Introduction to Quantum Field Theory

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Lecture 38 : Phi-4 Theory: Interaction Picture -1

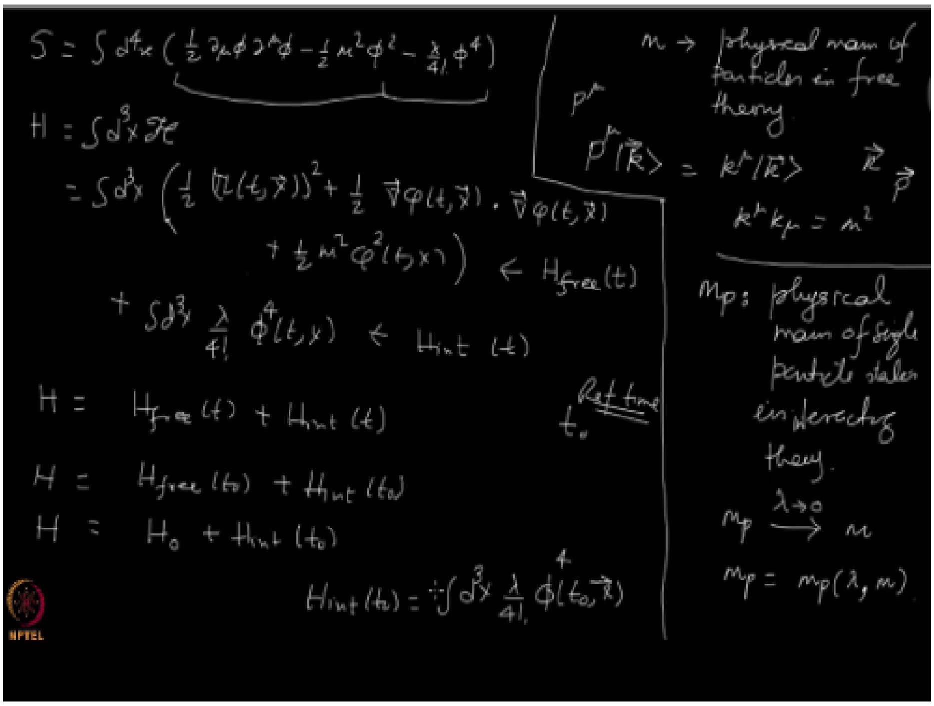

Figure 1: Refer Slide Time: 00:14

Okay we were looking at this theory which is described by the following action, okay. So as you know that this is a interacting theory and if we keep terms only which are quadratic and do not include the quartic term in here then that theory is a free theory. So if you have only up to, terms up to here in the action then that is a free theory, okay.

Now also I would like to emphasize the fact that when we were studying free theory then this parameter m turned out to be the mass of physical particles in the theory, right. So m was physical mass of particles in free theory. And why do we say that? We say that because we were able to construct states which were labelled by the eigenvalues of momentum operator.

Now that you have seen that we can construct P mu which comes from which is a charge of translations, translation symmetry. So you know that if you act on the ket label by this vector k, then you get k mu k. So this state is a eigenstate of Hamiltonian and the momentum operator in free theory, okay. And further we saw that if you take k mu k mu this gives you m square, okay.

So this means that this state has, which has energy e because that is an eigenstate of Hamiltonian with value k 0 and has momentum k, k 3 vector k because it is a eigenstate of operator p, this operator, okay. And furthermore the energy and momentum eigenvalues here satisfy this dispersion relation which tells you that this state is a single particle state where the particle has mass m, right?

Now just because we have another term in here spoils this argument that now you cannot conclude that m is the mass of physical particles in this theory. So you have to similarly construct states and then find the dispersion relation and then only you can say what is the mass of the particles in that theory, right? So what happens is that the moment you add interactions in theory, this parameter m no longer remains physical mass, okay.

And we will have to carefully find out what physical mass is in this theory. Okay that we will unfortunately not have time to do in this course. I will do it in the next one, next part, second part of quantum field theory course. But here let me just write it. So if I say m p as the physical mass in interacting theory, okay. So in phi-4 theory let me say that m p is the physical mass of single particle states.

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}(\phi(x))^4 \right) \tag{1}$$

m $\rightarrow$ physical mass of particle in ϕ^4 theory

$$P^\mu |\vec{k}\rangle = k^\mu |\vec{k}\rangle \tag{2}$$
$$k^\mu k_\mu = m^2 \tag{3}$$

$m_p =$ Physical mass of single particle state in interacting theory

$$m_p \xrightarrow{\lambda \to 0} m \tag{4}$$
$$m_p = m_p(\lambda, m) \tag{5}$$

Then we expect that if you were to take the parameter lambda to 0 okay, then the physical mass should tend to m, right? Because after all if lambda goes to 0, that term goes away and you are back to free theory. So it is obvious that the physical mass should go to the parameter m because you are going to free theory in that case. So as lambda goes to 0, this should be true okay.

So in general the physical mass would be a function of lambda and m and because these are the only two things in here. So what are the things which parameterize this section? That is lambda and m, right? So the physical mass has to be parameterized by lambda and m, okay. But that is a story that we are not going to discuss in this course, but one should be aware of this fact, okay.

Now we also wrote down the Hamiltonian for this theory which, let me write down here. So the Hamiltonian which we have written down several times is d cube x times the Hamiltonian density, let me write like this, times the Hamiltonian density, okay. It is density because integrating this thing over volume gives you Hamiltonian. So that is called Hamiltonian density. That is the curly H, okay.

And Hamiltonian density is this. So we have half pi, let me make the time in x space coordinates or space labels and time labels explicit. So you get this plus half gradient of phi dot gradient of phi okay plus half m square phi t, x square. Let me put a square here plus one more term, which is lambda over 4 factorial phi t, x to the 4.

So I can put a 4 here or if you do not like that you can write like this okay, whichever way you feel comfortable, okay. So that is the Hamiltonian you will get. And note that the first part here is what you would get if your theory was free and the second part is coming because you have interaction. So this part is H free okay. And this part is H interaction.

So what you have is that the Hamiltonian I am writing as H free plus H interaction, okay. Note that this Hamiltonian here is a conserved charge, right? It comes from translation symmetry. And by translation, I mean translation under time, okay. And so H is a constant and H does not depend on time. So the left hand side is not really time dependent okay, it is a constant.

But if you look at the individual parts, they will depend on time. And because if you look at this first part H free of t, that is not following from some symmetry principle, right? That is not a conserved charge under following from translation. So there is no reason for this to be independent of time. Similarly, this part H interaction, this is not following from some symmetry, it is not a conserved charge. So that will not be time independent. But of course, the sum of these two which is the full Hamiltonian will be time independent, okay. Now what I do is I choose some time t naught as a reference time, okay. It is arbitrary, but I choose some time t naught and write the Hamiltonian as H free of t naught plus H interaction of t naught. That I can do. I can put any time in here because the sum will always be a constant.

So I choose some time t naught, okay. So now H free of t naught I will write as H naught just as a shorthand notation; that is standard. So this part I will write as H naught. So H naught is a constant, plus H interaction at t naught. By definition that is also a constant because this is evaluated at some time t naught, okay. So these two are constants. And of course, left hand side is constant, right? So that I can do.

So if you look at H interaction now or H naught then the fields and the phi's and the pi's which appear in here they have time argument S t naught now because I am evaluating them at a specific time t naught, right. So if I look at, for example let me write H interaction, that is easier to write. So H interaction I will also suppress, let me write it once, but later I will suppress the t naught argument here.

So this is your d cube x, these are the space coordinates. And then you have lambda over 4 factorial phi t naught x to the fourth okay. So they are 4 factors of phi t naught which I multiplied in here, okay. So that is the H interaction okay. Now as I said that you cannot integrate this theory, interacting theory and get answers. So we want to do perturbation theory, okay.

So what we want to do is be close to free theory and treat the interaction term as perturbations, okay. So that is the idea. So that tells that everything I am going to do is going to be formulated in terms of free theory, right? Because I know how to exactly solve free theory okay, and that is what I am going to utilize.

$$H = \int d^3x \mathcal{H} \tag{6}$$

$$= \int d^3x \left(\frac{1}{2}(\Pi(t,\vec{x}))^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))(\vec{\nabla}\phi(t,\vec{x})) + \frac{1}{2}m^2(\phi(t,\vec{x}))^2 + \frac{\lambda}{4!}(\phi(t,\vec{x}))^4 \right) \tag{7}$$

$$H_{free} = \int d^3x \left(\frac{1}{2}(\Pi(t,\vec{x}))^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))(\vec{\nabla}\phi(t,\vec{x})) + \frac{1}{2}m^2(\phi(t,\vec{x}))^2 \right) \tag{8}$$

$$H_{int} = \int d^3x \, \frac{\lambda}{4!}(\phi(t,\vec{x}))^4 \tag{9}$$

$$H = H_{free}(t) + H_{int}(t) \tag{10}$$

$$H = H_0 + H_{int}(t_0); \quad \text{some reference time } t_0 \tag{11}$$

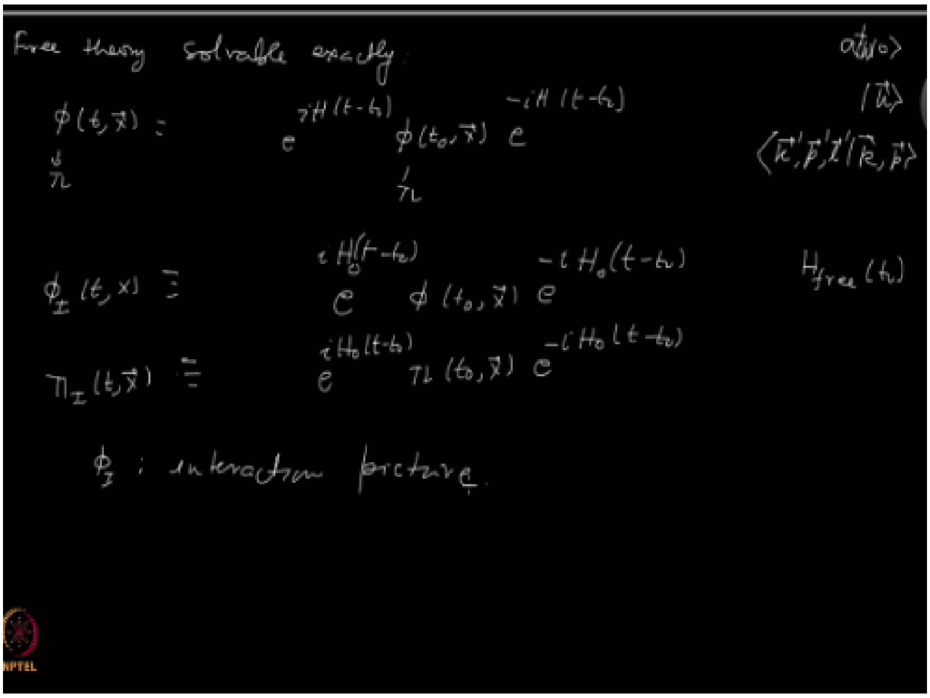

Figure 2: Refer Slide Time: 12:39

So because I know how to solve exactly a free theory and by solving exactly I mean I can calculate the correlation functions exactly in this theory, right because I could construct the states by acting with a daggers on vacuum, okay. So that gives you some apart from some overall normalization k and any question which you want to ask of, let us say this sort, so you can have any number of k's and p's in here.

And you want to know the inner product of this state with some other state that you can easily find because you know the commutation relations of a's and a daggers and then you can construct this. So you can evaluate all such correlation functions in this theory. And then you can answer any question that can be asked in quantum theory, okay. So that is what I mean by saying that free theory you can solve exactly.

And because interacting theory you cannot, we want to be expressing things in terms of free theories and do a perturbation around it, okay. So that is what we want to do. So first thing in free theory, our fields pi and phi, these operators evolve according to the free Hamiltonian, right? So as time changes the field, these operators change in time, because you are in Heisenberg picture and things change according to free Hamiltonian, right?

But here in our present case, because phi is an interacting field, if you start with phi at t naught at position x, I mean if you are given this field at time t naught, then field at phi at t, x is given by the following operation. Here H is a constant and similarly for pi. You can replace this by pi and the same thing holds true, okay. But this is a complicated thing, we do not know exactly how to deal with such object and proceed.

So what we will do is we will start with this field configuration phi at t naught, x. So some field configuration is given to me, okay. See giving a field configuration at some time t naught is not telling a lot, you just choose some field configuration. It is the evolution with time that knows whether you have an interacting theory or a free theory.

If a given field configuration were to evolve with a full Hamiltonian which is interacting Hamiltonian, then you know that the evolution is happening with the field will know about interactions at a later time t whatever you get knows about interactions. But if you were to evolve it the same field configuration with H naught which is free part of the Hamiltonian, right?

You remember H naught is just H free at time t naught. So that is the constant we have chosen. But the important point is that H naught is only quadratic, right? So this evolution of the field phi is according to what would happen in free theory. So I am constructing this and of course, this is not phi because phi evolves with H. So I give it a name, I say phi I, okay. So that is the way I will define phi I.

$$H = H_0 + H_{int}(t_0) \tag{12}$$

$$H_{int}(t_0) = \int d^3x \, \frac{\lambda}{4!}(\phi(t_0, \vec{x}_0))^4 \tag{13}$$

Free theory is solvable exactly

$$a^\dagger(k)\,|0\rangle = |\vec{k}\rangle \tag{14}$$

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\,\phi(t_0, \vec{x})\,e^{-iH(t-t_0)} \tag{15}$$

$$\phi_I(t, \vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0, \vec{x})\,e^{-iH_0(t-t_0)} \tag{16}$$

$$\Pi_I(t, \vec{x}) = e^{iH_0(t-t_0)}\,\Pi(t_0, \vec{x})\,e^{-iH_0(t-t_0)} \tag{17}$$

Where ϕ_I; interaction picture

Now the reason I do it because this will be useful. And the reason it will be useful because I am starting to construct things in terms of free theory right, this evolution is in terms of free theory. And similarly I construct another quantity in the following manner. So I take pi at time t naught x and evolve it with free Hamiltonian H naught, okay. It is important that H naught is taken at a time t naught okay, is H free at t naught.

And this I define as pi I (t, x), right? So these are definitions. Okay, very good. And these fields phi I and pi I, they are called fields in interaction picture, okay. Now we will start working with these objects and try to express everything in terms of phi I's and pi I's.