Introduction to Quantum Field Theory

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Lecture 4 : Symmetry and Normalization of the States

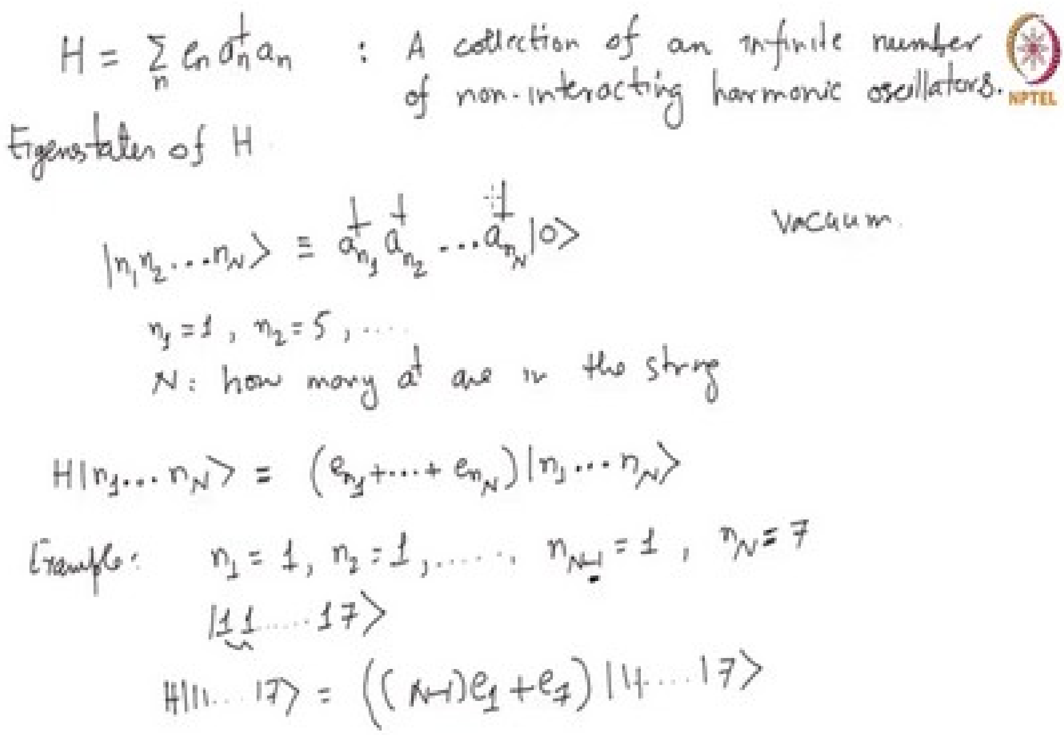


Figure 1: Refer Slide Time: 00:20

So we were looking at this system which is a second quantized system I mean the description is seconD quantization and it is equivalent to that of a collection of infinite number of non-interacting harmonic oscillators. So, each sum represents a harmonic oscillator in this sum and each oscillator has its own natural frequency. So, here this is basically h bar omega and.

So, omega n is different for different oscillators and we started looking at if you recall let me go back if I can here we started looking at the eigenstates of this Hamiltonian and that is what I am going to do today a little bit more generality. So, let us find out the eigenstates of this Hamiltonian. And we found that if you take the vacuum that is the ground state of your theory you recall if you act on this with the Hamiltonian you ket zero or if you have if you put a half h bar omega term here then it gives a half h bar omega ket zero.

$$H = \sum_n e_n a_n^\dagger a_n \tag{1}$$

A collection of infinite number of non-interacting harmonic oscillator. Eigenstate of H

$$|n_1 n_2 \cdots n_N\rangle \;=\; a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_N} |0\rangle \tag{2}$$

N $\rightarrow$ how many $a^\dagger$ are present in the string

$$H|n_1 n_2 \cdots n_N\rangle \;=\; (e_{n_1} + \cdots e_{n_N})|n_1 n_2 \cdots n_N\rangle \tag{3}$$

Example: $n_1 = 1, n_2 = 1 \cdots n_{N-1} = 1, n_N = 7$

$$|11 \cdots 17\rangle \;=\; ((N-1)e_1 + e_7)|11 \cdots 17\rangle \tag{4}$$

So, that is the ground state and it is also called vacuum. So, a most general state which will be an eigenstate of the Hamiltonian would be of this form. So, I take a n with the subscript capital N and dagger and I act with several of them a n 2 dagger a n 1 dagger and let me denote this system by sorry this state by the symbol sorry. So, your n 1 could be 1 n 2 could be 2 or 5 or whatever and so forth.

And your capital N is basically telling how many daggers you have or how many oscillators are not really let us say capital N denotes how many of daggers you have and there is r in the string this string of letters how many of a daggers have there. So, that is what it is counting. So, if you look at the action of Hamiltonian on this state you can easily compute the way we did last time that it will turn out to be the following.

So, it will be in of course an eigenstate and the eigen value would be the sum of these energies that is evident from the discussion of last time a specific example of such a state could be that you take n 1 to be 1 n 2 to be 1 and capital N - 1 to also be 1 and let us say this one is state number oscillator number response oscillator number seven. So, you have a state 1 1 1 7. So, this is one of the eigenstates of the system.

So, you will have here a 1 dagger a1 dagger so on and so, forth a 1 dagger and finally a 7 dagger. So, this is what it means and if you have such a system then of course the energy of the system would be using the formula above you can find that it will be just you know let us say these are let us say these are I have already written capital N – 1 so this will be capital N - 1 times e 1 + e 7 that is what you will get.

So, that is fine now you can see that there is a special symmetry of these states which is obeyed because these are a daggers and a daggers all commute with each other. You remember the computation relation let us go back and see if we can find it? Here see a n a m dagger is delta m n and what is not written here I thought I have already said here no not here also let me check. So, somehow I missed to write that i can you know the a daggers commute with themselves even for one particular oscillator the a daggers commute.

The a's also commute and a and a dagger anyway they do not commute they commute only when they belong to different oscillators. So, here this is saying that a and a dagger commute if they belong to different oscillators if n and m are different then you gets a zero here. So, they commute the only issue is when they belong to the same oscillator the index m and n then only they do not commute and the a's and a daggers anyway commute.

So, I can write here a and dagger am dagger is equal to a and a m is equal to zero you can put a hat these are operators this anyway holds true. So, we can use that here where are we here. So, all these are going to commute right it does not matter what is here. So, if they are all going to commute then we have a symmetry let me write that down.

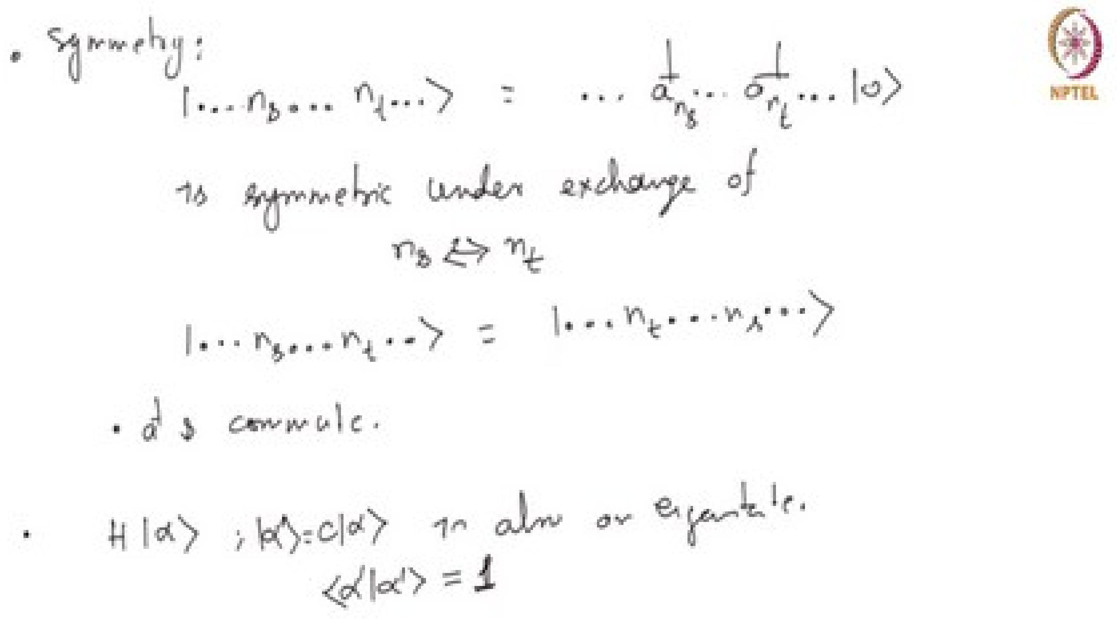


Figure 2: Refer Slide Time: 08:13

Symmetry

$$|\cdots n_s \cdots n_t \cdots\rangle \;=\; \cdots a^{\dagger}_{n_s} \cdots a^{\dagger}_{n_t} \cdots |0\rangle \tag{5}$$

is symmetric under exchange of $n_s \leftrightarrow n_t$

$$|\cdots n_s \cdots n_t \cdots\rangle \;=\; |\cdots n_t \cdots n_s \cdots\rangle \tag{6}$$

$$|\alpha'\rangle \;=\; c\,|\alpha\rangle \tag{7}$$

$$\langle\alpha'|\alpha'\rangle \;=\; 1 \tag{8}$$

If $|\alpha\rangle$ is an eigenstate then so is $|\alpha'\rangle$

So, if you take a state which is some things are here then you have n s then you have n t. So, it is the same kind of notation as before I am just omitting what could be written here because I just want to let us make it a bit nicer. So, this is what this is corresponding to these dots you have semi daggers and then you have a and s dagger corresponding to this n s then you have a and t dagger corresponding to this n t and then corresponding to those dots you have other a daggers acting on the vacuum of this theory.

Now you see that because I can interchange these two nothing changes because they commute. So, this one will go through all the a daggers you have here because it can come here and this guy can go backwards because it commutes it can sit here there is no problem which means that this state or all the eigenstates of the Hamiltonian they are symmetric under exchange of n s with n t.

So, if you exchange these two the result does not change which is just saying that I can interchange these two positions. So, I can say everything remains the same and here I interchange n t and n s that is what is the statement about the symmetry of the states of the system that is good and as you saw the reason was that a dagger commute that is fine, nice and how about the symmetry sorry the normalization of these states.

So, if you are given a state which is an eigenstate of the Hamiltonian. So, if you take a state let me do a shorthand to avoid writing too much. So, let us say capital phi sum state which could be something like this if that is the state of the Hamiltonian then that is the state of this the eigenstate of the Hamiltonian then C times get alpha is also an eigenstate. And you would like to normalize the states.

Meaning fix the C such that if I call this as beta or not beta let us call it let us call it alpha prime or let us call it alpha yeah alpha prime. So, what I have to do is I have to fix C such that alpha prime alpha prime this inner product is 1 that is when you do this you call that the states are normalized. So, I have to find c such that this happens. So, let us look at normalization now.

0.1 Normalization of the state

So, let us look at this, let us look at the state as we wrote before the states n 1 to n N I am looking at the normalization of these states. So, let us check. So, let us see what I get for the inner product if I take the inner product of this get with its corresponding bra and let us see what we get and depending on what we get we can divide accordingly to get a normalized eigenstate. So, let us do that and recall what this was? This was a n 1 dagger a n] let me see yeah a n 2 dagger so on so, forth a n N - 1 dagger a n dagger acting on ket zero.

So, if I am looking at inner product of this one. So, first I should calculate this quantity and that is easy the ket turns into a bra and because all these a daggers commute and correspondingly a is commute I do not have to worry I can write here what should I write here; let me check and this is how I want to write I want to write it like this a n N that is what you can write.

Now let us look at this inner product. So, the inner product is. So, you write this one 0 a n 1 a n subscript 2 a n with a subscript N and then you have a n subscript one dagger a n subscript 2 dagger and a n N - 1 dagger a n dagger ket zero we are almost there. So, let me write it like this 0 and 1 up to a n N - 1. So, you have a subscript N - 1 here that is the one I am writing here now this guy I am going to push it through these a daggers.

Now you can make these two commute. So, sorry these two commute if n 1 and n 2 correspond to two different oscillators. So, if n 1 is not same as n 2 then they commute anyway sorry I meant these two for this mistake. So, what I am trying to say is these two guys the a and a daggers they commute if this subscript is different from this subscript right because the a digest corresponding to different oscillators commute anyway.

They do not commute and you get a constant term if a n N and a and a n 1 dagger they correspond to the same oscillator. So, for the moment let us take all the indices to be different. So, take the case where n 1 is not equal to n 2 is not equal to n 3 and none of them are equal. So, I am looking at a state where none of these are equal just to avoid the complication in counting good. So, if none of them are equal then this will commute with this.

So, it sits here then but this commits with this one also. So, it can go through it will go through up to here. So, up to here it will be able to come. So, let me write that down. So, here and then it will sit next to this before this because this cannot commute if they do not commute. So, I have to use the commutation relation. So, let me do it in steps and then you have I am just writing in the next line for ease of visualization a n 1 dagger a n 2 dagger a n subscript N - 1 dagger and then you have a n n and a n dagger sorry this was a n.

So, this is n subscript N. So, let me write that down in the next line you have a n N a n N dagger acting on the vacuum good. But now here you can use the commutation relation and this will be equal to a n dagger a N you remember I want to bring this next to the ket because then it can annihilate it. So, I am using commutation relation and then you get a + 1. So, this let me write this entire piece becomes this which is which is just this right because this guy kills, it kills the vacuum and this one is left.

So, what I have effectively is a and one to a n n minus one and then a n 1 dagger up to a n N - 1 dagger and these gives me just to ket 0. So, this is same as what you have here right except for the fact that this term this fact this a and this a dagger they are gone. So, it is looking exactly the same. So, you can repeat the same steps again and take the one here a n N - 1 dagger and try to bring it next to it and do the same thing as we did here and again you will get pick up a factor of 1 here this will kill and you will be left with ket 0.

So, if you keep repeating that you will end up with 1 right because let me do it in one two steps you will end up with this 0 and you know the last one will be what a n 1 a n 1 dagger this will be the last one which will be left and then again the same step you will have this and if you say that my get 0 is normalized meaning this product is 1 then I get 1 which means that the state which with which we started this one this is already properly normalized.

So, I do not have to do anything now you can already imagine that if some if this is not true if some of the n 1's and n 2's and n 3's are same then you have to take care of the commutation relations and you will get a normalization factor which will not be one. So, that will be left as an exercise.

$$|n_1 n_2 \cdots n_N\rangle = a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_{N-1}} a^\dagger_{n_N} |0\rangle \quad (9)$$

$$\langle n_1 n_2 \cdots n_N| = \langle 0| a_{n_1} a_{n_2} \cdots a_{n_{N-1}} a_{n_N} \quad (10)$$

The inner product will be

$$\langle n_1 n_2 \cdots n_N | n_1 n_2 \cdots n_N\rangle = \langle 0| a_{n_1} a_{n_2} \cdots a_{n_{N-1}} a_{n_N} a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_{N-1}} a^\dagger_{n_N} |0\rangle \quad (11)$$

take the case when

$$n_1 \neq n_2 \neq n_3 \cdots \neq n_N \quad (12)$$

$$\langle n_1 n_2 \cdots n_N | n_1 n_2 \cdots n_N\rangle = \langle 0| a_{n_1} a_{n_2} \cdots a_{n_{N-1}} a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_{N-1}} a_{n_N} a^\dagger_{n_N} |0\rangle \quad (13)$$

$$a_{n_N} a^\dagger_{n_N} |0\rangle = (a^\dagger_{n_N} a_{n_N} + 1) |0\rangle \quad (14)$$

$$= |0\rangle \quad (15)$$

If we keep on repeating this we get,

$$= \langle 0| a_{n_1} a^\dagger_{n_1} |0\rangle \quad (16)$$

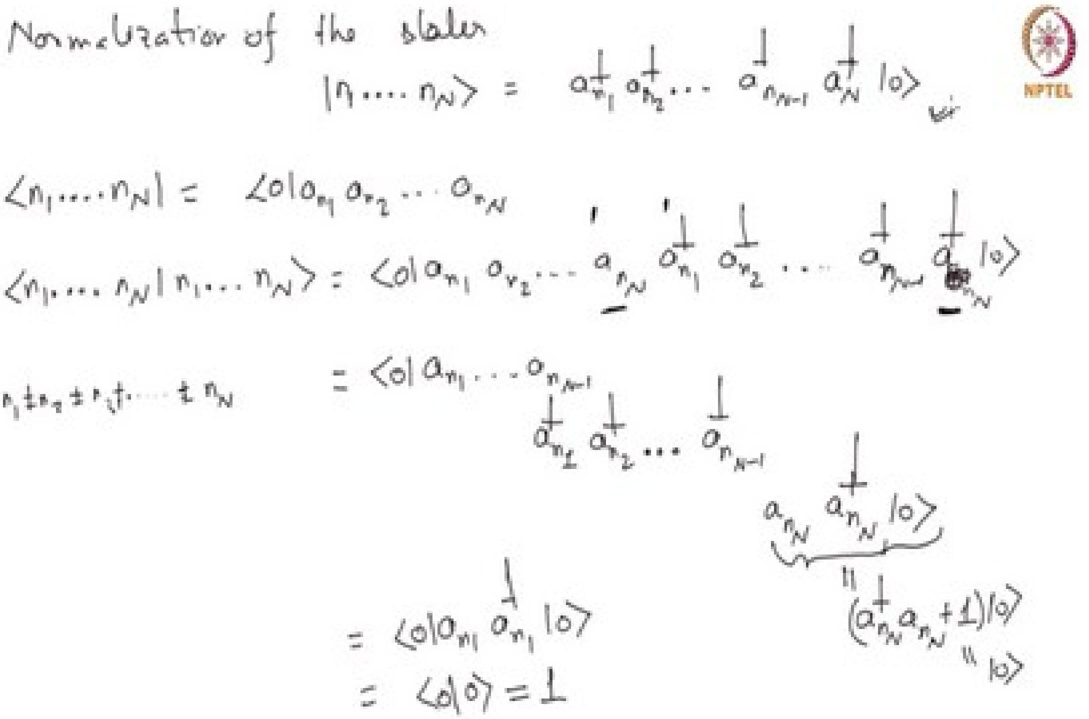
$$= \langle 0|0\rangle \quad (17)$$

Exercise: Find the normalozation of the states

$$|\alpha'\rangle = c\,|\alpha\rangle = (a^\dagger_{n_1})^{m_1} \cdots (a^\dagger_{n_N})^{m_N} |0\rangle \quad (18)$$

Find 'c' such that

$$\langle \alpha' | \alpha' \rangle = 1 \quad (19)$$



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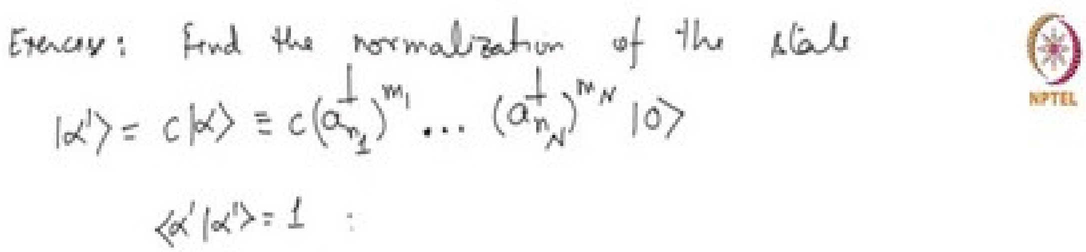

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So, exercise is find the normalization of the state let me denote the state by alpha it is not a good notation but I do not want to spend time writing all the indices. So, I will use alpha but the definition of the state is this so you have a n 1 dagger m 1 time. So, you have a n 1 dagger a n 1 dagger a n 1 dagger it appears m 1 times and you have a n subscript capital N this thing appearing m subscript capital N times and that is your vacuum.

Now of course this is an eigenstate of the Hamiltonian just as you saw before and the goal is to find a C a constant C such that C times alpha let us put a C here, C times get alpha this is let me define it to be alpha prime. So, that this state alpha prime is a properly normalized state meaning find c such that alpha prime alpha prime is one this. So, that is what you do final normalization of the state this one now that I have put all the C's it sounds strange but you understand right you have to just find the C.

So, that will be an exercise and in the next video I will start looking at another system that will have states which will have one-to-one correspondence with the states that you have seen in this theory. So that is what I am going to do in the next video.