Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 37 : Phi-4 Theory: Manipulating the Ground State

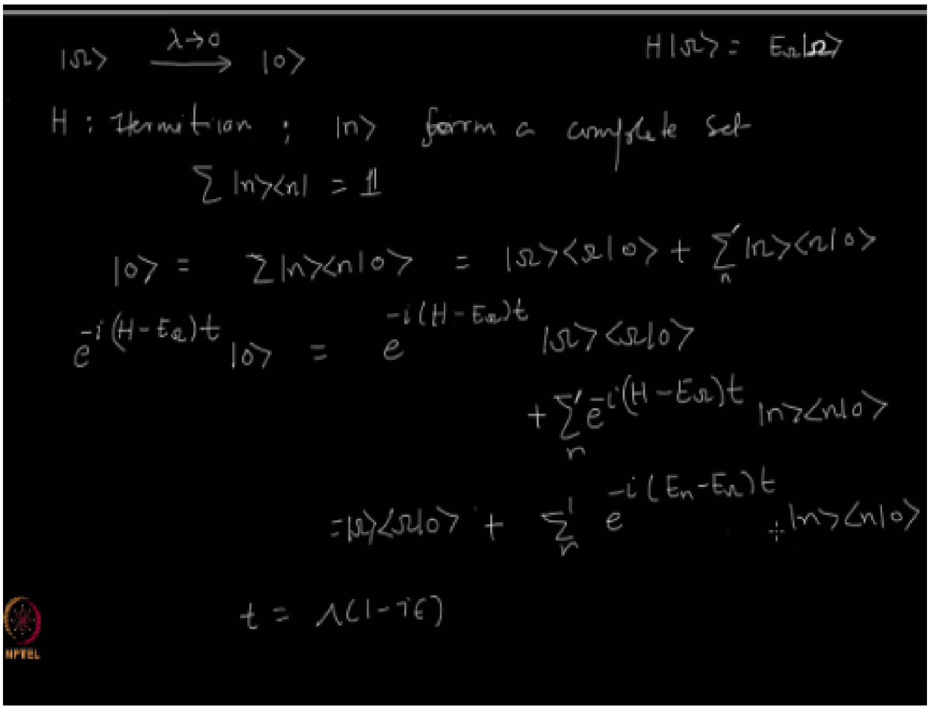

Figure 1: Refer Slide Time: 00:19

Okay, so let us try to write down the ground state of interacting phi-4 theory, ket omega in terms of the ground state of free theory, okay. So that is what we want to do. And we know that this is our expectation. It has the coupling constant in front of the phi-4 term in the action goes to 0, this ground state should go to the free ground state, okay. So let us try to get an expression for that.

Now remember that the Hamiltonian H of the full theory, the phi-4 theory is Hermitian. So its eigenstates which I will label as ket n will form a complete set, okay which basically means that this is identity, right? Now here although I am denoting the sum and putting a sigma here, but the states could have continuous labels. So in that case sigma will get replaced by integral.

And in fact, there are states which are labelled continuously, but nevertheless I will denote by I will express the completeness relation in this form, okay. So also another thing I should say before I proceed that since it is Hermitian and we know that the eigenvalues of Hermitian operators are positive, so the eigenvalues of these states will be positive, okay, positive real numbers.

Okay, so let us try to express the vacuum of the ground theory in terms of the vacuum of the interacting theory. And to do that, I start by writing this. So I insert a complete set here okay, so I just have vacuum on both sides, and I have inserted this complete set, okay. This I further write as the following. So this is the ground state. So all I am doing is breaking up this sum into two pieces.

One piece, the first state is the ground state and then of course, I have this vacuum here plus all other remaining states, okay. Let me put a prime to signify that I have out of this already removed this piece, okay. So the ground state contribution has been already taken care of, and this has all other excited states in the theory, okay. So now what I will do is, I will take this equation, okay.

This one, so actually I should have written this down, but anyway. I will multiply both sides by the following factor, okay. So this is the operator with which I want to multiply this entire equation. So I multiply it on the left hand side, I get this. And I multiply it on the right hand side here, I get the following. And I think I did not say e omega is the energy of the ground state.

So if you take Hamiltonian and act on the ground state, it has energy omega. So there is the definition of e omega. So I have just subtracted e omega from the Hamiltonian and constructed this operator, okay. And you know this is time translation operator. So that operator I am acting on the ket 0 okay or the free theory. So here the same thing. This is some complex number, okay.

Okay, so then what? Now here I cannot do much, because this is the ground state of vacuum and this is the full Hamiltonian. So I do not know what this is. So I leave it like that. But on the right hand side, I can do something. So if H minus e omega, this operator acts on omega okay, it will give you omega minus e omega, right? So this object gives you 1 because it becomes e to the minus i times 0 t. So only this piece is left and this piece is left right, because this is an eigenstate of the Hamiltonian. So that gives you e omega there. So you get the following. You get, so this is this piece what I have written here and then ket omega. It does not matter I could have written there also, it looks a bit nicer here. Plus e to the minus i E n minus E omega t, okay. Okay, so we are almost there.

$$|\Omega\rangle \xrightarrow{\lambda \to 0} |0\rangle \tag{1}$$

H: hermitian $\quad |n\rangle$ forms a complete set

$$\sum_n |n\rangle \langle n| = \mathbb{1} \tag{2}$$

We see that I can express the ground state ket omega in terms of ket 0. There is some factor here, which is I am not so much concerned with that. So we are almost there except for the fact that I have these additional terms, right? So what I really want is this, this expression, right? Because then it relates omega to ket 0. But I am also having this baggage that I have all other states appearing in here.

But I can kill this with a nice trick, okay. So what I will do now is take the following, do the following. So I will write t as lambda times 1 minus i epsilon, okay. Let me go to the next page and then write. So I will rewrite this. I think there is a way of doing it. Okay. Let me not try. I will go to the next page.

$$|0\rangle = |n\rangle \langle n|0\rangle = |\Omega\rangle \langle \Omega|0\rangle + \sum_n{}' |n\rangle \langle n|0\rangle \tag{3}$$

Multiplying both side by $e^{-i(H-E_\Omega)t}$,

$$e^{-i(H-E_\Omega)t}|0\rangle = e^{-i(H-E_\Omega)t}|\Omega\rangle\langle\Omega|0\rangle + \sum_n{}' e^{-i(H-E_\Omega)t}|n\rangle\langle n|0\rangle \tag{4}$$

$$= |\Omega\rangle\langle\Omega|0\rangle + \sum_n{}' e^{-i(H-E_\Omega)t}|n\rangle\langle n|0\rangle \tag{5}$$

Where $H|\Omega\rangle = E_\Omega|\Omega\rangle$

So we have e to the minus i E n minus E omega t and then ket 0. Then you have summation over n. And let us go back and check. This is correct. And the trick that I want to play is this. I want to take t to be the following okay, where epsilon is small number and positive, okay. i is this imaginary piece so what I have done is I have taken t and given it a imaginary component, okay.

So keeping epsilon fixed okay a small number, I am assuming epsilon to be small, keep epsilon fixed okay, and take lambda going to infinity limit, okay. And at the end take epsilon goes to 0. So I will take two limits, one is lambda going to infinity limit, second is epsilon going to 0 limit. But first I will take lambda going to infinity limit and then epsilon going to 0 limit.

And which limits I should take first, second and do what kind of manipulations is dictated by what I am trying to achieve, okay. These are mathematical tricks. And why we are doing is only justified by the outcome, okay. If they serve the purpose, they are justified. If they do not, then they are not justified. So as you will see that these steps and tricks are justified in a moment.

So if I do this, then let us see what happens here. Then this thing becomes the following. t I should replace by capital lambda, 1 minus epsilon. And then you have these objects anyway. Now let us take first limit lambda goes to infinity going to infinity, okay. In that case what will happen is the following. So look at this term, look at this exponential. This is e to the minus i E n minus E omega. I am just opening the factors lambda.

So this is I have taken care of this. Now let me take care of -i epsilon piece also. This gives you e to the minus i times minus i gives you a -1. So we have minus epsilon from here. Then you have a capital lambda, this one. And then you have E n minus E omega, okay this space. And this is multiplying all the states, okay. And of course and you sum over.

$$t \to \Lambda(1-i\epsilon): \quad \epsilon > 0 \tag{6}$$

Keeping ϵ fixed, take $\Lambda \to \infty$ then at the end take $\epsilon \to 0$

$$\sum_n{}' e^{-i(E_n-E_\Omega)\Lambda(1-i\epsilon)}|n\rangle\langle n|0\rangle \tag{7}$$

$$\lim_{\Lambda\to\infty}\sum_n{}' e^{-i(E_n-E_\Omega)\Lambda(1-i\epsilon)}|n\rangle\langle n|0\rangle \tag{8}$$

$$e^{-i(E_n-E_\Omega)\Lambda(1-i\epsilon)} = e^{-i(E_n-E_\Omega)\Lambda}\,e^{-\epsilon\Lambda(E_n-E_\Omega)} \tag{9}$$

Now I see that as you take lambda to infinity, keeping epsilon fixed, so epsilon is some small number, but it is a fixed number, okay. So you keep it fixed there, and you take lambda to infinity.

When you do so this piece gives you an exponential suppression, because E n minus E omega is positive. And the lowest energy state was having energy E omega that is the ground state.

All other states E n okay, all other states labelled by n, they have energies, which are higher than E omega, right? That is because it is the ground state, it is the lowest energy state. So this is always positive. So this is positive, epsilon I said is positive. Lambda, I am going to take to positive infinity. Then this entire factor is going to a large positive number.

And then because of the minus sign, this gets exponentially suppressed. Okay, so it goes to 0 in this limit. So what we have achieved is that the contribution coming from all these terms, which are all the excited states in the theory, they give you a vanishing contribution, if you were to take this limit. And after that, you can take epsilon going to 0 limit at the end.

You can do that, because there is no problem, things have this term has already disappeared, okay. So now I can take epsilon going to 0 limit without disturbing anything, okay. So now if I do so then I am left with only these two things. So on the left you have this piece, on the right you have only this piece. And this piece we have killed by this tray. Okay, so let me write it down.

So we get then the following. Limit lambda going to infinity, followed by epsilon going to 0. So that is the second thing you should do after this and e to the minus i H minus E omega lambda 1 minus i epsilon. So what is this? This is this piece, the left hand side. Instead of t I have put capital lambda times 1 minus i epsilon, okay. So as you see that I am taking time to have an imaginary piece by doing this.

But eventually I take epsilon going to 0. So I do come back to real time, but I do so only after I have taken lambda to infinity. Now you should not worry about why what is the meaning of imaginary time here, time has to be real, because the way to see it is that I am doing a mathematical manipulation with the help of which I can get rid of these expressions and I can write omega in terms of ket 0, okay.

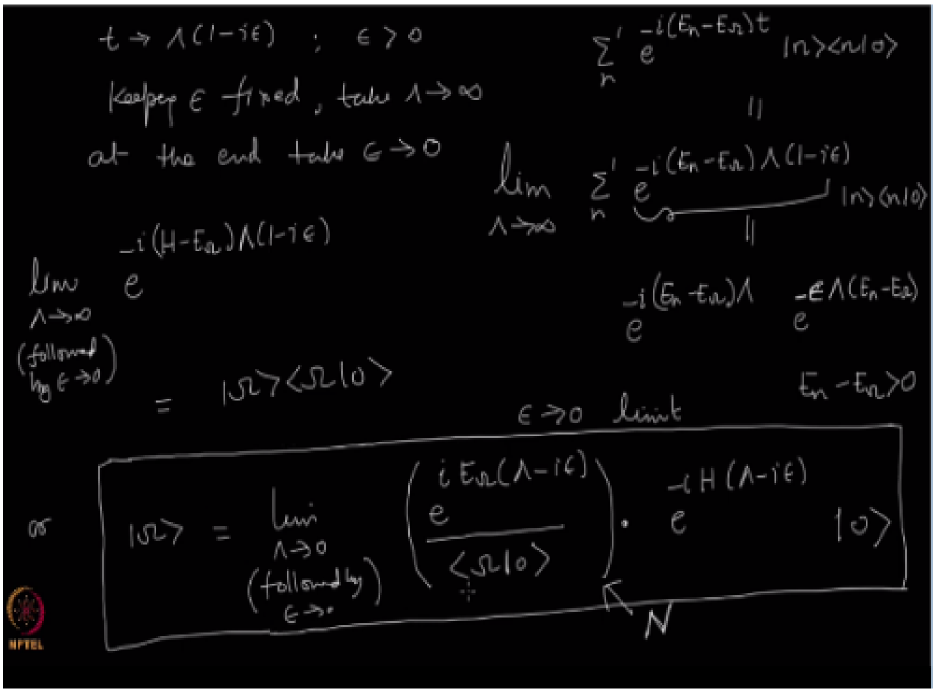

Figure 2: Refer Slide Time: 07:55

$$(E_n - E_\Omega) > 0 \tag{10}$$

In $\epsilon \to 0$ limit

$$\lim_{\Lambda\to\infty} e^{-i(H-E_\Omega)\Lambda(1-i\epsilon)} = |\Omega\rangle \langle\Omega|0\rangle \tag{11}$$

Or

$$|\Omega\rangle = \lim_{\Lambda\to\infty} \frac{e^{iE_\Omega(\Lambda-i\epsilon)}}{\langle\Omega|0\rangle} e^{-iH(\Lambda-i\epsilon)} |0\rangle \tag{12}$$

let's say

$$\mathcal{N} = \frac{e^{iE_\Omega(\Lambda-i\epsilon)}}{\langle\Omega|0\rangle} \tag{13}$$

Then

$$|\Omega\rangle = \lim_{\Lambda\to\infty} \mathcal{N}\, e^{-iH(\Lambda-i\epsilon)} |0\rangle \tag{14}$$

But now this will stay with us okay and we will have to pay attention to this change in this time. But that is something of that we will take care of later. But for now we have almost achieved our goal. So let me write it. So we have this is equal to okay or I will just put it out. Ket omega is limit lambda going to infinity followed by epsilon going to 0 e to the minus sorry e to the i E omega lambda 1 minus epsilon, this one minus minus plus, so this piece.

And I divide it by this complex number whatever that inner product is. I do not need to know it, I do not need to know its value or its expression. So this is here times I still have to take care of H. So you have e to the minus i H lambda minus i epsilon. This acting on the ground state or the free theory. Okay let me try to see if I can do. Okay, cannot do it. Fine, so I will do it with the hand.

So apart from this overall constant okay, we have actually this piece where it is the full Hamiltonian. We will do something to this later. But at least for now we have achieved our goal, okay. And this will be some constant, okay. So let us continue our discussion further in the next video. Maybe I should give it a name before I quit. Let us call it, what do I call it? Let us call it N, some normalization factor we have.

Okay some factor N. And we will, anyway we are going to get rid of this later. So for now we have achieved our goal, okay. So see you in the next video.