Introduction to Quantum Field Theory

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Lecture 39 : Phi-4 Theory: Interaction Picture -2

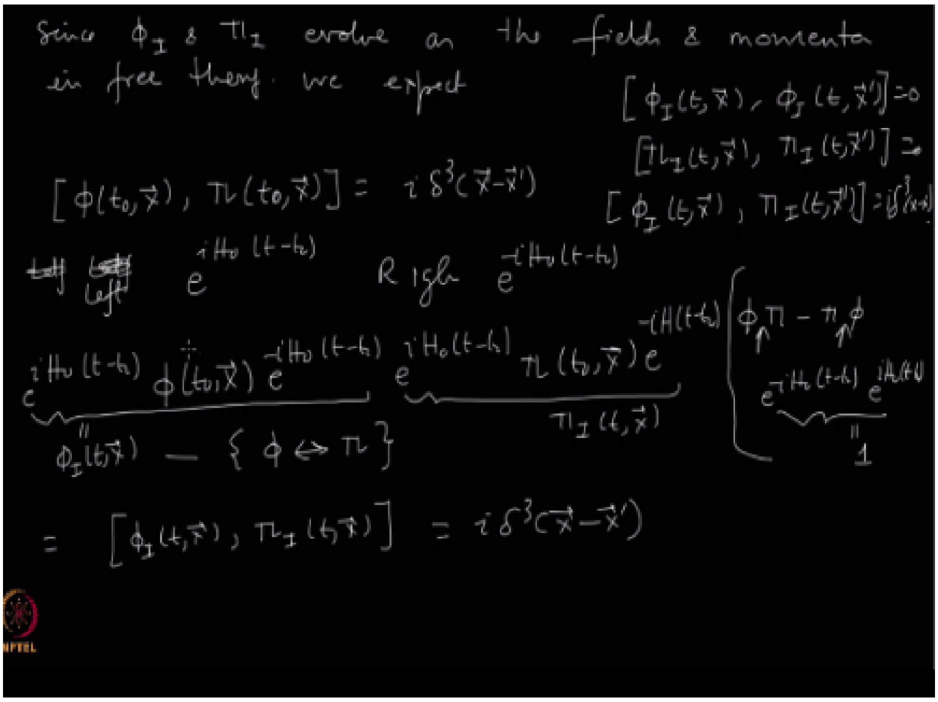


Figure 1: Refer Slide Time: 00:15

So let me write since phi I and pi I behave like or evolve like evolve as the field and momenta, conjugate momenta with conjugate momentum I mean I do not mean physical momentum, momenta in free theory, we expect that they will satisfy the following commutation relations that if I were to construct and that is the expectation that we will have. So this we should expect to be 0.

So these are equal time commutation relations and I should expect that if I take phi I (t, x), pi I t, x prime I should get i delta cube x minus x prime. And let us check whether the expectations are met. So let us check this one and the others will follow similarly. So let us check the third one. So I start with the field phi at t naught x and pi t naught x. So these are the operators in full theory, okay.

And these are, this is an equal time commutation relation and that equal time commutation relation has to be this because that is how we quantize our theory, okay. Now we will turn it into expression of this form. And that is easy, all I have to do is bring in these factors. See phi I has a e to the i H naught t minus t naught on the left and e to the minus i H naught t minus t naught on the right, okay.

And this the inverse of this factor is just the same thing with the plus sign, right? So the inverse of this is e to the plus i h naught t minus t naught. So I will use this and in this expression I will multiply on left with e to the, what was there, i H naught okay. And on okay again I have written wrong, f t. And on the right I will multiply with e to the minus i H naught t minus t naught, okay.

So here you have phi times pi minus pi times phi. So let me write here you have phi pi minus pi phi. And in the middle I will insert e to the minus i H naught t minus t naught e to the i H naught t minus t naught which is 1, right? This is 1. These two factors cancel and give you 1. So I will insert in between 1. So when I do all these things I get e to the i H naught t minus t naught, okay.

Then you get a phi t naught, sorry okay. Then I have, I will insert this. H naught t minus t naught H naught t minus t naught. Then you have the factor of pi, okay. And then on the right you have multiplied with this one so you get this, okay. And minus the same thing with phi and pi interchange, okay. Now you see this factor here this is precisely phi I, right? We can go and check.

This with a plus i minus i with a phi at t naught. So that is phi I, this piece. And this piece is pi I, right? And if this the thing which you are subtracting is exactly the same thing in reverse order. So what you get is phi, sorry I t, x pi I t, x and this commutator, okay? And what has happened on the right hand side? On the right hand side I have multiplied from left this object and right this object.

Delta cube is a, this entire thing is a complex number. So that does not interfere with these, these operators. So these two operators can be brought together and they just kill each other and give you 1. So this factor remains, okay. So indeed the commutation relations are exactly the ones which the free theory satisfies or even your full theory this the same commutations relations are satisfied, okay.

Since ϕ_I, Π_I evolve as the field and momentum in free theory we expect

$$[\phi_I(t,\vec{x}), \phi_I(t,\vec{x}')] = 0 \tag{1}$$

$$[\Pi_I(t,\vec{x}), \Pi_I(t,\vec{x}')] = 0 \tag{2}$$

$$[\phi_I(t,\vec{x}), \Pi_I(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \tag{3}$$

Let's take the third commutator

$$[\phi_I(t,\vec{x}), \Pi_I(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \tag{4}$$

Multiply left by, $e^{iH_0(t-t_0)}$ and right by, $e^{-iH_0(t-t_0)}$

$$e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{x})\,e^{-iH_0(t-t_0)} \tag{5}$$

So this is good. And this is good because we know how to deal with fields which evolve like free fields okay and the operators which evolve like free operators. So we will be able to do exactly what we did in free theory. We will be able to define a vacuum which is annihilated by a's and a daggers, we will be able to introduce them okay, because our fields are behaving like free fields.

And the entire machinery that was developed in the case of free field theory can be used here provided we can express everything in terms of phi I's and pi I's. If we can do so then we can get some answers. So now our task is to look at the operator phi and try to express it in phi I and pi I, okay. So that is the goal. So as I said we want to express everything in phi I and pi I

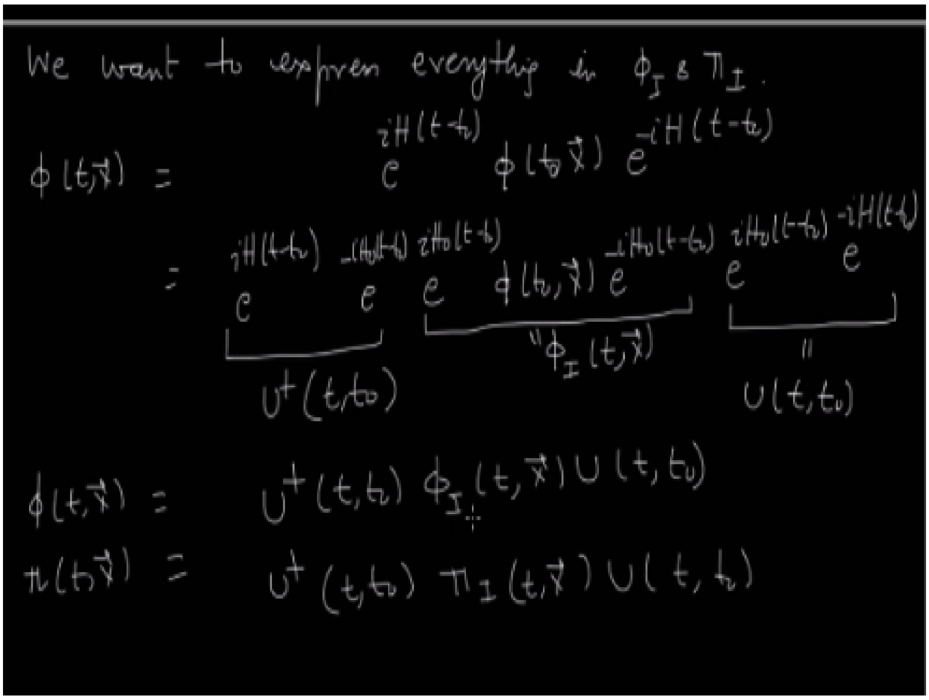


Figure 2: Refer Slide Time: 07:15

and because these are the objects we know how to deal with, okay. So let us see. Okay now that is not difficult. Let us start with phi okay, the field in the interacting theory and you know how it evolves, okay. That we wrote some time back, let me write it again. That is full Hamiltonian, okay. Now what I will do is I will insert in here and in here identity in this form, okay. That is what I am going to do. So this becomes, let me start with here. So let us see what we have to write. So I want to have plus i H naught in the left. So I write e to the i H naught t minus t naught, but then I should kill it. And this guy is there, okay. And similarly here I should put minus i H naught.

But then I should kill it. And you have this guy here, okay. Now this is our phi I, okay. And this is fine and I define this to be, I give it a name, this operator U (t naught, t). And this you can then see that this is just the dagger of it, okay. See H naught and H are both Hermitian. So when you dagger only the sign of i is going to change and the order is going to change.

And when you dagger this one, this guy goes to the left with a plus sign and this is exactly what you get here. And this guy becomes the second one, okay. So this is U dagger t naught t. And as you can see U is unitary. That is why I am using the symbol U, because if you were to multiply this factor with that factor you will get 1, okay.

So what I have achieved is that I can write phi t of x as U dagger which is not appearing to be something very spectacular because we wanted to express phi's in terms of phi I and pi I. But what we have here is phi is in terms of U and phi I. But U is made out of the H naught and H. And H naught and H are constructed out of phi's, right? H naught is, H naught is this.

So it is made out of pi and phi and H is also the full thing, so it is also made out of pi and phi. So we have not really gotten what we wanted to. But then you will see that I can express U also in terms of phi I's, okay. And that will solve our problem and we will be able to express phi completely in terms of phi I's, okay. So let me write here. In the same fashion you will get the following.

So now our issue is to do something to U to turn it into an expression involving only phi I's okay, so that is the task. Okay, so let us try that. Let me write what U is. U is this i H naught and then minus i H, i H naught and then e to the minus i H t minus t naught. Okay, remember these two exponential they do not commute. You cannot just take this one and put to the left. See these are operators, right? The H naught is made out of field operators phi's and pi's and similarly the H, okay. And if these both were H, then it is another thing but right now they do not commute, right?

Because H naught is different operator from H operator. So you cannot reverse the order. That you should keep in mind. Okay, so anyway, that is the expression of U. Now what I am going to do is I am going to find out a differential equation that the U satisfies, okay. And I will provide a boundary condition and to solve for U in a different way.

$$[\phi_I(t,\vec{x}), \Pi_I(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}') \tag{6}$$

We want to express everything in terms of ϕ_I and Π_I

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH(t-t_0)} \tag{7}$$
$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\;e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\;e^{-iH(t-t_0)} \tag{8}$$

But as you will see that when I write down the differential equation, I will have something nice. I will be able to turn the entire U into only expression involving only phi I. So let us do this. So let us multiply with i and take a derivative of U with respect to time t, okay. I am putting a partial derivative because U is a function of both t and t naught. And I am taking a derivative with only t.

So let us see what it gives. So when I take a derivative here, it will give you e to the i H naught t minus t naught i H naught. So I am differentiating this one. I get this H naught. Now this H naught you can put that side or this side, it does not matter. But it has to be before this factor okay, because they do not commute.

Where we define

$$U(t,t_0) = e^{iH_0(t-t_0)}\;e^{-iH(t-t_0)} \tag{9}$$
$$U^\dagger(t,t_0) = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)} \tag{10}$$

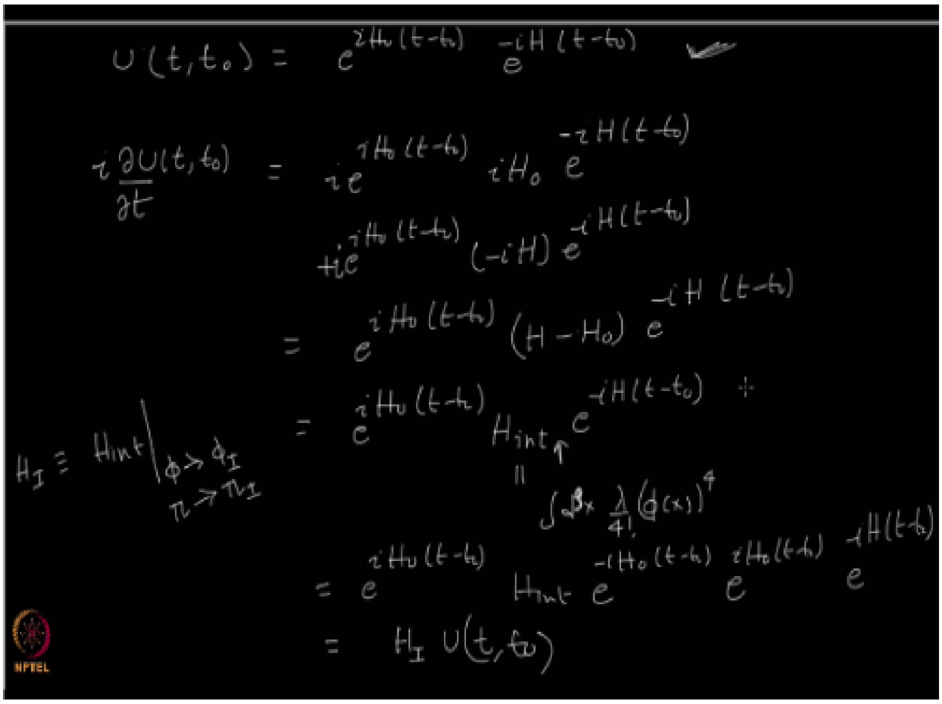
and we already have

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)} \tag{11}$$

And then when I take the derivative on the second term, it gives me i H naught t minus t naught. That pulls out minus i H. And e to the minus i H t minus t naught and I have been forgetting to multiply factor of i, okay. So i square is -1 and this one will have +1 because -i times +i gives you +1. So I can write the above expression as follows. Now this is getting very useful. You will see immediately why.

So this is e to the i H t minus t naught. This is the interaction part, right? This is actually the following d 4x lambda over 4 factorial phi x to the fourth, right? That is the difference of H and H naught. And you have this factor e to the minus i H t minus t naught, okay. So till now it does not look like much has happened.

But now what I will do is at this location I will insert e to the minus H naught t minus t naught and into the i H naught t minus t naught. So that is the factor I am going to insert in here and I get the following. H interaction the interacting part of the Hamiltonian. And you have e to the minus i H naught t minus t naught i H naught t minus t naught. So that is identity.



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I have done nothing that is multiplying 1. And you have then e to the minus i H t minus t naught. Okay, now if you see this part, this is containing H interaction which is just this. So you have field, four factors of field phi and there is some lambda or 4 factorial d 4 sorry it should be d cube x, okay. So what these factors these operators on left and right will do is they will turn your phi into phi to the fourth, okay.

That you can check. So this changes H interaction into something we write as H I where H I is defined to be H interaction and where when you take the expression of H interaction, interacting part of the Hamiltonian you replace the field phi by phi I. And if there were any pi's in the H interaction, you replace them by pi I. But of course, in our case we do not have a pi here. So this is the only relevant piece. But in general that is what you would do. So you get H I. I hope you agree that this gets turned into H i times this piece, right? And let us see what this piece is. This is exactly what you have in here. This is U (t, t naught) or t naught t whichever way you read. So what I have got? I have got a differential equation pi del U over del t equal to H I U(t, t naught) t naught t, right?

So it is a differential equation for U, which involves H I. And H I is constructed out of phi I's. So when you are going to solve for U using this equation the solution will involve only phi I's, right? And that is what we wanted to do. See here if I get a solution of U in terms of phi I's and then of course U dagger will also in terms of phi I's, okay?

And then the entire right hand side will be completely in terms of phi I's, okay. And that is what we wanted to achieve. And of course, you will need a boundary condition to solve the expression for U. So let me also give that to you.

$$\phi(t,\vec{x}) = U^{\dagger}(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0) \tag{12}$$

$$\Pi(t,\vec{x}) = U^{\dagger}(t,t_0)\,\Pi_I(t,\vec{x})\,U(t,t_0) \tag{13}$$

$$U(t, t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)} \tag{14}$$

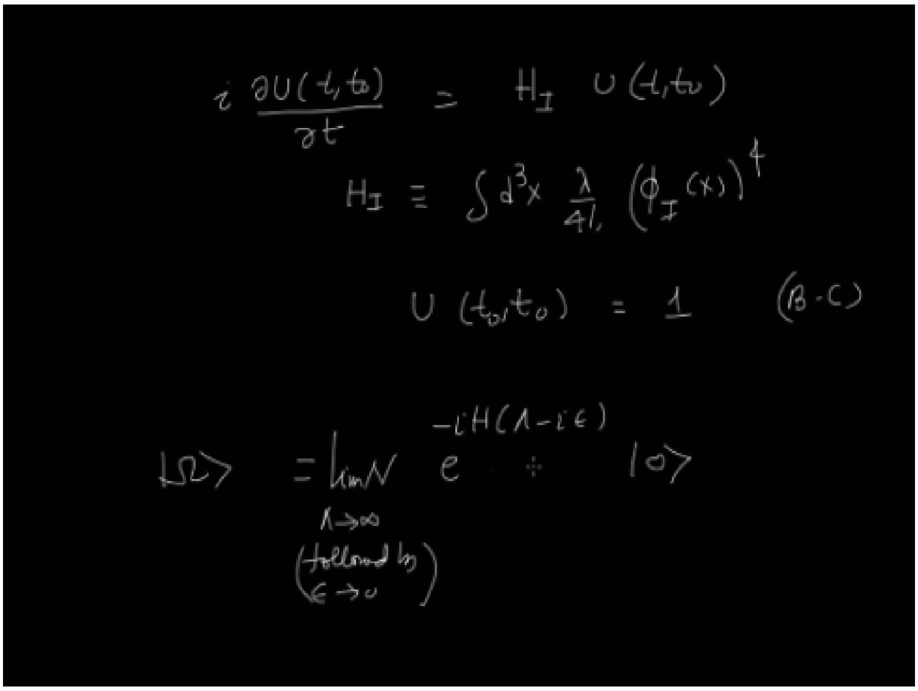
$$i\frac{\partial U(t,t_0)}{\partial t} = ie^{iH_0(t-t_0)}\,(iH_0)\,e^{-iH(t-t_0)} + ie^{iH_0(t-t_0)}(-iH)e^{-iH(t-t_0)} \tag{15}$$

$$i\frac{\partial U(t,t_0)}{\partial t} = e^{iH_0(t-t_0)}\,(H - H_0)\,e^{-iH(t-t_0)} \tag{16}$$

$$i\frac{\partial U(t,t_0)}{\partial t} = e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH(t-t_0)} \tag{17}$$

$$i\frac{\partial U(t,t_0)}{\partial t} = e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)} \tag{18}$$

$$i\frac{\partial U(t,t_0)}{\partial t} = H_I\,U(t,t_0) \tag{19}$$



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Where

$$H_I \equiv H_{int}|_{\phi\to\phi_I,\Pi\to\Pi_I} \tag{20}$$

$$\frac{\partial U(t,t_0)}{\partial t} = H_I\,U(t,t_0) \tag{21}$$

$$H_I = \int d^3x\,\frac{\lambda}{4!}(\phi_I(x))^4 \tag{22}$$

$$U(t,t_0) = 0 \qquad \text{(boundary condition)} \tag{23}$$

and the ground state of our theory

$$|\Omega\rangle = \lim_{\Lambda \to \infty} \mathcal{N} \, e^{-iH(\Lambda - i\epsilon)} \, |0\rangle \tag{24}$$

So where we are we have found that i del U over del t okay, where H I as I wrote just now in this case is d cube x lambda over 4 factorial phi I x to the fourth, okay. And to uniquely determine the solution you need boundary condition, and that is easy. You can check that U (t naught, t naught) is 1. And let us see. Where is it, yeah. If you put t naught here, that becomes 1 this becomes 1.

So U (t naught, t) is 1. Okay, so we are almost there as far as solving fine terms of phi I is concerned. And we will continue from here in the next video, okay. So this is one thing and yeah and I should remind you that it is not the only phi which we have to worry about. It is also the omega, right? This is also not yet in terms of free vacuum and free operator. You still have an H here.

So we still have to do something but we were almost done because we had free vacuum here, okay. So yeah I think I should write e to the minus i H lambda minus i epsilon e to the minus i H lambda minus i epsilon times some factor here N and this is what we have found till now, okay. Let us check if it is correct.

This is correct and of course, I should put lambda 1 to infinity followed by epsilon going to 0, okay. So this is where we stand right now. Okay, that is our boundary condition, okay. When you have a differential equation to get a unique solution, you need to have a boundary condition okay, and that is our boundary condition. This is first order differential equation.

So you need one boundary condition which is here, okay. So things have started looking good now. And in the next video we will try to get everything in terms of phi I's and free vacuum and then we will be in business. Okay, see you in the next video then.