Introduction to Quantum Field Theory

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Lecture 46 : Momentum Space Feynman Rules

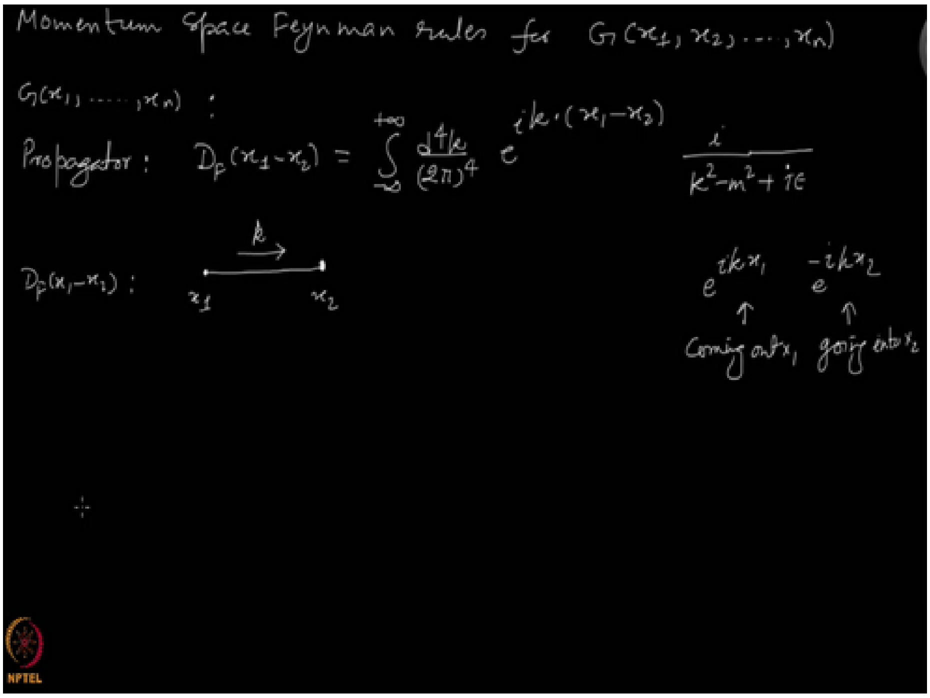


Figure 1: Refer Slide Time: 00:20

We will derive what are called Momentum Space Feynman Rules for greens functions. So, let me write that down. So, they are still looking at these Green's functions which are functions of the space time points and we want to write down Feynman rules in terms of in momentum space. So, we will see soon what I mean by this. So, earlier what we have done is we have written down Feynman rules for these objects. And they were I can hopefully I can go and see. So, they were listed here for example And now, we are going to derive another set of Feynman rules for the same object. So, let us recall that when we had a propagator in the expression so, the propagators which we denoted by x 1 – x 2, D F F x 1 – x 2. This has an expression of the following form. Where this m is the parameter that appears in the Lagrangian, not the physical mass, this is not physical mass necessarily.

If the theory is interacting theory this is not the physical mass but anyway there is the mass parameter that appears in the Lagrangian and this is the Feynman propagator that you have and integral run from minus infinity to plus infinity. And we were denoting earlier the propagator D F between space time points x 1 and x 2 by line connecting these 2. Now, what we will do is we will of course still denote the propagator by this.

So that was whenever we drew this we meant this object. But now, what we will do is, we will keep this expression in mind and it is of course a function of x 1 – x 2. So that information is here. But then I also want to use this variable k which is a dummy variable because it is integrated over. So, D F does not depend on k. Nevertheless, I will put a k here and give it a direction.

And by doing so, I am just reminding myself that in this expression of D F, the dummy variable that I am using that which is k in this case, I will just put an arrow and I give it a direction and put that k here to remind myself that the integration variable used in this expression is k. You see, if you have an expression which has several Feynman propagators appearing.

They all will come with different integration variables, k 1, k 2, k 3 and so forth. And I will just put k 1 here or k 2 on another propagator and so forth to remind myself that is the variable which appears in here. And also I am going to always keep this form in mind. So, the way it is written here is right now, if x 1 is the, if the momentum k, this k is a momentum, not a physical momentum, it is a loop integration variable but it is a momentum variable.

If that is directed from x 1 to x 2 in this then the exponential that I have here the x 1 from which the momentum is coming out as a positive sign. So, it is e to the i k dot x 1 and the 1 into which it is going the momentum is going into x 2 that guy that factor comes with a minus sign. So, all I am saying is this exponential you can write as e to the i k x 1 times e to the minus i k x 2.

And we will remember all this by noticing that the way I have drawn here k is coming out of x 1 and going into x2. So, here x 1 is the one which comes to the positive sign and x 2 is the one which comes to the negative sign. So, this way we will remember this. So, if it is coming out of x 1, it will have a positive sign and if it is going into x 2 then it will have a negative sign.

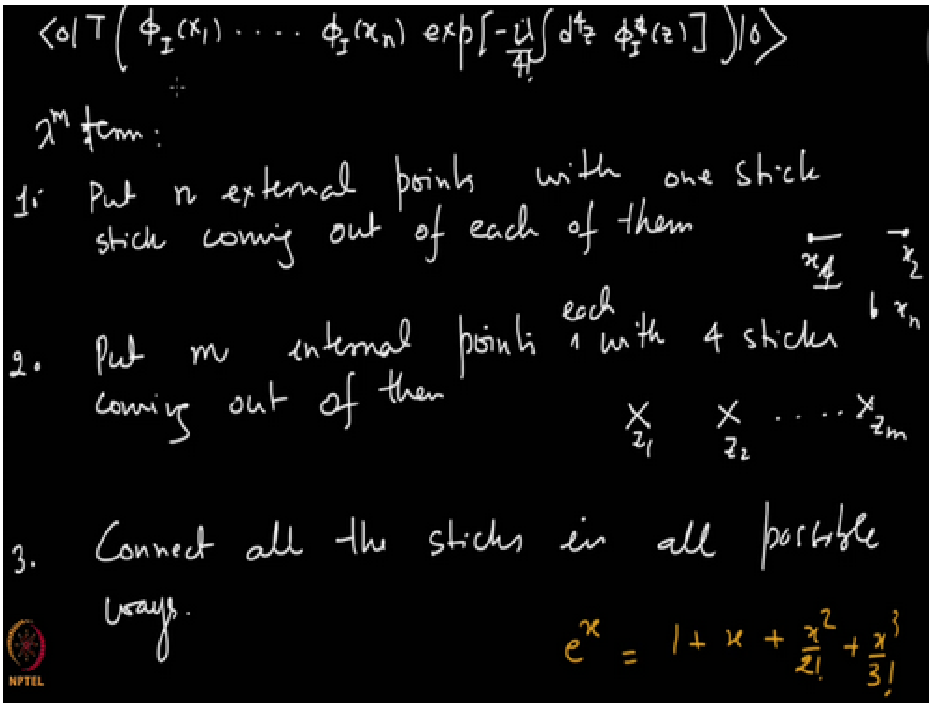
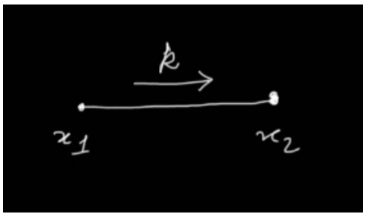


Figure 2: Refer Slide Time: 01:13

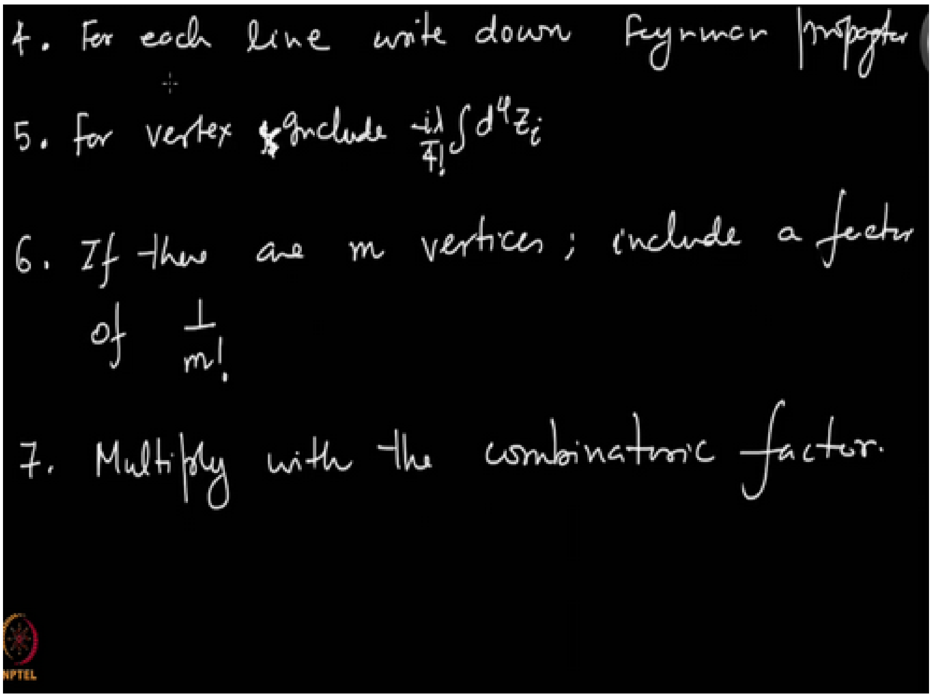
Momentum space Feynman rule for $G(x_1, x_2, \cdots x_n)$

$$D_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x_1-x_2)}\, \frac{i}{k^2 - m^2 + i\epsilon} \tag{1}$$



$$\begin{aligned} e^{ikx_1} &\rightarrow \text{ Coming into } x_1 \\ e^{-ikx_2} &\rightarrow \text{ Going into } x_2 \end{aligned} \tag{2}$$

Now, whether you should put in this direction or in the opposite direction you choose. When you are drawing the diagram that will be your choice but once we have drawn it drawn the propagator and put a directed momentum, I will always follow this. So, now let us look at one of the diagrams which I think we have already analyzed and try to write down the expression for that Feynman diagram.



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So, let me do that. So, here x 1, x 2, x 3, x 4, this is the one which is given to us and we want to find out we want to write down the expression for this Feynamn diagram. So, you recall what Feynman rules we had found, you just put the propagators for each of these lines and include –i lambda over 4 factorial for each of these vertices integrate over z 1 and z 2.

And then you should multiply with 1 over 2 factorial because this is an order lambda squared diagram. There are 2 vertices. So, the exponential in the master formula is being expanded to second order. So, it has 1 over 2 factorial and then you multiply with the combinatoric factor. So that is what we are supposed to do and that is what we are going to give to do.

Let us call this F of x 1, x 2, x 3, x 4. I am giving it a name instead of calling g of these things because g by g i would usually mean all the diagrams that appeared at order lambda square. So, because I am looking at one particular diagram, I am just calling it by this name. So, this is a Feynman diagram and I have this name. So, this as I said, is the following. I should include so, I should first have all the propagator. So, let me call this propagator as propagator number 1, this propagator is propagator number 2, this propagator as propagator number 3 and this as 4, this one 5 and this one 6. So, what I really have here is D F of x 1 – z 1, D F of x 2 – z 1, D F of x 3 – z 2, D F of x 3 – z 2, D F of x 3 – z 4 then D F of z 1 – z 2 for this line, line number 5 and D F of z 1 – z 2 again for line number 6.

And these are all multiplied and then I should integrate over the z 1 and z 2. Then I should include –i lambda over 4 factorial for each vertex. Then as I said 1 over 2 factorial coming from the expansion of the exponential times the combinatoric factor which is we have determined earlier. So that is the expression we have for this Feynman diagram. Now, we will use the explicit expression here, maybe I should just bring it there.

So, let us say, I will just write that rule here. So, it is easy to read. So that x 1, x 2 have a momentum that is coming flowing from x 1 to x 2 and the expression we have is D 4 k over 2 phi to the 4, e to the i k dot x 1 – x 2, i over k square minus m square plus i epsilon. So, now, let us look at all the terms all the factors D F's that contain z 1.

So, look at this term and this term and this term, not term, these factors. So, this will have first I should what I should do is assign momentum to each propagator in this diagram. So, let me assign it this way. So, I am assigning k 1 flowing like this, k 2, k 3, k 4 and of course, you can make a different choice. You do not have to assign moment flowing like this because the k is, the k 1, k 2 and all these are dummy variables.

So, you can choose whatever you wish but let us assign it in this manner. So, if I look at this D F, this z 1 is the one in which the momentum is going. And if momentum is going into x 2, for example, here, it comes with a negative sign. So, this one will contribute e to the i k 1 because that is what is here –z 1. So, this one, this factor gives e to the minus i k 1 dot z 1.

This one will give you e to the minus i k 2 because I am looking at this one the momentum is k 2 times z 2. Then this one which corresponds to k 5 line, line number 5, it will give you what e to the i k 5 z 1, sorry, this was supposed to be z 1 and then this one. Which is corresponding to the line 6, it will give you e to the i k 6 z 1. So, I have just collected all the exponential factors which I will get from these ones, these 4 and which have z 1.

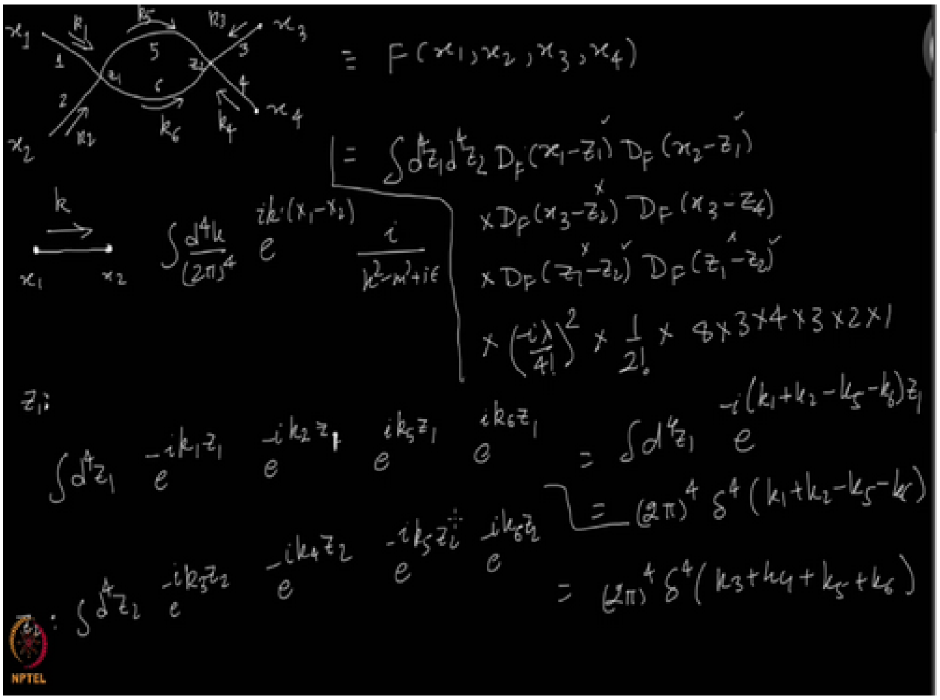
Let us also now, maybe I will just do it first. So, here I have taken care of these vectors and now, I can easily do the integral over z 1. So, I am just doing this entire thing in parts. So, I am first collecting only the z 1 pieces because they are just exponentials. And I can easily do the integral without disturbing anything else. So that is what I am doing. So, first I have collected all the pieces which contains z 1.

And I do the integral d4 z1 and this is easy because this is simply d 4 z 1, e to the minus i k 1 + k 2 – k 5 – k 6 z 1. And this you know, this is just a delta function. Let us do the same thing for z 2. So, let us look at this one, z 2 is here, z 2 is here also and hear at this place. So, here you will get d 4 z 2, let us look at this line x 3. So, again, it is entering into this one so, you will have a minus sign.

So e to the minus i get 3 z 2 e to the minus i k 4 z 2, e to the i k 5 z 2 again with the minus sign because it is entering into this vertex and e to the minus i k 6 z 2. And this will give you again as before and a delta function which will be this. So, you will have k 3 + k 4 + k 5 + k 6. There is no minus sign because all of them have the same sign in this case because all the momenta are entering into the vertex here into this point.

So they all come with the same sign. So, you see what has happened here is that at each vertex now, see we have taken care of what was happening at z 1. So, we have done the integral over z 1, we have done the integral over z 2 and we have taken care of all the factors that contains z 1 and z 2. And the result of those integral has given us these delta functions.

And what are these delta functions saying? These delta functions are saying the following. If



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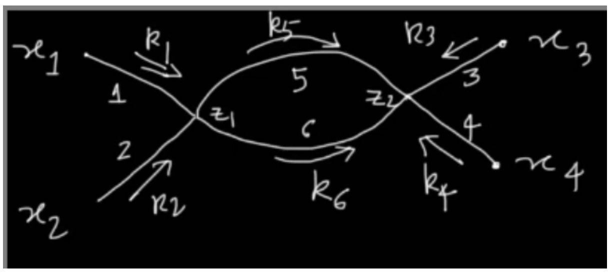
you look at z 1, the integral over z 1 has given a momentum conservation at the vertex. So, what is the total momentum that is entering the vertex z 1, it is k 1 + k 2, k 5 is exiting so, you can think of it as –k 5 entering, k 6 is exiting so, you have –k 6 entering. So, the delta function you get is k 1 + k 2 – k 5 – k 6 which is the total momentum that enters into z 1. And that is what is here and what do you have got it adds it to you have got k 3 + k 4 + K 5 + K 6 which is the total momentum that enters into the vertex. So, k 3 enters k 4 enters k 5 k 6 enters and k 5 enters. Remember still that there is no physical momentum in the problem. Because the object that you have is only a function of x 1, x 2, x 3 and x 4 there are no momenta in the object that you are calculating.

So, these are just the dummy variables. But nevertheless, you see that this is how it comes out. So, this is good and let me write that down.

$$F(x_1,x_2,x_3,x_4) = \int d^4z_1 \int d^4z_2 \; D_F(x_1-z_1)D_F(x_2-z_1)D_F(x_3-z_2) \times D_F(x_3-z_4)D_F(z_1-z_2)D_F(z_1-z_2) \times \left(\frac{-i\lambda}{4!}\right)^2 \times \frac{1}{2!} \times 8\times 3\times 4\times 3\times 2\times 1 \tag{3}$$



$$\int \frac{d^4k}{(2\pi)^4}\; e^{ik\cdot(x_1-x_2)}\; \frac{i}{k^2-m^2+i\epsilon} \tag{4}$$

$$\int d^4z_1\; e^{-ik_1z_1}e^{-ik_2z_1}e^{ik_5z_1}e^{ik_6z_1} \;=\; \int d^4z_1\; e^{-i(k_1+k_2-k_5-k_6)z_1} \tag{5}$$

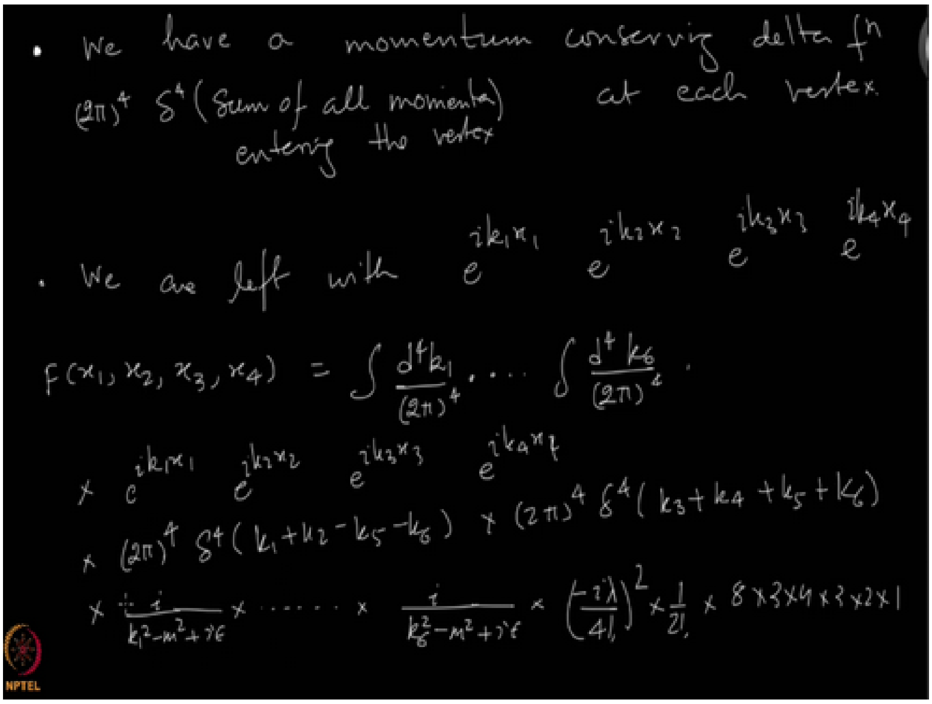
$$\int d^4z_1\; e^{-ik_1z_1}e^{-ik_2z_1}e^{ik_5z_1}e^{ik_6z_1} \;=\; (2\pi)^4\delta^4(k_1+k_2-k_5-k_6) \tag{6}$$

So, what we have seen is the effect of the vertex during the default z integral at the vertex is this that we should include. We see that we have a momentum conserving delta function or more precisely 2 phi to the 4 times delta function at each vertex. And what is the argument of the delta 4 here? It is sum of all momentum, all momenta entering the vertex.

Remember, if some momentum are flowing out of the vertex then you say that negative of that momenta is flowing in and that is how you make this. That is good, the integrals are gone z 1 and z 2 then all z 1 and z 2 are taken care of. What is left is just the external points x 1, x 2, x 3 and x 4, these 4 external points. And let us see what these 4 external points leave behind.

So, they are leaving behind is coming from this exponential because you have e to the i k 1 x 1 where k 1 is leaving x 1. And e to the i k 2 x 2 and e to the i k 3 x 3 and e to the i k 4 x 4. So, we are left with e to the i k 1 x 1 times e to the i k 2 x 2 times e to the i k 3 x 3 times e to the i k 4 x 4 these 4 factors. So, all this is now taken care of up to here; the integrals are done.

The external point contributions are written and what is left is just this piece – i lambda 4 factorial squared and these factors and of course, you are still left with these factors and these integrals to be done and because each propagator comes with these factors. Till now, we have taken care of only the exponential factors. So, eventually now, we have at the end the following.



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So, we have F of x 1, x 2, x 3, x 4 this is equal to so, each propagator has an integral over the this variable k 1, k 2 and so forth. So, you get these 6 integrals because you have 6 propagators then we are left with these factors corresponding to each external point. So, let me write that one. Let me edit here this side into the i k 1 x1, k2 x 2, k 3 x 3, x 4 sorry to much, only for external points. Then we have calculated the z integrals and z1 and z 2 integrals and they give this.

And then what we are left with? We are left with these factors for each external for each propagators so, I have i over k 1 square minus m square plus i epsilon times the same thing 6 times and the last one is K 6 square minus m square plus i epsilon. And then our –i over lambda 4 factorial squared times 1 over 2 factorial times this combinatoric factor. So, the same thing I can write in this form.

So, we can say that we have found new Feynman rules now, for writing down the Greens functions. And so, what we should do is we should for each propagator assign a directed momentum. So, let me list down first. So, momentum space Feynman rules: So, the first rule is to each propagator in your diagram, you assign a momentum. So, first label the propagators by 1 2 3 4 so, you know how many of them are there. Assign a directed momentum. You can choose the direction, there is there is no one telling you which way you should put the arrows, it is up to you, you will get the same answers always.

Then what should we do? We should do the following; we have to have this integral for each of the propagators. So, include for each propagator. d 4 k i over 2 phi to the 4 and you have to integrate over this. Also, for each propagator you should include i over k i square minus m squared plus i epsilon because you see that you have this thing coming for each propagator.

So, what I am basically doing is right now, it is just trying to make a rule which is easy to remember. So that whenever there is a Feynman diagram, I can just write down the expression of it without doing a lot of work. So, I have taken care of this piece now. The integrals have taken care of this piece. Now, let me make this has a rule that at each vertex I should have a momentum conserving delta function.

At each vertex include 2 phi to the 4 times delta 4 of all momentum that is entering and that will take care of these 2 factors. What is left now? This is left. So, what is the rule here? That I should make the rule should be this. For each external point x I, if the external point is x i then for each of them assign a factor, yeah may be here I can make a k give better this way. So, for each external point x k assign a factor of e to the minus l k x k. If l k enters the vertex or enters the external point so, in our case let us see what we should expect and what we are getting. So, here k 1 is not entering x 1, it is exiting it so, k 1, – k 1 is entering x 1. So, if – k 1 is entering x 1, I should get from my rule e to the minus i k 1 sorry minus so, I should get e to the minus i minus k 1 x 1. Which is e to the i k 1 x 1 and which is precisely what I have.

So, this rule is also correctly formed. Now, what is left is only these factors. All else I have taken care of. So, I will turn also them into rules. (Refer Slide Time: 30:16)

So, the rule is include minus lambda our 4 factorial for each vertex and finally, not finally yet, include i m c l u d e include 1 over V factorial, if they are you have V number of vertices in this case you have 2 vertices so, V is 2. 1 over V factorial, if there are V vertices in the Feynman diagram. And finally, multiply with the combinatoric factor. So, we have the fullest list of momentum space Feynman rules for writing down these Greens functions.

And I think they are quite straightforward. What you should remember is that we have just created a mnemonic I think it is called mnemonic. So, that we can easily remember what we should be writing down as an expression for the Feynman diagram. But if you ever have a doubt why this rule or that rule has to be there, you go back and start writing down the expression from scratch.

$$\int d^4 z_2 \, e^{-ik_3 z_2} e^{-ik_4 z_2} e^{-ik_5 z_2} e^{-ik_6 z_2} = (2\pi)^4 \delta^4(k_3 + k_4 + k_5 + k_6) \tag{7}$$

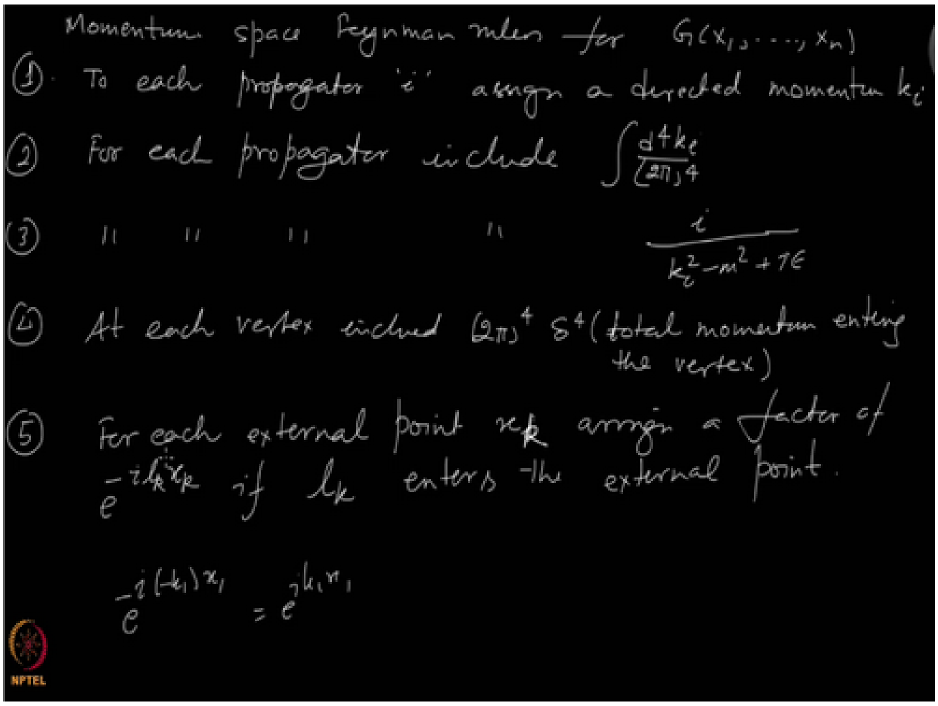

Figure 6: Refer Slide Time: 25:31

•We have a momentum conserving δ function

$$(2\pi)^4\delta^4(\text{sum of all momenta entering the vertex})\,,\ \text{for every vertex}$$

• We are left with $e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}e^{ik_4x_4}$

$$\begin{aligned} F(x_1,x_2,x_3,x_4) &= \int\frac{d^4k_1}{(2\pi)^4}\cdots\int\frac{d^4k_6}{(2\pi)^4}e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}e^{ik_4x_4} \\ &\quad\times(2\pi)^4\delta^4(k_1+k_2-k_5-k_6)\times(2\pi)^4\delta^4(k_3+k_4+k_5+k_6) \\ &\quad\times\frac{i}{k_1^2-m^2+i\epsilon}\times\cdots\times\frac{i}{k_6^2-m^2+i\epsilon}\times\left(\frac{-i\lambda}{4!}\right)^2 \\ &\quad\times\frac{1}{2!}\times 8\times 3\times 4\times 3\times 2\times 1 \end{aligned} \tag{8}$$

Momentum space Feynman rule for $G(x_1, x_2, \cdots x_n)$

1. To each propagator $'i'$ assign a directed momenta k_i.

2. To each propagator include $\frac{d^4k_i}{(2\pi)^4}$

3. For each propagator include $\frac{i}{k_i^2-m^2+i\epsilon}$

4. At each vertex include $(2\pi)^4\delta^4(\text{total momenta entering the vertex})$

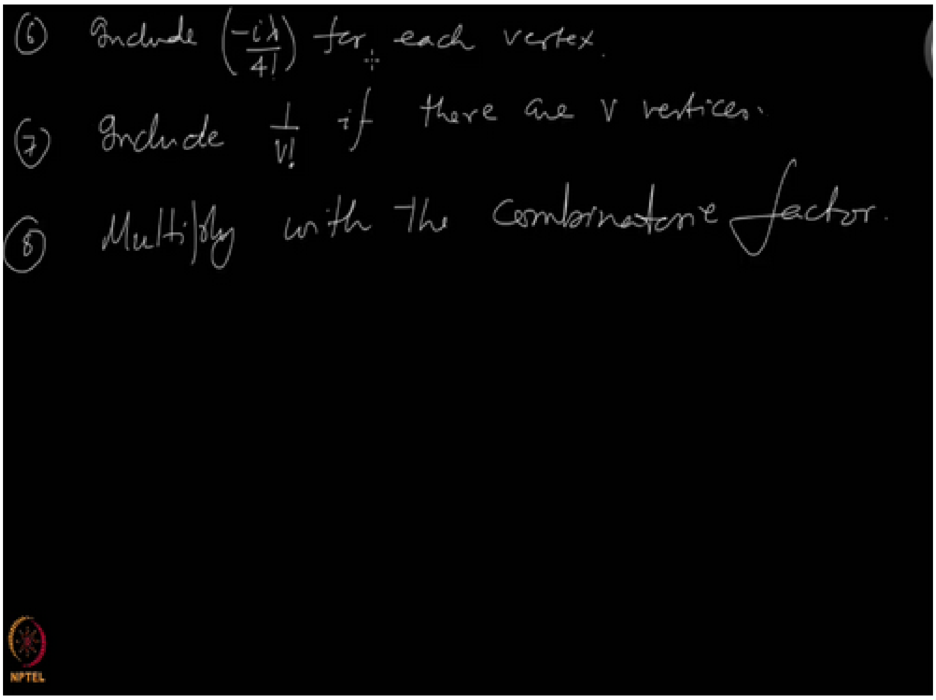

Figure 7: Refer Slide Time: 30:16

5. For each external point x_k assign a factor of $e^{-il_k x_k}$, if l_k enters the external point $e^{-i(-k_1)x_1} = e^{ik_1 x_1}$.

6. Include $\left(\frac{-i\lambda}{4!}\right)$, for each vertex.

7. Include $\frac{1}{V!}$, if there are V vertices.

8. Multiply with the combinatorial factor.

And do all this and convince yourself that indeed that is what should be termed as a or what should be made as a rule so that the entire expression gives you exactly the same answer. So, this is not something some deeper understanding we are just trying to create some memory devices. So that we can write expressions directly without going through the full computation starting from the beginning. So good. And we will stop here and we will continue our discussion on Feynman diagrams and Feynman rules in the next video.