Introduction to Quantum Field Theory

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Lecture 44: Feynman Diagrams

Feynman Diagram

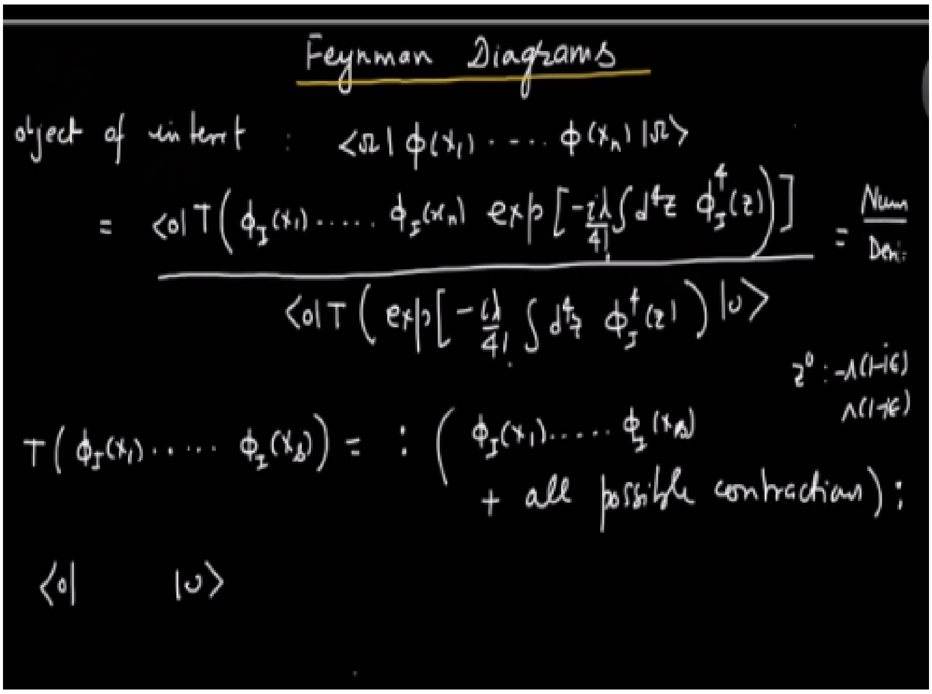

Figure 1: Refer Slide Time: 00:13

Okay, so let us recap what we have been doing. So the object of interest to us are these correlation functions in the interacting theory, okay. And we have then written down everything in the interaction picture where the objects involved are objects in free theory. So for example, this is free vacuum and then we showed that this expression or this correlation function can be written as this.

So I am writing for phi-4 theory right now. So d 4 z and phi I z to the 4, okay. And there is a lambda over 4 factorial here. And this all sits within the time ordering. And then of course, we remember that we have to divide this entire thing with almost what you have in the numerator except for these fields, okay you just have this exponential. So time ordered, okay.

And also remember that the time component here is not running from minus infinity to infinity. It runs from okay I should not write here. So you know that the time component z naught is running from minus lambda 1 minus i epsilon to lambda 1 minus i epsilon where lambda goes to infinity and eventually we have to take epsilon going to 0 limit. But the space components run from minus infinity to plus infinity, okay.

That is what we had. Now we have also seen in the last video that time ordered product of these fields whatever you have in numerator or any number of fields, I am writing x s with a subscript s to distinguish from this n, okay. So this we saw that can be written as normal order product like this plus all terms plus all possible contractions, okay.

And this is very useful because the moment you sandwich this time order product between vacuum states, between vacuum state because everything is normal ordered. All terms that have one or more operators get killed, right? Because it is an operator, it will have either an a or an a dagger or it will be phi plus or phi minus and that will kill the vacuum. And only those terms will survive that have no operators in them.

Meaning all the fields are fully contracted. Only those out of these contracted contractions will survive, everything else will vanish, okay. And this remember this is what we call is our master formula. And I will just write numerator and denominator of the master formula. So let us look at numerator.

Object of interest

$$\langle\Omega|\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \frac{\langle 0|\,T\left(\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\text{exp}\left[\frac{-i\lambda}{4!}\int d^4z\,\phi_I^4(z)\right]\right)|0\rangle}{\langle 0|\,T\left(\text{exp}\left[\frac{-i\lambda}{4!}\int d^4z\,\phi_I^4(z)\right]\right)|0\rangle} = \frac{\text{Numerator}}{\text{Denominator}} \tag{1}$$

$$z^0 \to \Big(-\left(\Lambda(1-i\epsilon)\right)\,,\;\left(\Lambda(1-i\epsilon)\right)\Big)$$

$$T\Big(\phi_I(x_1)\cdots\phi_I(x_s)\Big) =: \Big(\phi_I(x_1)\cdots\phi_I(x_s) + \text{all possible contraction}\Big) : \tag{2}$$

Numerator

$$\lambda^0 \quad : \quad \langle 0|T\Big(\phi_I(x_1)\cdots\phi_I(x_n)\Big)|0\rangle$$

$$\lambda^1 \quad : \quad \langle 0|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\int d^4z\,\left(\frac{-i\lambda}{4!}\right)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\right)|0\rangle \tag{3}$$

$$\lambda^2 \quad : \quad \langle 0|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\int d^4z_1\,\left(\frac{-i\lambda}{4!}\right)\phi_I^4(z_1)\int d^4z_2\,\left(\frac{-i\lambda}{4!}\right)\phi_I^4(z_2)\right)|0\rangle \tag{4}$$

let n=4

Numerator

$$\lambda^0 \;:\; \langle 0|T\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big)|0\rangle \tag{5}$$

$D_F(x-y)$: Feynman propagator

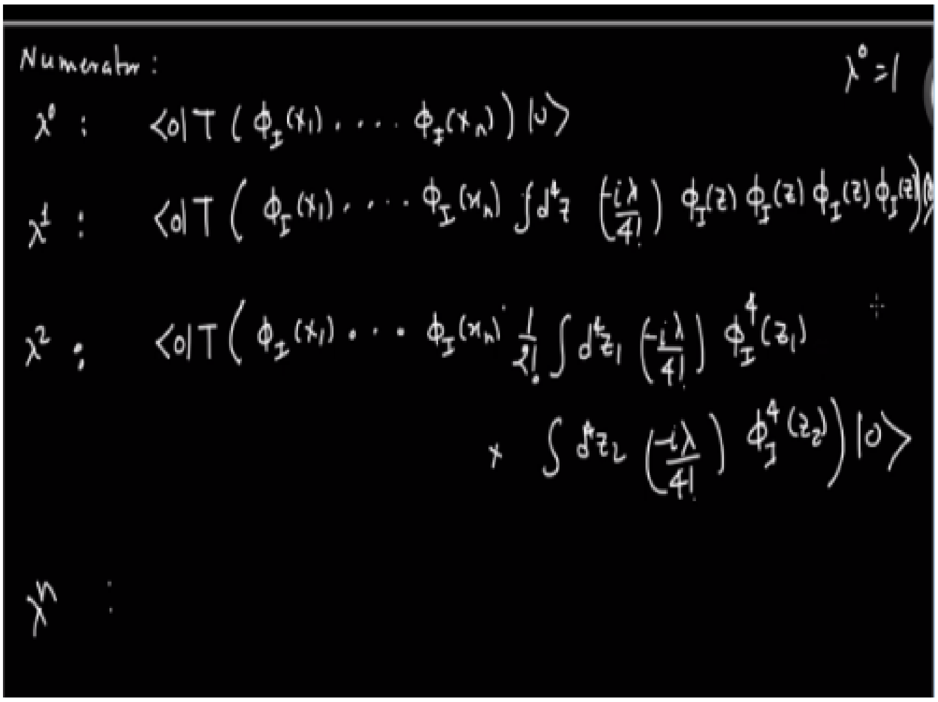

Figure 2: Refer Slide Time: 04:34

So what I am going to do is now just expand the exponential. So the first term is 1 which is ordered lambda to the 0 term, lambda to the 0 means 1, right? Lambda to the 0 is 1. So first term is anyway 1 and I write lambda 0 term is we have an external fields, n fields. So this is what numerator gives. So imagine if you are working in free theory, then the lambda would be 0 and this exponential will not be there.

Then this term first term coming from the exponential, that first term will be the prediction for a free theory, right? So when you look at the first term that corresponds to free theory, which is what I have written here. But now let us keep order lambda 1. So now you have effects of interaction. And this is the following. And then you have, let us go back.

You have a -i lambda over 4 factorial d 4 z of this object. So this is what you will get. Integral d 4 z –i lambda over 4 factorial. And you have phi I z, sorry. I am writing it four times. Because it is phi-4 z. So I am just writing it four times, okay. So this is what we have. Let us write down one more. If you are at order lambda square, then the result would be this, okay.

Then you will have two factors of this piece coming by expanding the exponential. So you will have first 1 over 2 factorial. And that 1 over 2 factorial comes because of the exponential. So exponential, e to the x is 1 plus x plus x square over 2 factorial. So that 2 factorial I am talking about, okay. So that is this 1 over 2 factorial. And then you have two pieces of this piece appearing twice, right?

So I write it like d 4 z 1 phi I z 1 4. So you can write four factors like this. And then d 4 z 2 phi I to the 4 of z 2, okay. And I forgot. Let me do it again. Minus i lambda over 4 factorial phi I z 1 to the fourth times d 4 z 2 minus i lambda over 4 factorial phi I z 2, okay phi I to the 4 z. And then you have this which is closing this bracket and the vacuum state.

So you see the moment you have lambda square term, the additional thing is 1 over 2 factorial here which appears. And if you go to order lambda to the n apart from all these things, we will have here 1 over n factorial and n such factors like this, okay. So that is the numerator that we have.

And we can use Wick's theorem now to convert this time order product to objects involving only contractions or only the Feynman propagators, right? So let us do that.

So let us take n equal to 4. I am looking at one example. And I look at the numerator for this case, okay. And at order lambda to the 0 that is at order lambda 1 sorry at order lambda to the 0 we have this object, okay. All the terms which are containing lambda they have they did not contribute, okay. This is what we have. Now we have calculated this already. I think we have done that, let us check. Yeah, here. You see this is what we calculated. This phi 1, phi 2 these are just phi I's at x 1, x 2 and so forth. And this we saw that when sandwiched between vacuum, only these terms survive in which all the phi's are contracted. There is no phi which is uncontracted. So here the ones which were uncontracted they got killed by the vacuum, okay or rather they killed the vacuum, okay.

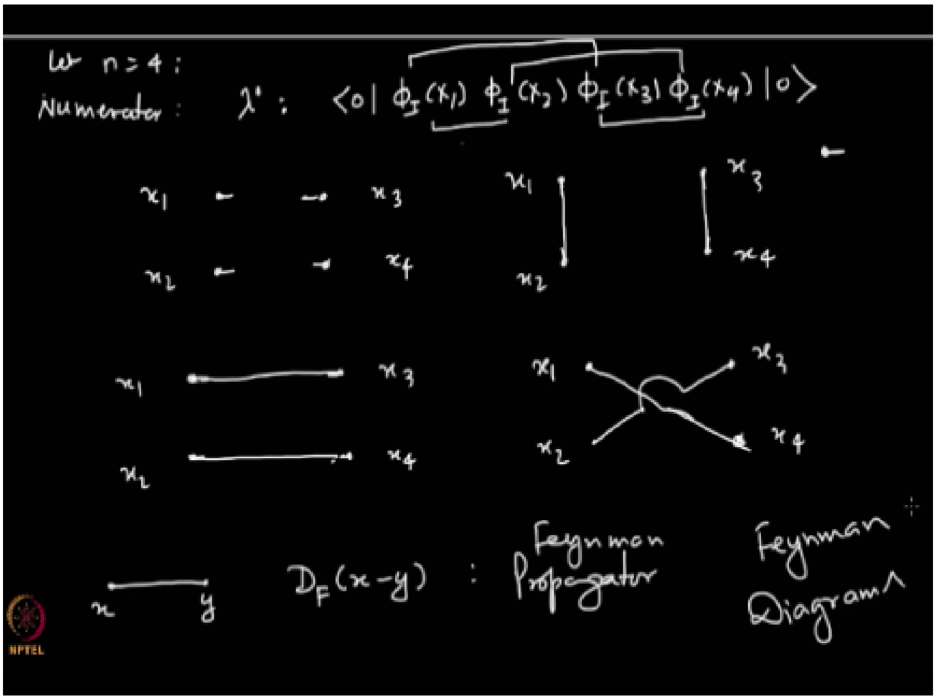

Figure 3: Refer Slide Time: 09:18

Numerator: λ^1

$$\langle 0|T\left(\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\int d^4z\left(\frac{-i\lambda}{4!}\right)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\right)|0\rangle \tag{6}$$

4 external points and 1 internal point. The 4 contractions between $x_i's$ and z will give us,

$$\begin{aligned}\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\int d^4z\left(\frac{-i\lambda}{4!}\right)\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\\ &= D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)\\ &+ D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)\\ &+\\ &.\\ &.\\ &.\end{aligned} \tag{7}$$

So only these three survived and these three of course, we wrote in terms of the Feynman propagators. So we have exactly three terms here. First term here, second term this, third term that one, okay. So that is what we are looking at again, okay. So what I will do is I will try to have a diagrammatic way of representing the same calculation that we did before, okay.

So the way we will do is this. To each point x 1, x 2 and x 3 x 4 I will associate a point like this and a stick coming out, okay. One stick coming out would mean there is one field at that point. So let me represent here x 1, x 2, x 3, x 4. So these are our external points. By external I mean the things which appear in here, the arguments which appear in here, okay.

And these ones which are integrated over, them I will call internal points. So z will be an internal point in this and x 1, x 2 and x n these are external points, okay. That is just a name that I give. And as I said, because at x 1 there is one field. So I take out one stick at field, at point x 2 I have again one field so I have like this, okay. So we have one stick coming out here, okay.

Now our Wick's theorem says that I should take this and do, I mean the result will be all possible contractions, okay. So all those terms which are fully contracted, those ones. So what are those ones? So you will have one term in which phi I x 1 is contracted with phi I x 2 and x 3 is contracted with x 4. So this gives a propagator D F (x 1 - x 2). And this gives you a propagator D F (x 3 - x 4), right. So this propagator D F (x 1 – x 2) this contraction I will denote by just connecting these two, okay. I wish I had drawn that x like this in this case. It does not matter, but let us do it this way nevertheless. So x 1 should be contracted with x 2, so I connect x 1 and x 2. And then I could contract this, which means I connect it here, okay. So I am representing these contractions by these diagrams.

So that is one diagram. There is another possibility that I contract this with that one and this with that one, okay. So I will show that one by the following diagram. So again I start by first putting down the external points, x 1, x 2, x 3 and x 4, okay. And each of them has one stick coming out of them, okay. Now I should contract here x 1 with x 3, which corresponds to joining these two.

And x 2 with x 4, which corresponds to joining these two. So you have got another diagram for this expression. And there is third one. Remember we have had, we had a total of three terms like this. So this is what exactly we are writing down and just making pictures and but the pictures represent exactly the same expression. So this one is D F (x 1 – x 2) times D F(x 3 - x 4), okay.

So here again I should start by writing the external points. And then there is one possibility remaining which is x 1 getting contracted with x 4, okay. And then x 2 with x 3, I just to, because it is in a plane it will look like it is touching this line. So I just draw it like this. This just means that it is going above it, okay. But mathematically it means the same. It is D F (x1 – x 4) times D F (x 2 – x 3), okay.

So each term sorry each line in this is a propagator. And if there are two lines you should multiply the two propagators, right? That is what we have seen, right? We have to multiply these two and it is just this expression which I am showing as a diagram, okay. So then as I said, if you have two points x and y and joining these two points by a straight line we mean this okay. And this is called a propagator or a Feynman propagator, okay. And you have to construct, you have to draw all these diagrams by joining all the points in all possible ways, because once you have connected all the points okay, all the operators have now combined to form propagators and there is no operator left to kill the vacuum, okay. So these are the only ones which will give a non-vanishing contribution in this example, okay. Now let us go to order lambda terms in the numerator. So again we are looking at numerator and now we are interested in order lambda 1 term. So what we have now is this. Okay, that is what we have. Now there is in addition to these, in addition to these four external points we have one internal point here which is z, okay. Now

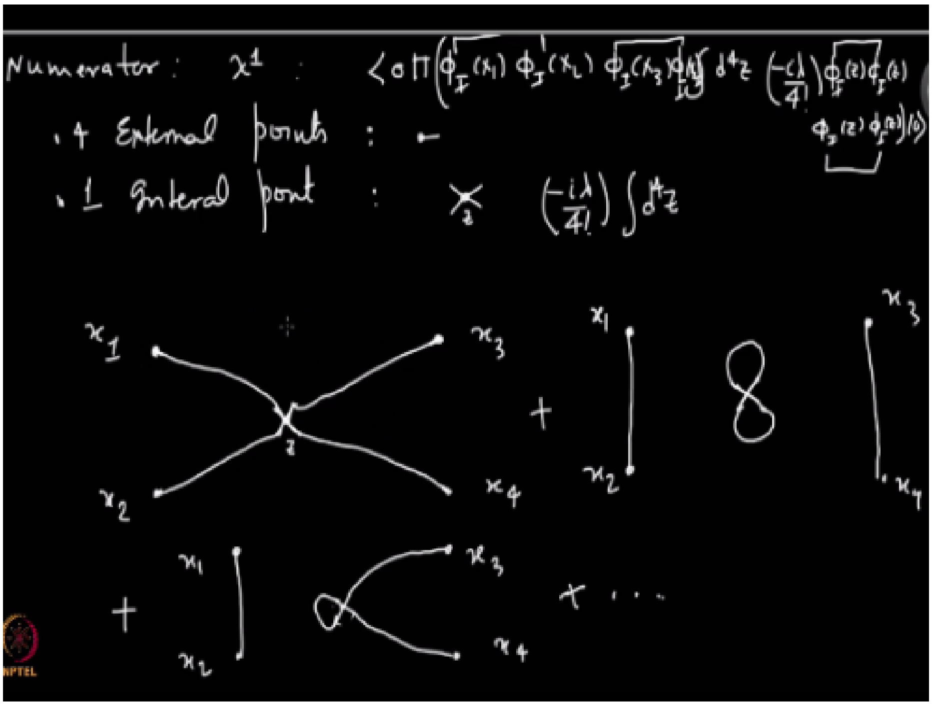

Figure 4: Refer Slide Time: 17:08

as always external point, you put the point and then you take a take one stick out of it because there is one field at that point which is phi I, okay.

But look at the internal point at point z. So again, I draw one point here for z. But then there are four fields at point z. This one is also at z. This one is also at the z, this one and that one. So instead of one field, I have four fields, so I take out four sticks. See these sticks just remind me that they are four fields at the point z. That is what the purpose of the sticks is, okay. So that is one thing.

And also remember that whenever you have an internal point, okay, it comes with a factor of minus i lambda over 4 factorial. So when you are writing the expression of this object, whatever you do, this piece will come and in fact you will also have integral d 4 z. That integral is still there right, and this piece. So this we have to keep in mind. So let us see what will be the Feynman diagrams.

I think I did not say that. These are called Feynman diagrams, okay. When you start joining these ways, these are called Feynman diagrams. They are just the diagrammatic representation of these correlation functions, okay. So what would be a Feynman diagram of this? So before drawing the diagram, just let us see what it would mean to write down an expression of this.

So we will use again Wick's theorem and convert this into a normal order product of fields, and then do all possible contractions and keep only those terms in which all the points are contracted or all the all the sticks are connected, right. And when you do so you can connect phi x 1 with this phi (z), this phi (x 2) with that phi (z), this phi (x 3) with this phi (z), this phi with this phi.

And you could do other way around. So you could take this one and contract with this one. So x 1, x 2 get contracted. So there are many possibilities. And it will be easier to list them out if we draw diagrams. So let us draw a diagram for this order lambda 1 term, okay order lambda term. So let us see. So I start by again, putting the external points. So always fix your external points.

So you label which one is which. So this is x 1, this is x 2, this is x 3, this is x 4, okay. We have one stick coming out of them, okay. I am not drawing the stick now. Or maybe I can and then you have a point z, which is internal point. And therefore sticks coming out of them, out of it, okay. So now let us do contraction. There are many contractions and let us write one of them first.

So I could contract, okay before that, so let us say this stick corresponds to this field phi. This stick corresponds to that field phi. This one to this field and that one to that field, to this one, okay. So one possibility is I contract the phi at x 1 with the first phi here, which means I contract this and that. So that is one Feynman propagator. Then I could contract x 2, phi x 2 with this phi.

So you get this. Similarly, you can contract here, okay. And you can contract there. So that is one possible term. And of course, there are many more contractions. So let me draw one more. And of course, you have to add all of them, right? Because that is your Wick's theorem. You have to add all of them. So here let us again start like that, x 1, x 2, x 3 and x 4 and then you have this. Now I could connect this with that.

So I contract these two, which means I am connecting like this. And I am connecting these two, which means this. And then I connect this and I connect this, which means two of these are contracted and other two are contracted. So this is another possible Feynman diagram for this expression. And maybe one more I will show and you should check if there are any, which other diagrams are left, okay.

So what we could do is, we could contract x 3 and x 4 with the vertex here, okay. This internal point is also called a vertex. So you could connect x 3 with this. So here I am doing what I am doing is x 3 with one of the phi's, x 4 also with one of the phi's, okay. And then the remaining two phi's in the phi I z's they, let us contract them, okay. So I join this one with itself, okay.

Even though it is circular, but it is just a propagator, right? This is, what is that object? That is contracting one of the phi's here with another phi here. So that is a propagator. So even though it goes circular, it is just D F (z – z), okay. And then this is the only remaining thing, you contract x 1 with x 2, okay. And then there are other terms. So good, we are able to draw Feynman diagrams for this expression.

But there is still more to say. So here, if you look at this one, I have drawn this diagram and here x 1 has gone to one of the lines here, x 2 has gone to another and so forth. But there are many more diagrams of this variety. Let us understand this.

Combinatoric factor $4 \times 3 \times 2 \times 1 = 4!$

$$\langle 0| \, T\left(\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\mathrm{exp}\left[\frac{-i\lambda}{4!}\int d^4z\,\phi_I^4(z)\right]\right)|0\rangle \tag{8}$$

So let us, and then we have integral d 4 z okay and you have of course, the factor minus i lambda over 4 factorial. So when you are contracting, you have this contraction. This is the one corresponding to our diagram that we have drawn, okay. And what is the result? This is just D F(x 1 – z), right? So this is contracted between this and this one. So D F(x 1 – z) D F(x 2 – z) D F (x 3 – z) D F(x 4 – z) okay.

And then the other factors are of course there, let me not worry about them right now. But then you could have done something else. You could have contracted this one with not this field, but that one. So let us say we are looking at another contraction. So this is another possible contraction, okay. And this one goes, gets contracted with this one, and so forth. So this is a different contraction.

But the expression is again identical because these are getting contracted with these fields which are all at the same z. So again you are going to get the same thing, right? It is identical

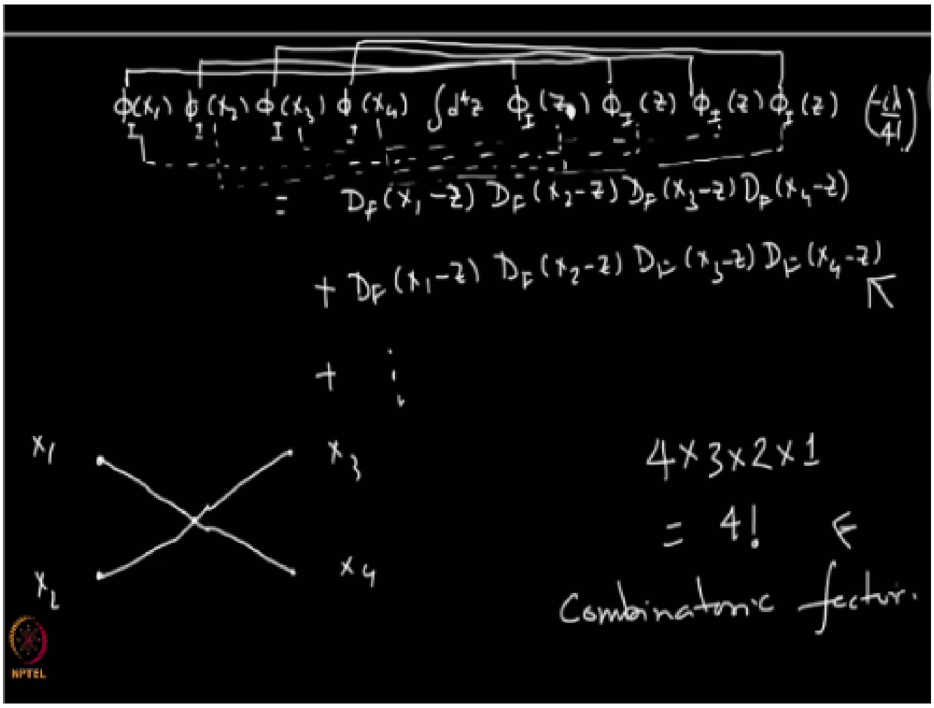

Figure 5: Refer Slide Time: 25:44

result. And there are still more possibilities. Instead of taking x 1 and contracting it with this phi at z, you could contract this one with the third factor, and you get another term.

So you get several terms, which in the language of diagrams is this. So let us start again with x 1, x 2, x 3 and x 4. And here is our internal point or the vertex. So x 1 could go to could contract with any of this. So it could go to any of the four. So there are four possibilities here. Let us say it goes to this one, okay. So you have made one particular choice and you have contracted with let us say the first one.

Then this phi could contract either with the second factor or with the third factor or with the fourth factor. So it has three possibilities, okay. So let us say x 2 goes to one of them. And we have three possibilities. Now the third field phi x 3, this has still two possibilities where it could go to any of the remaining two. So it could go to any of the remaining two lines here. Let us say it goes to 1.

So there were two possibilities that I have counted. Now once three of these are contracted, the fourth one will have only one option left and it can go to only 1. So you see you have four factorial diagrams or contractions, which give identical result. They all will give exactly this expression, okay. So instead of drawing the same thing again and again 4 factorial times, we will just draw it once.

And to count all these possibilities, we will include in the expression, this 4 factorial, okay. And this we call combinatoric factor, okay. Okay. So let us see what we have done till now.

So if I am, I want to write an expression of this form, I am just summarizing, okay. And you have okay, and you want to get the term of order lambda to the m. Let us say

λ^n **term**

1. Put n external points with on stick, stick coming out of each of them.

2. Put m external points each with sticks coming out of them.

3. Connect all the sticks in all possible ways.

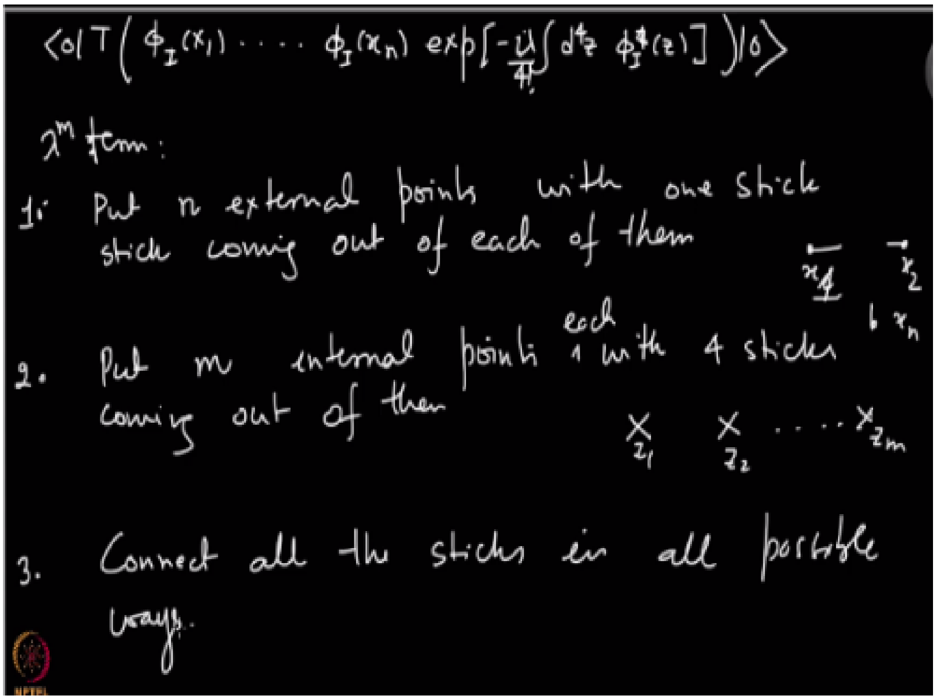

Figure 6: Refer Slide Time: 30:21

4. For each line write down Feynman propagator.
5. For vertex include $\left(\frac{-i\lambda}{4!}\right) \int d^4 z_i$
6. If there are m vertices include a factor of $\frac{1}{m!}$.
7. Multiply with the combinatoric factor.

Then for each vertex or the internal point include d 4 z i okay. So if they are m of them, d 4 z 1, d 4 z 2, d 4 z 3 so forth up to d 4 z m. And each time you write d 4 z you have a factor of minus i lambda over 4 factorial also with you, okay. Because that is always going to be there, right? Whenever you have one such piece, there is going to be a minus i lambda over 4 factorial because when you expand this, entire thing will always come together.

Then one more piece because when you are expanding the exponential at nth, the nth term or lambda to the n term, we will have 1 over n factorial, right. So that we have to include. So if you have, if there are m vertices or equivalently if you are at order m, then you should include a factor of 1 over m factorial, okay. And the last thing, we should count the combinatoric factor for each diagram.

Because some of the diagrams one can draw in many ways as we discussed and then we should multiply that factor. So multiply with the combinatoric factor, okay. So in principle we have now expression for these objects the numerator in terms of Feynman propagators and these coefficients some numbers and this Feynman propagators have to be integrated over the internal points, okay.

At least in principle we have the result for this object at any ordering lambda. So whatever order lambda you want, you can write down those expressions or you just draw the Feynman diagrams which contains the same information, okay. So we will continue discussion more on Feynman diagrams in the next video.

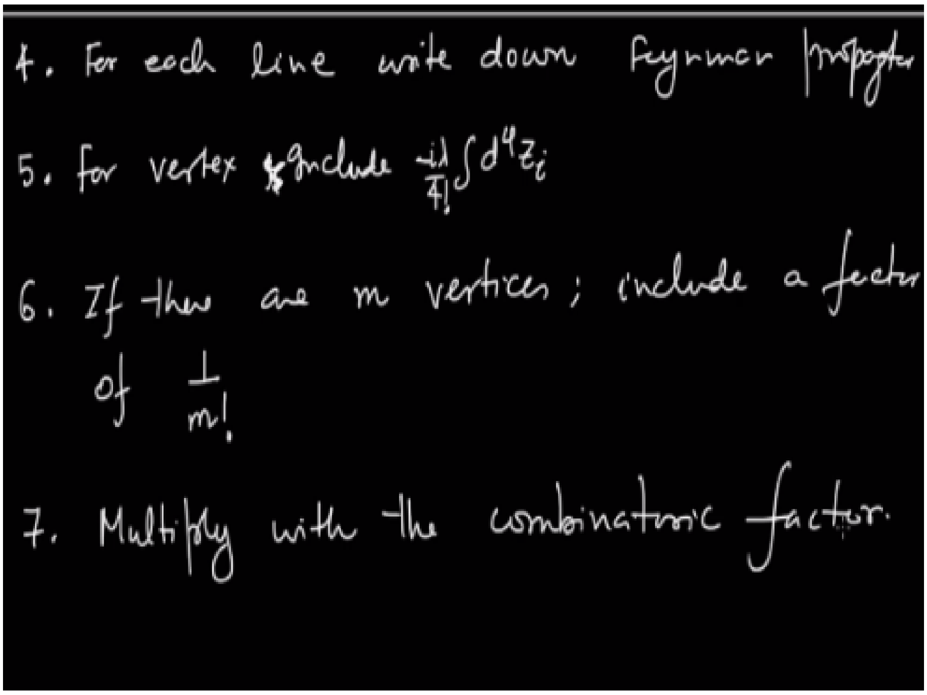

Figure 7: Refer Slide Time: 33:23