Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 40 : Phi-4 Theory: Interaction Picture Continued

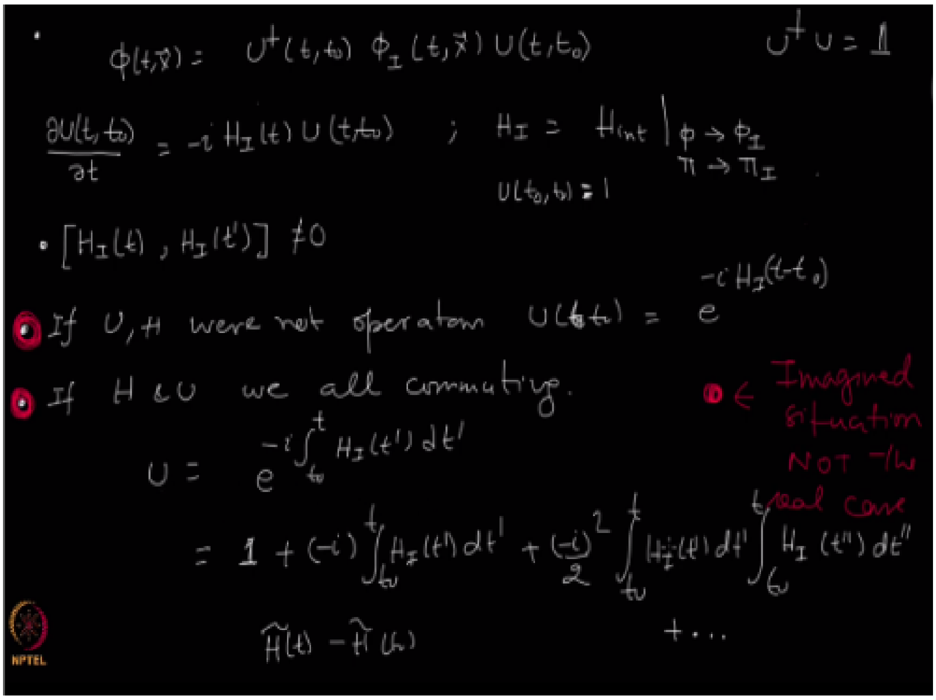

Figure 1: Refer Slide Time: 00:26

Okay, let us continue from where we stopped last time. And let me first quickly summarize what we have done so far. So we have written down that the fields in the interacting theory we can express as the following in terms of fields in the interacting picture, so phi I and you have to sandwich between these operators, okay. And where I had yeah okay I think I wrote down the expression of U earlier. Yeah. So here is the expression of U and U dagger. Okay and we saw that U is unitary. So U dagger U is 1, okay.

And then we had also written down the differential equation that U obeys okay, where we had defined what H I is and that is just the interacting part of the Hamiltonian written using phi I's and pi I's and where you take the Hamiltonian, interacting part of the Hamiltonian and make the replacement phi going to phi I and pi going to pi I. That is what we had found earlier, okay.

And our task is now to find a solution for this differential equation together with the boundary condition that you U(t 0, t 0) is 1 okay because from the definition it is clear that U(t naught, t naught) is 1, okay. Now that is what we are going to do in this lecture, we are going to find the solution. Okay and the difficulty arises from the fact that these are operators which do not commute with each other.

In fact H I's at different times also do not commute. See H I's are made out of fields and these fields do not commute okay, unless they are at equal times. So in general H I's at different times will not commute, maybe I should write. So this is not true in general, right? Only when you are looking at equal time commutation relations then the fields commute but H I's when constructed this way, I mean because they are fields.

And if you are looking at H I with fields at time t and H I at field at different time, fields at different times okay, then they will not commute. Okay, so this is the source of difficulty in finding a solution to this. So we will go in steps so that things become easier. So to begin with, if these were not operators okay, these are ordinary functions then the solution is easy, okay.

Then you would expect then you expect that U of sorry t will be like this. That is what you would expect, right? But given that these are operators which do not commute, we will have to do more things. Now if H I's were commuting, if U's and H I's all these things were commuting, let us say even then even when they were operators okay or think of them as matrices and but they were commuting.

Then we can still easily find a solution. So these are all imagined scenarios. If H and U were all commuting then the solution would be this. So just to have a red flag I am saying that these are imagined scenario. So these are not the real, this is not the real situation. Okay maybe what I will do is I will write in red for a while so that there is no issue or maybe. Okay, so let us see whether indeed this would be a solution if these were all commuting. So first I should tell you what it means to write in this form if H is an operator. So the meaning of this expression is as follows. So this you should, this is formally defined as the following. So 1 minus i t naught to t H I t prime, dt prime plus -i square over 2 factorial, and then you have t naught to t and so forth, okay.

That is the meaning of this exponential. This expression, the second line is what is meant by writing this exponential, okay. And let us verify that indeed, this is the case. Maybe I will leave it as an exercise to or maybe I can do it, not difficult. So let us check that indeed what I am saying is true.

$$\phi(t,\vec{x}) \quad = \quad U^{\dagger}(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0) \tag{1}$$

$$\begin{aligned} \frac{\partial U(t,t_0)}{\partial t} &= -iH_I\,U(t,t_0) &(2)\\ U^{\dagger}U &= 1 &(3)\\ H_I &\equiv H_{int}|_{\phi\to\phi_I \& \Pi\to\Pi_I} &(4)\\ U(t_0,t_0) &= 1 &(5) \end{aligned}$$

So let us take that expression here this one and differentiate it with respect to time t, with respect to this t. So when it takes on the first term it gives you 0. When it acts on the second term, it gives you H I (t). Okay, so you can think of it like this. So you do the integral okay and you get whatever. So let us say H tilde represents the integral of H I, okay. And then you put the upper limit, so that will give you H I (t).

And the lower limit will give you t naught. Now when you differentiate this, you get back again H I (t), okay. Because this is the integral of H I. So differentiating it will give you again H I. So it is clear that if I were to differentiate this term, I am going to get H I (t), okay. And the similar situation is here, they are two factors, okay. So you use the chain rule, okay and you generate two terms.

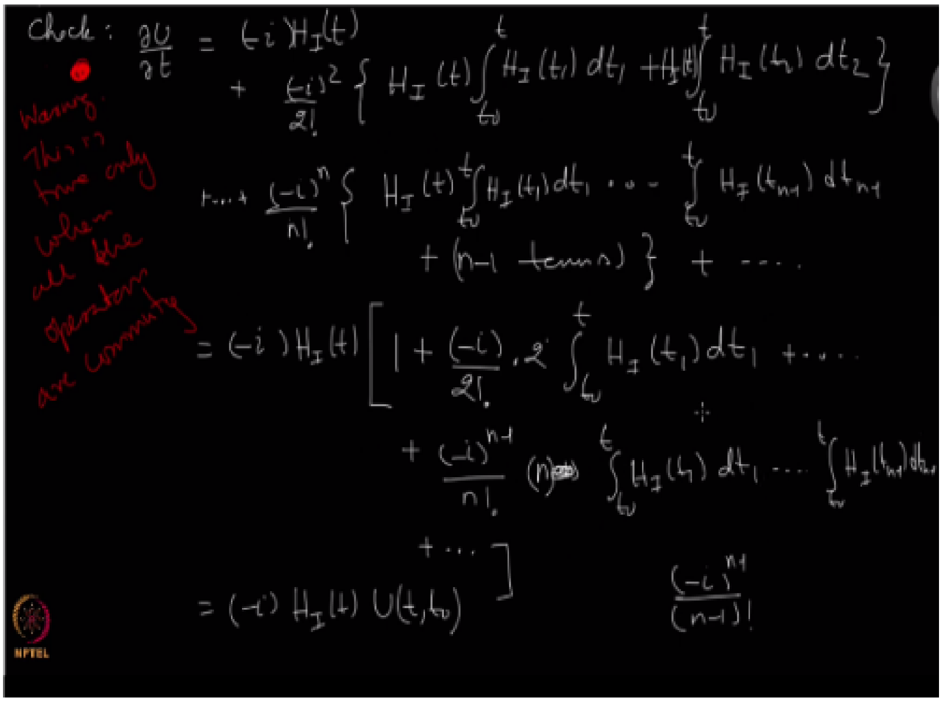

Figure 2: Refer Slide Time: 08:19

Okay and then you can again just find the derivative easily. So let me give you what you will get. So check that you get the following. So the first term will give you -i times H I (t). The second term will give you -i whole square over 2 factorial that you already had. And then as I said we will get H I (t) from the first integral and then you have the second integral left with you.

Okay, I have changed the name of the variable. Instead of calling t prime and t double prime I am calling t 1 and t 2. So let us go back and check. So this one I have taken care of, I am working on this one. So first I take derivative of this term times this one and then this one times the derivative of this.

And using the argument which I gave for this first term, you see that this will just give you H I (t) times this integral plus this integral times H I (t), okay. So that is what I have written here plus and you get -i to the n over n factorial. And then you will have H I (t) and then instead of one such factor you will have n - 1 factors. Okay, this n - 1 is telling you that they are such n - 1 factors.

And then you will have other term. Just like you had two terms here you will have in total n - 1 terms okay and just like the above. Now you can pull out H I (t) from each of the terms. So I will do that. So I pull out -i H I (t) and I get the following. So this one gives me 1. This one gives me so I have pulled out -i. I have a 2 factorial here, okay. And what? This is also pulled out, okay.

And these two are same. And you can change the dummy variables and these two are identically the same. So there is a factor of 2 coming from here which cancels this 2 but I will keep it so I get a factor of 2 from these two terms because they are identical. And then you have integral t naught to t H I (t 1) dt 1, okay. So and so forth and this term you get -i power n - 1 over n factorial, okay. And then you get here I pull out this. So you have n - 1 such factors, okay. I am writing this term and you will get n – 1. No sorry n, you will get n. They are total n terms, not n - 1 terms, okay. So that is what you will get. And now I can cancel this 2 with this 2 factorial here. And this n when I cancel with n factorial, it will give me n - 1 factorial. Let me write it cleanly.

So this is this will become let me write the coefficient of this one. It will become -i to the n - 1 over n - 1 factorial. And this will have n - 1 such factors, right. So you see that this is just an exponential again, okay and which is what we said the answer is. So and this is our beginning thing. This is answer I gave, this expression. And you see we have recovered this one. So this is again U.

So I get -i H I (t) and then U (t 0, t), right. So clearly this is a solution provided H I's commute, okay. But as I said this is not the true situation. So let me put a red color to warn. This is true only when all the operators are commuting, okay. But nevertheless whatever we have done will be useful in finding the true solution. So let us go to our real situation and this is our real situation.

So what do we have? We have t naught to t delta t –i H I t U (t 0, t). Okay, that is the one we want to solve with the boundary condition that equal to 1. Now there is a nice way of doing it. It just some algebra that I should do. So what I do is I integrate on both sides, okay. And first I will change the arguments here. So instead of t I will write t 1. You can write it there is no problem, okay.

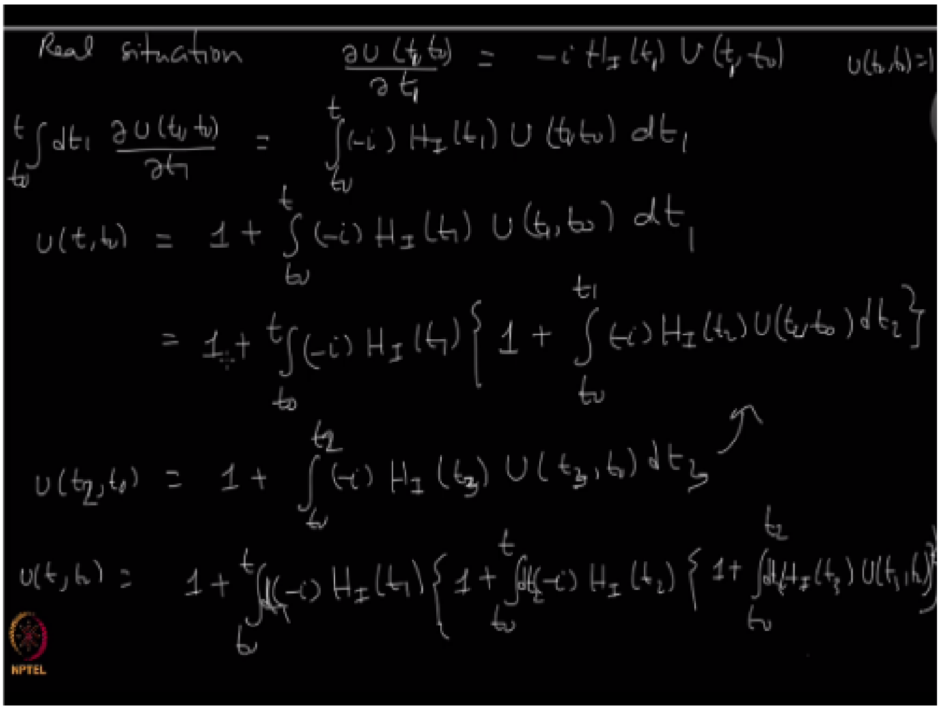

Figure 3: Refer Slide Time: 17:29

$$[H_I(t), H_I(t')] \neq 0 \tag{6}$$

- If U, H were not operator,

$$U(t, t_0) = e^{-iH_I(t-t_0)} \tag{7}$$

- If U, H were all commuting,

$$U \;\; = \;\; e^{-i\int_{t_0}^{t} H_I(t')dt'} \tag{8}$$

$$U = 1 + (-i)\int_{t_0}^{t} H_I(t')dt' + \frac{(-i)^2}{2}\int_{t_0}^{t} H_I(t')dt' \times \int_{t_0}^{t} H_I(t'')dt'' + \cdots \tag{9}$$

And now I do an integral over t 1 from t naught to t. So I do this, t naught to t dt 1 del U over del t 1, okay. And this gives me t naught to t and dt 1. Maybe I can write it in the right side, okay. So all I have done is I have integrated on both sides, okay. Now when you integrate the left hand side you just get U of, doing the indefinite integral gives you U (t 1, t naught). Then you put the upper and lower limits.

You get U (t, t naught) minus U (t naught, t naught) which is 1 okay and I can take that 1 to the right hand side. So -1 becomes a +1 and then you have this term, okay. So this is fine. But I still have U here, okay. So I do not have a solution yet. But what I can do is I can take the left hand side and put t equal to t 1 okay and substitute in here. So what I am saying is let us take this expression and substitute in here.

So what do I get? I get 1 plus t naught to t -i this piece is H I (t 1). But now the first argument is t 1. So I should put t 1 here okay and t 1 here. So let me do it. So what is this? This is 1 plus integral t naught to t 1 -i H I okay. Here it should be dummy variable, so I should write t 2 U of, let us check whether everything is correct. So maybe it will help to write this for U (t 1, t naught).

This will be 1 plus t naught to t 1. So I am replacing t by t 1 in here -i H I. Now this was a dummy variable, integration variable. So that I change and you see this is what I have put in here, okay. So up to here this was the same. And when I got U I replaced 1 plus this this and this correct, okay. So now what I will do is I will take this one and again make this replacement. But this time I want t 2.

So this becomes t 2. This is t 2. This dummy variable I should change to t 3 okay and then substitute in here. So if I do that I get the following, okay. I am always careful that I am never changing the order in which these operators appear because as I said things do not commute and I cannot change the order. So I have to be careful. Now we substitute this thing. And when you substitute this you get 1 plus t naught to t 2 H I t 3 okay and U t 3 t naught. So this is how it will go. So you keep substituting for U this expression. So what you get eventually is the following. I am not sure if this is okay. Yeah, this and this one. Okay, so what you eventually get is this.

Check:

$$\begin{aligned}\frac{\partial U(t,t_0)}{\partial t} = (-i)H_I(t) + &\frac{(-i)^2}{2!}\left\{H_I(t)\int_{t_0}^{t} H_I(t_1)dt_1 + H_I(t)\int_{t_0}^{t} H_I(t_2)dt_2\right\} \\ &+ \cdots + \frac{(-i)^n}{n!}\left\{H_I(t)\int_{t_0}^{t} H_I(t_1)dt_1 \cdots \int_{t_0}^{t} H_I(t_n)dt_n + (n-1)\text{ term}\right. \\ &+ \cdots \end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial U(t,t_0)}{\partial t} = (-i)H_I(t)\Bigg[1 + \frac{(-i)^2}{2!}\cdot 2\int_{t_0}^{t} H_I(t_1)dt_1 + \cdots + \frac{(-i)^{n-1}}{(n-1)!}\cdot n \\ \times \int_{t_0}^{t} H_I(t_1)dt_1 \cdots \int_{t_0}^{t} H_I(t_{n-1})dt_{n-1} + \cdots\Bigg]\end{aligned} \tag{11}$$

Let me just copy from my note book. 1 plus –i dt 1 H I (t 1), that is what we had. Let us go back, this piece. So when you are multiplying this with 1 okay that is what you get. And then you have other term -i t naught to t dt 1 H I (t 1) and t naught to t 1 dt 2 H I (t 2). Okay, so

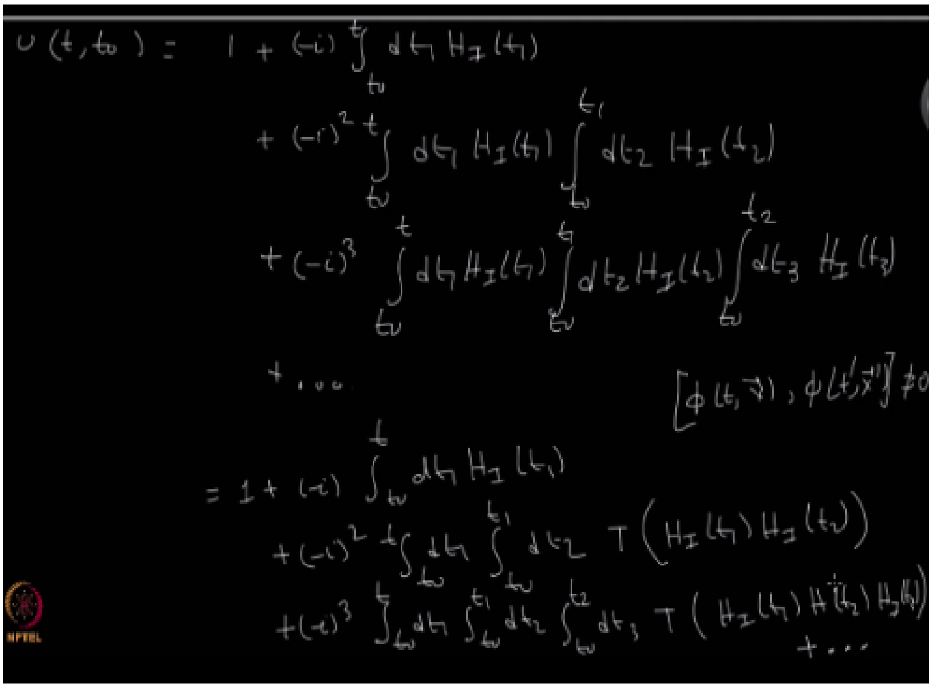

Figure 4: Refer Slide Time: 23:55

you should read it like this, because here you have to do the t 2 integral first and the limits go from t naught to t 1.

But t 1 is the integration variable here, right. So now you are doing integral over t 1. So this has to be maintained and then let me write one more term. Here it was -i square. Now –i cube t naught to t dt 1 H I(t 1) integral t naught to t 1 dt 2 H I (t 2) integral t naught to t 2 dt 3 H I(t 3) and you continue like this, okay. So I think I said that already several times that you cannot interchange the order in which H I appear in the integrals, okay.

Because in general fields do not commute. You know this that this is not true in general. So they do not commute in general, okay. And of course, they commute if t prime and t are same. So that is the equal time commutation relation. But here as you see H I's are at different times. For example this one, this one has H I at a time, I mean the time argument of H I is always smaller than the time argument of H I here, right?

Because here the integration limit of t 2 goes from t 0 to t 1. So the maximum value t 2 takes is t 1, okay. And then you are doing integral over t 1. So t 1, the t 2 always remains below t 1. So H I of t 2 and H I of t 1 they are always at different time arguments, okay. So they do not commute in general and similarly here, okay. And that is why we have to be careful with the order in which we have written things, okay.

So good then. Then what? Yeah, so I have already said something here which, so you see the H I's appear in a particular time order. So if you look at this term for example, here t 3 goes from t naught to t 2, right. So the maximum value t 3 takes is t 2. And t 2 goes from t 0 to t 1. So the maximum value t 2 takes is t 1. And then t 1 runs from t naught to t.

So this is the H I's, these three H I's they are ordered in a particular time, in a particular order. So to the right you have the one with the lowest time. Then the next lowest time and then the highest time, okay. So it is running this way. So you always have the operator H I at lowest time to the right. And then you successively go towards left ending up with the highest time, okay. So I will define a time order product of these H I's then, okay. So I will write maybe I will write here and then go to the previous sheet.

Thus we have

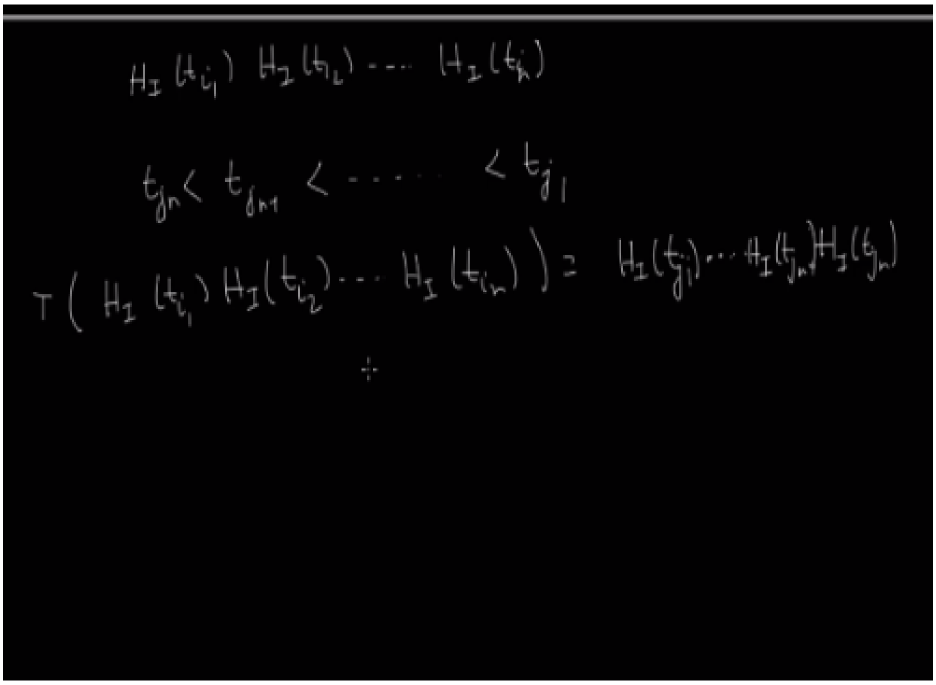

Figure 5: Refer Slide Time: 28:26

$$\frac{\partial U(t,t_0)}{\partial t} = -iH_I\,U(t,t_0) \tag{12}$$

Above is only true when all the operators are commuting

Real situation

$$\frac{\partial U(t,t_0)}{\partial t} = -iH_I\,U(t,t_0); \quad U(t_0,t_0) = 1 \tag{13}$$

$$\int_{t_0}^{t} dt_1 \frac{\partial U(t_1,t_0)}{\partial t_1} = \int_{t_0}^{t} (-i)H_I(t_1)U(t_0,t_0)dt_1 \tag{14}$$

So we have H I (t i 1) H I (t i 2) i n. You are going to have for the nth term n such H's okay. And suppose that t n sorry t i t j n is the smallest one okay. So this is the smallest time followed by this one and so forth and then eventually, okay. So suppose this is how these different times are related. So this is the smallest one, this is the next one, this is next one.

Then I define a time order product of these H operators. H I (t i 1), t i 2, H I (t i n) is this. So you should take the smallest one and put it first. And so I am saying that this is the smallest one. So this comes first. Then H I (t j n-1) and so forth. And then you end up with H I (t j 1) okay. So this is what is called time order product of these operators, okay.

So what I can see is that this one this expression, I can write using the new definition that I have written, 1 plus -1. There is nothing to be done here. Second term is -i square t naught to t dt 1, t naught to t 1 dt 2. And these two H I's I can write using the time order product, okay. This is exactly the same thing. It is naturally ordered here. Now I have just written the same thing explicitly.

Now if you were to interchange H I t 1 and H I t 2, so if you put this way and that way, it is still the same. Because time ordering operator knows that it has to put the smallest one first and the second one should be the one with the larger time. So no matter which way you put these H's, the time ordering operator will always make this happen. So it will always put in this correct order, okay.

So that is the advantage. And maybe I can write this one more term, t naught to t dt 1, t naught to t 1 dt 2, and t naught to t 2 dt 3, okay. And then you have time order product of H I (t 1) H I (t 2) and H I (t 3) okay. And you can interchange the order within the t and nothing changes for the reasons I explained above. So that is what I have got, okay. That is good.

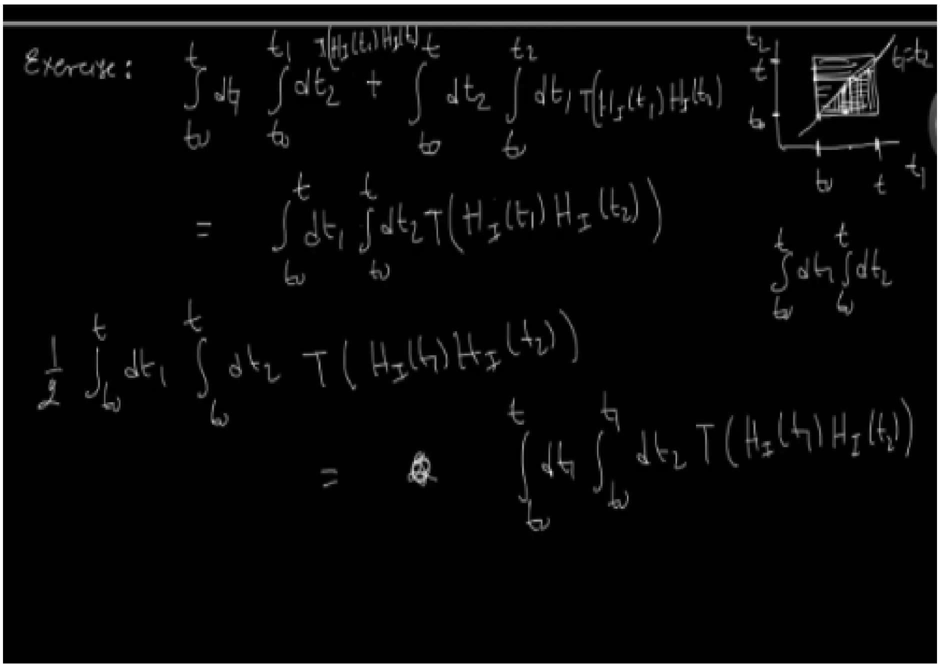

Figure 6: Refer Slide Time: 34:03

But you see, we were expecting something like an exponential here, right. So where was it? Yeah, this was this is what we had written, okay. And here, the difference from what we have right now in our calculation is that the integration limits are all going to t okay. And then you have all these nice 1 over 2 factorial and 1 over 3 factorial, which produce an exponential.

And what we have sorry, I think I said opposite. So we had seen here that in this context, you get 1 over n factorials at each order at each power. And then the integration limits were going from t naught to t for all the variables, okay. And you get a nice exponential. But what we have now is the integration limits are going from t naught to t 1, t naught to t or in this case t naught to t 2 then t naught to t 1 then t naught to t.

So that is one difference. Upper limits are not all t. And you also do not have n factorials in the denominators, okay. So that is the difference we are seeing. But a small exercise can show us that we can bring n factorials in the denominators and put all the upper limits of integration to be the same as T. So that is what we are going to do. So here is a, let me write it as an exercise though I am going to solve it but let us put on the heading, exercise. So what we have here is let us look at this term the second one from so you have integral over t 1 and t 2 and t 2 goes from t naught to t 1 and t 1 goes from t naught to t okay. So let us look at that space.

So you have t 1, you have t 2 okay and from t naught to some value t both the I mean I am just constructing this. So consider a square whose lengths are t minus t naught, whose length is t minus t naught, okay. So it is a square. Now the full volume here in this t 1, t 2 space okay is this. It is the full volume but I can divide it into two parts like this. So the division is in the following way.

So if you are in this region okay, then for any value of t 1 let us say this t 1, for any value of t 1, t 2 goes from t naught. So here t 2 takes the value t naught. So t 2 goes from t naught to maximum to this, right? So when you are doing integration over t 2, this is how you are integrating, okay. And what is this line? This is this diagonal line is t 1 equal to t 2 line, okay.

So you see that in this region when you are integrating your t 2 is always below t 1, right? Because this is the line t 2 is equal to t 1. And any point below this, below this line in this region is t 2 less than t 1. So this diagonal line okay naturally divides this square into two regions. In one region one variable is always less than the other and in the other one other region it is just reversed, okay.

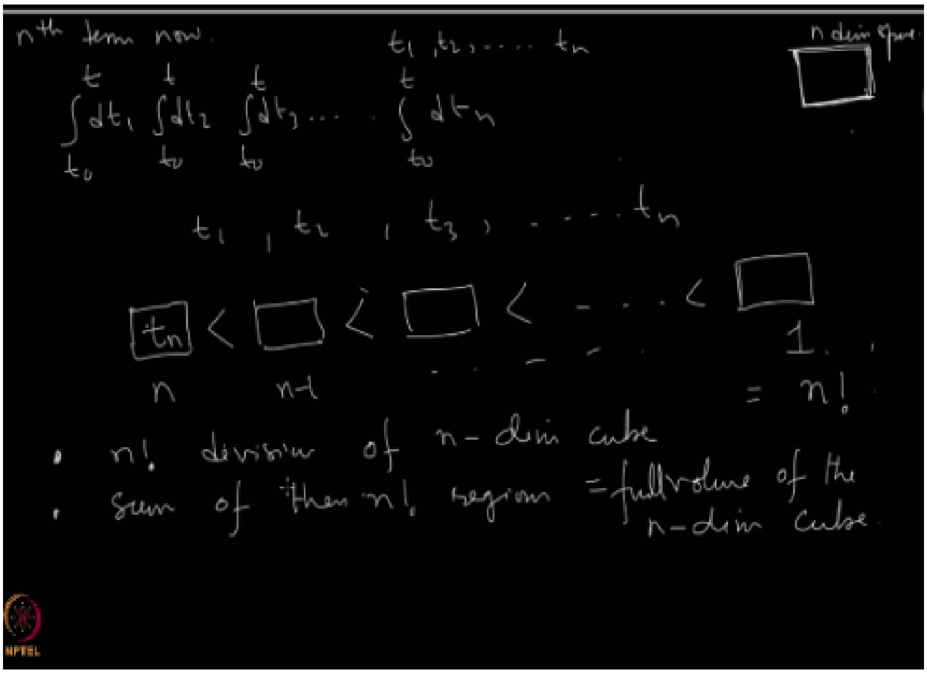

Figure 7: Refer Slide Time: 41:10

$$U(t_1, t_0) = 1 + \int_{t_0}^{t} (-i) H_I(t_1) U(t_1, t_0) dt_1 \tag{15}$$

$$U(t_1, t_0) = 1 + \int_{t_0}^{t} (-i) H_I(t_1) \left\{ 1 + \int_{t_0}^{t_1} (-i) H_I(t_2)\, U(t_2, t_0) dt_2 \right\} \tag{16}$$

$$U(t_2, t_0) = 1 + \int_{t_0}^{t_1} (-i) H_I(t_3)\, U(t_3, t_0) dt_3 \tag{17}$$

So that is the division that has happened in this. But you see that the sum of these two regions is the full region, okay. Which says integral dt 2 t naught to t 1 okay and then of course, you integrate over the full t 1. That is this region okay plus now t 1 goes from t naught to here okay and this is t 1 equal to t 2 line. So the maximum value t 1 takes here is t 2. And then you integrate over okay so dt 2, t 2 then goes from 0 to t.

So this is t, t naught to t sorry, not 0 okay. And how much is that? This is the full volume, okay. So that is good. Now what I have got in my this expression here is one of those regions, right? This is one of the regions which I have written, but it comes with a time order product of these two H. So let us then see.

So what I do is I write here H I (t 1), H I (t 2) time order product and here also time order product of H I (t 2), H I (t 1). So I am just writing this. Okay, in fact let me write the same thing and here also. So I am integrating over the same quantity in all the three, right?. So this is then allowed. And you see that this one and this one they are identical because I can change the dummy variable t 1 to t 2.

So this will become dt 2 and this one will become t 1. So this is exactly the same as here and here that does not that does not bother because it becomes H I (t 2) and H I (t 1). But I can write it the other way around exactly like this. Because the time ordering operator will take care of everything, okay. So you see that the left hand side is just, both the terms on the left hand side are same, okay.

So you see that this implies that integral t naught to t dt 1 integral t naught to t I am writing this one dt 2 T of H I (t 1) H I (t 2) is equal to two times of this object, okay. So integral t naught two t 1 dt 2 and time order product H I (t 1) H I (t 2) and this one is dt 1 t naught 2 t two times this, okay. So this is good. This is good because let me take the 2. Let me take this 2 to the other side and write it as 1 over 2. So you see this is what we had in this expression here. And if I now change the limits to, upper limits to from t 1 to t, so all upper limits are t now. Then I get the same thing, but I get 1 over 2 here. And which looks like becoming an exponential, okay. Now let us look at this term, third term or rather I will look at a general term. So let us look at the nth term now, okay. So now instead of having just t 1 t 2, I will be integrating over t 1, t 2 and t 3 and t 4 and t 5. So you are going to be in basically n dimensional t space, okay. So you have t 1, t 2, t n and the volume element in this space is dt 1 integral dt 2 integral dt 3, integral dt n. And all the integrals are from t naught to t, okay just the same thing as before but now for n dimensions, okay. Now what I want to do is I want to divide this volume which n dimension volume in similar fashion. And remember what was the way in which we divided. We divided into independent volume elements without any overlap and those volume elements were specified by saying that t 1 is always greater than t 2 and hear other way around, okay. So the same thing we will do.

So we will divide this entire volume n dimensional cube n dimensional cube into distinct volume elements without any overlap using the same criteria, okay. So how many are there? It depends on how many divisions we can make. So if I am given these times okay, then I can have the following divisions. So I make some boxes here and I say that this time is less than this one, this time is less than this one and so forth, okay.

So the first one I could have any of the n times from t 1 to t n. So there are n possibilities here okay, so n possibilities. So let us say I put t in here. Suppose I made that choice and now I have only n - 1 choices and I can choose, make those n - 1 choices together with these n choices, okay. And I can fill up till I end up with one choice. So you see that I have in total n factorial choices.

Meaning I can divide this entire n dimensional volume into n factorial smaller volumes which will not have any overlap with each other and together they will fill up the entire n dimensional sorry this n dimensional cube okay, which has side from t naught to t, length is t naught to t. Okay, so we see that we have n factorial such choices or such spaces. So let me write it down.

So we have n factorial divisions of n dimensional cube okay of in the t space, okay. And if I sum all these n factorial regions then you get the full volume element, full volume of the cube, okay. So that is good. Now if I argue as before, the argument I gave here, right? Now you have instead of these two a total of n factorial terms, okay.

And you can of course again re-change the labels and use the fact that time ordering will take care of everything irrespective of what arguments the H's have. And using that same logic you arrive at the following result. Maybe I should write on the next sheet.

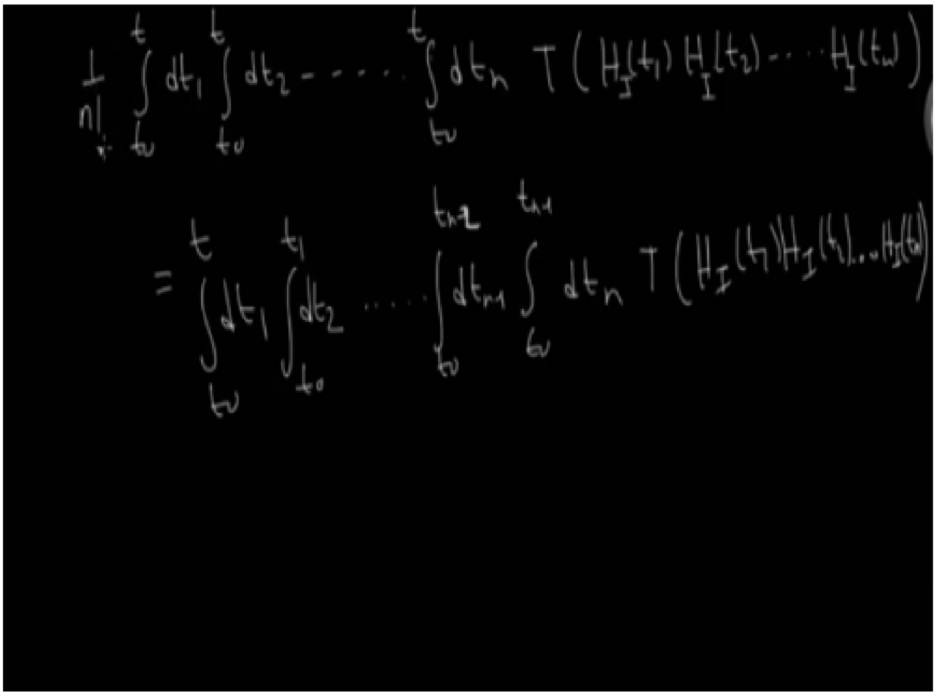

Figure 8: Refer Slide Time: 45:58

$$U(t_1, t_0) = 1+(-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 H_I(t_1) \int_{t_0}^{t_1} dt_2 H_I(t_2)+ \\ +(-i)^3 \int_{t_0}^{t} dt_1 H_I(t_1) \int_{t_0}^{t_1} dt_2 H_I(t_2) \int_{t_0}^{t_2} dt_3 H_I(t_3) + \cdots \tag{18}$$

$$U(t_1, t_0) = 1+(-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 T\Big(H_I(t_1) H_I(t_2)\Big)+ \\ +(-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \, T\Big(H_I(t_1) H_I(t_2) H_I(t_3)\Big) + \cdots \tag{19}$$

$$[\phi(t, \vec{x}), \phi(t', \vec{x}')] \neq 0 \tag{20}$$

Time ordered product

$$H_I(t_{i_1}) H_I(t_{i_2}) \cdots H_I(t_{i_n}) \tag{21}$$
$$t_{j_n} < t_{j_{n-1}} < \cdots t_{j_1} \tag{22}$$

$$T\bigg(H_I(t_{i_1})H_I(t_{i_2})\cdots H_I(t_{i_n})\bigg) = H_I(t_{1_1})H_I(t_{j_2})\cdots H_I(t_{j_{n-1}})H_I(t_{j_n}) \tag{23}$$

That integral t naught to t. So I am now integrating over the full cube and I should take H (t 1) H (t 2), okay. This entire thing if I divide by 1 over n factorial, so if I take one part of it okay, one division of this entire volume then that is the one which is equal to this one, okay. And this one dt n – 1, this one will go from t naught to t n – 2 and so forth. And this goes to dt 2 t naught to t 1 and this one will be dt 1 t naught to t, okay.

So now you see I have changed the upper arguments to t and I have got 1 over n factorial. So now if I take this result okay and substitute in our original expression that we had got in here okay then you see I will have 1 plus this one is fine minus i times this and this one will become -i square over 2 factorial times here it will be t here, it will be t and this will remain the same.

And similarly here I will have -i cube over 3 factorial and all the upper limits in this case will be t and the same thing. And this is what I will write as the following. So I write that my result is this. It is maybe this is the last sheet so I will write it explicitly, from t naught to t okay and this I define to be the following. I write it as exponential –i, okay slightly differently dt prime. Okay, I just brought in -i here and integration is from t naught to t okay and then I put a t here time ordering operator here.

So this T tells me that when I am expanding the exponential above okay and I should always ensure that the time, the operators H I's okay are correctly time ordered. Okay. So this is the definition of this object. So what is the meaning of this object? The meaning of this object is the above expression, okay. So that is just a convenient way of writing this bigger expression, okay.

So you see that we have succeeded at least in expressing the field phi in terms of phi I's only okay or phi I's and pi I's if our H I included pi, but for our phi-4 theory there is only phi I. So you see here now once I found the solution a few like this and you know H I is made out of phi I. So for our case H I is d 4 x, no d cube x phi x to the fourth, okay. And of course, lambda over 4 factorial, okay. So this is H I for us.

So you see this entire U is constructed out of only phi I's, okay because once you expand this will contain only phi I's powers of it, which means that I have now expressed phi in terms of phi I's only, okay which is one thing I wanted to do, okay. So that is good and the reason why I wanted to do was because phi I evolves with free field sorry free Hamiltonian. So this will be very useful.

Exercise:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; T\bigg(H_I(t_1)H_I(t_2)\bigg) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\; T\bigg(H_I(t_1)H_I(t_2)\bigg) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\bigg(H_I(t_1)H_I(t_2)\bigg) \tag{24}$$

$$\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\; T\bigg(H_I(t_1)H_I(t_2)\bigg) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; T\bigg(H_I(t_1)H_I(t_2)\bigg) \tag{25}$$

n^{th} term now

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3 \cdots \int_{t_0}^{t} dt_n \tag{26}$$
$$t_1, t_2, t_3, \cdots t_n$$
$$t_n < t_{n-1} < t_{n-2} < \cdots < t_1$$
$$n!$$

- n! divisions of n-dimensional cube
- Sum of these n! region = full volume of the n-dimensional cube

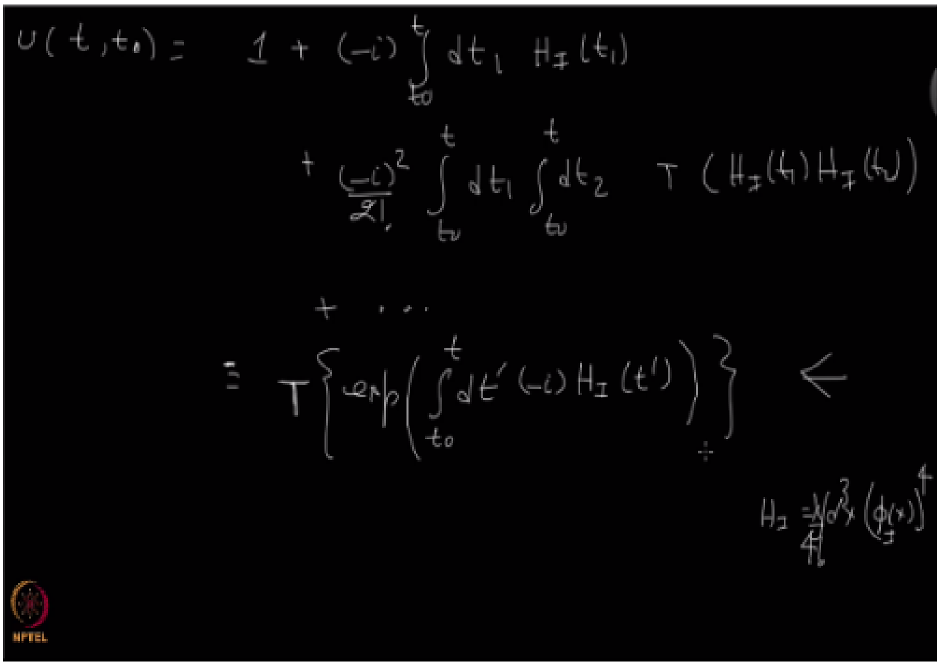

Figure 9: Refer Slide Time: 48:05

$$\begin{aligned}\frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3 \cdots \int_{t_0}^{t} dt_n \; T\Big(H_I(t_1) H_I(t_2) \cdots H_I(t_n)\Big) \\ = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-2}} dt_{n-1} \int_{t_0}^{t_{n-1}} dt_n \; T\Big(H_I(t_1) H_I(t_2) \cdots H_I(t_n)\Big)\end{aligned} \tag{27}$$

$$\begin{aligned}U(t_1, t_0) =& 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\Big(H_I(t_1) H_I(t_2)\Big) \\ &+ \cdots\cdots \\ =& T\Big\{\exp\Big(\int_{t_0}^{t} dt' H_I(t')\Big)\Big\}\end{aligned} \tag{28}$$

Where,

$$H_I = \frac{\lambda}{4!} \int d^3x \; (\phi_I(x))^4 \tag{29}$$

Now what I should do next is worry about our ground state, okay and do something here so that that also involves only objects which evolve in time with free field, okay. Sorry, evolve in time with free Hamiltonian. Okay, so good. We will stop here and we will meet in the next video.