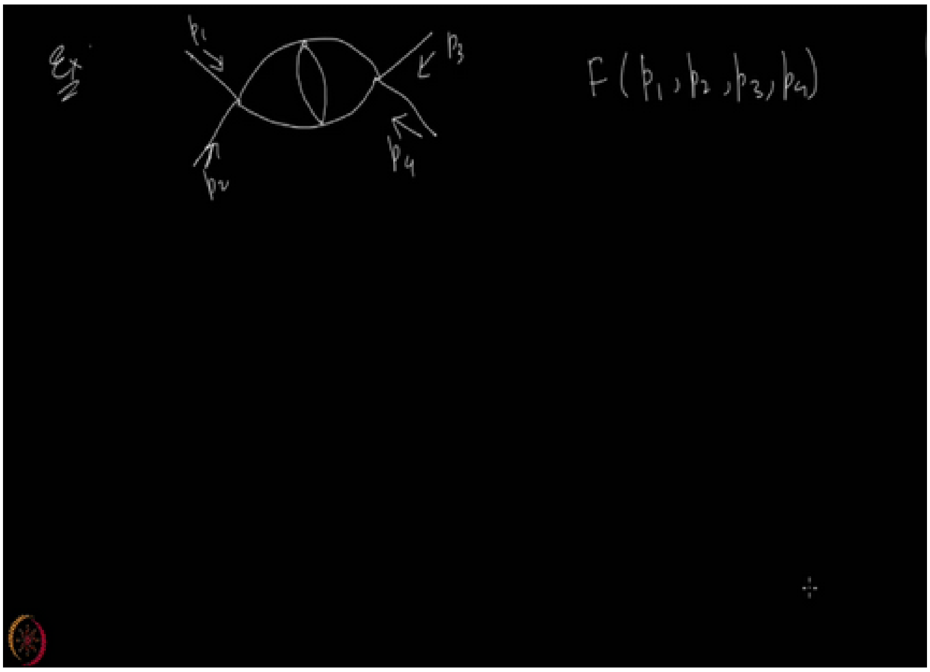


Figure 6: Refer Slide Time: 33:41

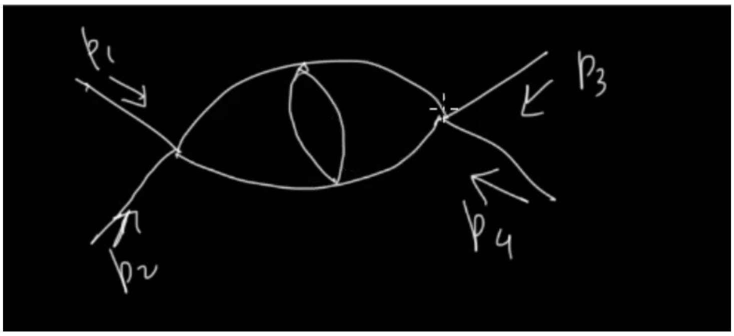
So, when we are looking at these Green's functions in Fourier space, the Feynman rules are the following: So, again you assign a directed momentum to each propagator and for each external propagator you have to write down this. These 4 pieces so, we have 4; so, we should include this piece for all the 4. Then we should include 2 phi 4 times and overall energy momentum conserving delta function.

And then include, d 4 q i over 2 phi to the 4 for each undetermined loop integral for each loop by undetermined I mean the following. In the case of 1 loop you see, we had these 2 internal lines, 2 internal propagators as opposed to 4 external propagators and you saw that the momentum flowing in this line got completely fixed in terms of the external momenta and this loop momenta.

So, you do not have any freedom left for this one. So, in general what will happen is certain propagators will have the momenta completely fixed in terms of all others. So, if you are looking at 2 loop, we will have 2 undetermined momenta q 1 and q 2 and all other propagators will have completely fixed values of the momentum that flows in there in terms of external momentum and the loop momentum.

So, this is what we should do and then assign momenta to all the loop in all the internal propagators using momentum conservation. Let me explain so, here you see, it is 1 loop so, these lines momentum is fixed p 1, p 2, p 3, p 4 because they are given to you. This one you say it is q and this one you use, you write down by using the momentum conservation. p 1 + P 2 – q flows in here.

And that is what I am writing as a rule that you should assign momenta to all internal propagators using momentum conservation and of course, some of them will remain free just like the one which is q. And then of course, include – i lambda over 4 factorial power v, if you have v vertices, 1 over v factorial, if you have v vertices and then multiply the combinatoric factor.



So, this is all I wanted to say about the Feynman rules for this Fourier transformed Green's functions and I would encourage you to play around with some other diagrams for example, you can try to write down the expression for this one. So, you are looking at Green's function or let us say Feynman diagram with p 1, p 2, p 3, p 4. And were p 1 is this, p 2 is this, p 3 this and p 4 is that. Now try to assign first the momentum here and see how many are undetermined and then fix all the remaining ones in terms of the external moment and the undetermined momenta and write down all the write down the expression using these rules.

Which you have made and in case you are feeling unsure about how to write or what it means, you go back further and you start from here. You start from here. So, you write the G n in terms of position space and then introduce this Fourier transform and then follow the steps. And then eventually you will end up at the answer either way whichever way you prefer. So, we will meet in the next video.