Introduction to Quantum Field Theory

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Lecture 43 : Wick's Theorem Continued

Wick's theorem

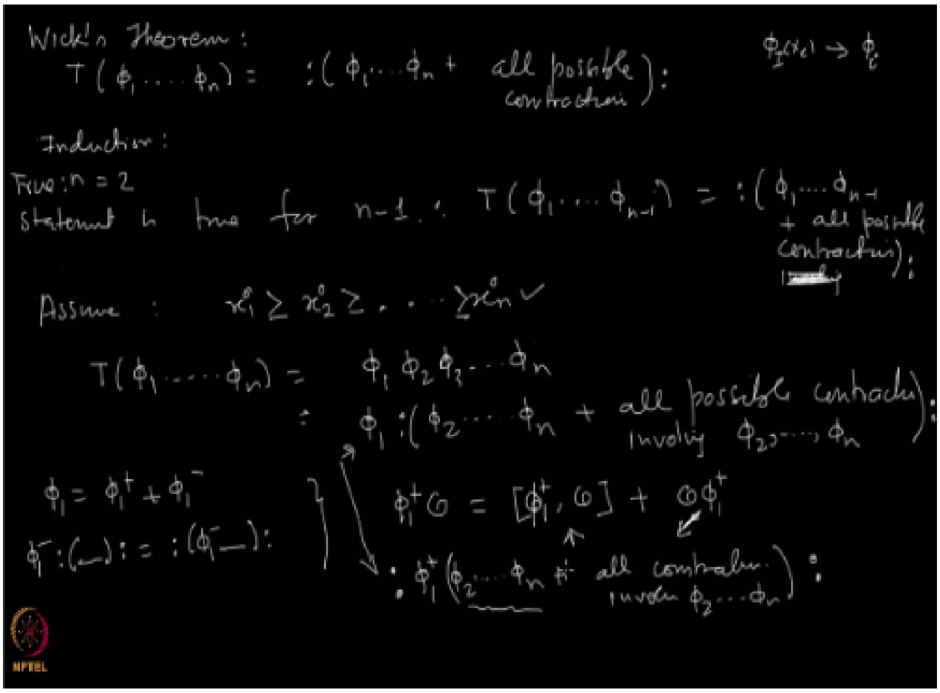

Figure 1: Refer Slide Time: 00:13)

Let us prove Wick's theorem and let me remind you what we saw last time. So we saw that if you take a time ordered product of fields phi I x 1 to phi I x n, so you have n number of fields here, okay. Then the time ordered product can be written as a normal ordered product of I mean normal ordered product of the fields phi I x 1 to phi I x n and also you have to sum up all the possible contractions.

Just like we saw in this case for the case of four fields, okay. So you have all these four fields here and the terms involving contractions. So let us prove it in general and the proof will be through induction and for ease of notation I will drop writing phi I x i and instead I will write phi I, okay. So if it is phi I x 1, it becomes phi 1 and so forth.

So the theorem we want to prove is that if you take a time ordered product of n fields, then that is just a normal ordered, normal order product of these fields. Okay, so you have these operators but you should normal order them. And the reason this is useful is because once you have normal ordered operators they kill the vacuum to the right and also to the left, okay.

So that is the reason why we want to do this. Now we have already seen that this is true for n equal to 2, okay. And let us say the statement is true for n - 1 fields and then we will prove that it is true for n fields also okay, if these two conditions are satisfied. So let us start. So I am given that and we are assuming that this is true for n - 1 fields.

So this is just all possible contractions involving, let us name it 2 here, it will be easier to do then or this one involving all possible contractions. That is right, that is sufficient, right. Okay. Now let us see what we get for n fields. So let me assume that x 1 0 the time component of x 1 is the largest one. The time component of x 2 is the second largest.

And the time component of x n is the smallest, okay. Assume this and then I can write T time order product of these fields as phi x 1, phi x okay. I was writing 1 and 2; phi 1, phi 2 to phi n. Okay just because I have assumed this time order, so the one with the largest time comes to the leftmost position and so forth, okay.

But if this is not really the time order, if it is something else, then we just write the corresponding relation here. So suppose x 2 naught was greater than x 1 naught and others remain the same. Then I will write here phi 2 phi 1 up to phi n, okay. And whatever I given the argument below does not really depend on this. So you can, you can choose the time order you want. And then the same thing will follow, okay.

So this is fine. So I write it as phi 1. Now see phi 2 to phi n I have chosen this time ordering right, which means that these are time ordered. These are correctly time ordered. So phi 2 has the argument, time argument which is larger than any of the other fields here. Similarly phi 3 here will have time argument larger than any of the fields which follow here, okay. So I can write this piece using this statement which is true for n - 1 fields. So phi 2 to phi n these are n - 1 fields. So I can write this right hand side, okay. Good. Now I write phi 1, phi 1 plus, plus phi 1 minus okay. And note that when I take this and put in here I will be multiplying phi 1 plus with these terms, this normal order product plus phi I minus times this normal order product. Now the one with phi 1 minus is easy.

Because if you take phi 1 minus okay and you multiply with this normal order product you can take the phi 1 minus within the normal ordering. Let me write it down. So whatever you have here, this you can write as, okay let us whatever was there. Okay, this is true because you see here these fields are operators, these are operators and this one is sitting to the left of everything, okay.

$$T\Big(\phi_I(x_1)\cdots\phi_I(x_n)\Big) =: \Big(\phi_I(x_1)\cdots\phi_I(x_n) + \text{all possible contraction}\Big): \tag{1}$$

By induction

Statement is true for $n-1$

$$T\Big(\phi_1\cdots\phi_{n-1}\Big) =: \Big(\phi_1\cdots\phi_{n-1} + \text{all possible contraction}\Big): \tag{2}$$

Assume

$$x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$$

$$\begin{aligned} T\Big(\phi_1\phi_2\phi_3\cdots\phi_n\Big) &= \phi_1\phi_2\phi_3\cdots\phi_n \\ T\Big(\phi_1\phi_2\phi_3\cdots\phi_n\Big) &= \phi_1 : (\phi_2\cdots\phi_n + \text{all possible contraction involving}(\phi_2\cdots\phi_n): \end{aligned} \tag{3}$$

$$
\begin{aligned}
\phi_1 &= \phi_1^+ + \phi_1^- && (4)\\
\phi_1^- : (\_) : &= : (\phi_1^- \_) : && (5)
\end{aligned}
$$

And what a normal ordering says, it says that put all the phi minuses to the left and phi pluses to the right, okay. And phi minuses commute among themselves of course and phi pluses commute among themselves. So you can take this phi 1 minus which sits to the left and if you put in the normal order, what will the normal order do? It will put all the minuses to the left which is exactly what you have here, right?

So this is true. Now let us look at the other term which is phi 1 plus okay times this operator. Now phi 1 times whatever operator I can write as phi 1 commutator with O plus O phi 1 plus, okay. Now here again you see this is the correct one, this is if the phi 1 plus is to the right then it is a normal ordered thing. So I can write this piece this term here as phi 1 plus, I am writing this term now.

Phi 1 plus and there is these all these operators, plus all contractions involving phi 2 to phi, okay. And this can be brought into the normal ordering, okay. So let us look at this term first, this times the phi 1 plus okay or phi 1 plus here times this normal order product. So I am looking at the first term here, okay. So you are looking at the first term, phi 1 plus and our first term involves no contractions, only these fields, okay. And as I said here, this times this operator so let us say O represents this operator. This I can write as a normal order product involving these two, okay. So I will have the same phi 2 to phi n and phi 1 plus, okay and in the normal, okay. And then you have these commutators.

So what you get here is okay, so because this is a normal order this will automatically put phi 1 plus to the right, to the right of all phi minuses and of course, if there are other phi pluses it will anyway you can commute them, it does not matter. But so it is clear that this is how you can write. And the commutators will give these terms okay and so forth.

Which is you can put all in one big normal ordered bracket. So you have phi 1 plus, phi 2, phi n. So this is the first term and then you have all these contractions, okay. So I am replacing this commutator by the contraction because they have been ordered correctly in the time. If it was other way around then you would have got the corresponding uh contraction.

So this and so this is what you are going to get. It is easy to check that this is true, okay. Now I mean if you do the same thing for these terms, other terms which involve less number of operators because they have some contractions, you will get a similar thing here, right. So all those operators that are missing from this, these terms, they will appear as contractions and other operators will get contracted with phi 1 plus and give you other propagators or contractions, okay.

$$
\phi_1^+ \mathcal{O} = [\phi_1^+, \mathcal{O}] + \mathcal{O}\phi_1^+ \tag{6}
$$

Looking at the first term,

$$
\begin{aligned}
: \phi_1^+ \phi_2 \cdots \phi_n : &= : (\phi_1^+ \phi_2 \cdots \phi_n) : + : \Big([\phi_1^+, \phi_2^-]\phi_3 \cdots \phi_n + \phi_2 [\phi_1^+, \phi_2^-]\phi_4 \phi_n + \cdots \Big) : \\
&= : \Big(\phi_1^+ \phi_2 \cdots \phi_n + \contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2\phi_3 \cdots \phi_n + \contraction{}{\phi_1}{\phi_2}{\phi_3}\phi_1\phi_2\phi_3\phi_4 \cdots \phi_n + \cdots \Big) :
\end{aligned} \tag{7}
$$

Use

$$
: \phi_1^+ \phi_2 \cdots \phi_n : + : \phi_1^- \phi_2 \cdots \phi_n := : \phi_1 \phi_2 \cdots \phi_n : \tag{8}
$$

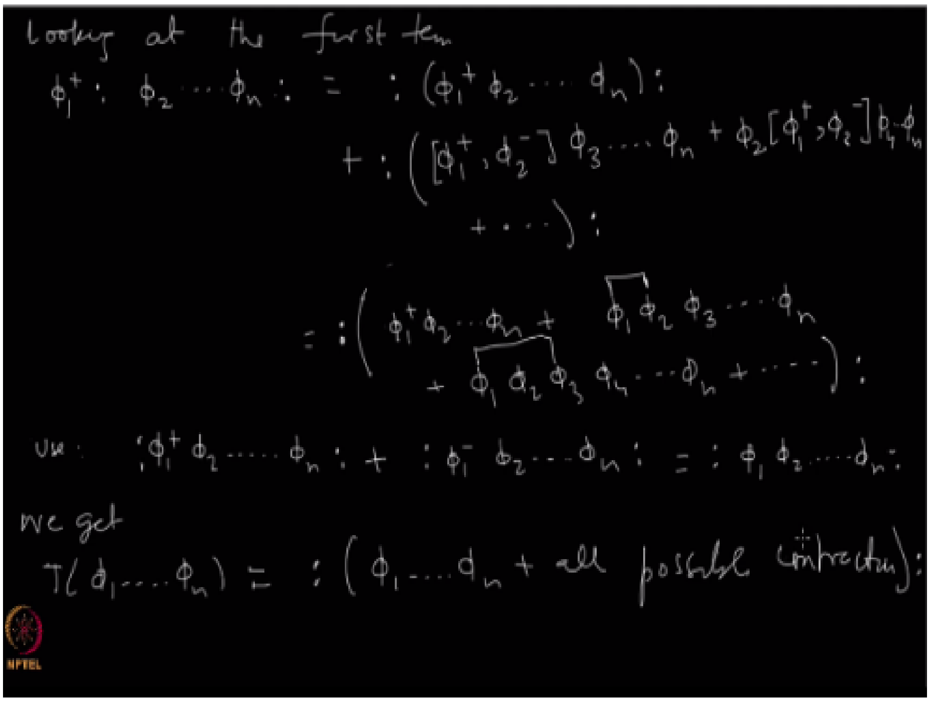

Figure 2: Refer Slide Time: 10:42)

We get

$$T\left(\phi_1\phi_2\cdots\phi_n\right) =: \left(\phi_1\phi_2\cdots\phi_n\right) : \tag{9}$$

Now let us see, so we are looking at this piece right now. I have taken care of phi 1 plus on this. And I can now combine with the other term. So I had a phi 1, see this phi 1 is phi 1 plus plus phi 1 minus. So now let us combine with phi 1 minus term also. And as you can see, you can write or use this. So you had phi 1 plus okay, this term I am writing right now, this one plus from the phi 1 and minus we had this, okay.

And of course, these two you can add and put the normal ordering outside because it gives the same result. And of course phi 1 plus plus phi 1 minus is just phi 1. So I get this, okay. And the same will, the similar things will hold for all the other terms. So you see that we finally get the time ordered product of n number of fields is just the normal ordered product of n fields and plus all possible contractions, okay. So let us continue discussion further in the next video.