Introduction to Quantum Field Theory

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Lecture 36 : Phi-4 Theory

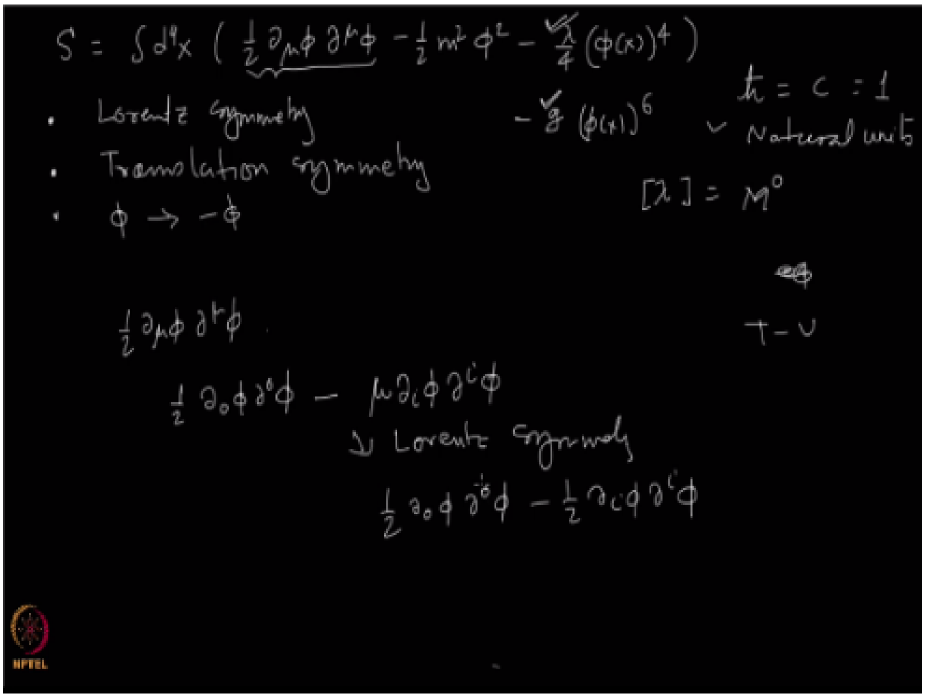


Figure 1: Refer Slide Time: 00:21

Okay, last time we decided that we will start looking at a world which is more interesting where things happen and we said that we will look at the following action.

For this last term I will write it more explicitly, I will put the arguments. This is what it is and others are already familiar to you, okay. So this is the phi-4 theory which we said we will look at last time and as I said that this has Lorentz symmetry. Okay, so the symmetry which was present in these two terms, I mean in the action which had only these two terms is still maintained by having a phi to the 4 term.

And of course, translational symmetry is also there okay for both space and time and we have also retained the phi into minus phi symmetry okay. So one thing you can do is find out the dimension of lambda. See right now you are we are working in h bar equal to c equal to 1 units. So these are called natural units, okay. And in these units you can try to find out what is the dimension of lambda. We usually say mass dimension of lambda because everything in these units is getting measured in the mass dimensions, okay. So you can ask what is the mass dimension of lambda and you should be able to show that lambda has mass dimension 0. Okay, that is something you can check.

And one thing I should remind is that, suppose you are given a Lagrangian or an action like this, okay with some fields, and some couplings okay, one should always start with this term, or equivalent of this term, okay the kinetic term. Because sometimes I have seen that you know we assume that m is mass so we can start from here, but that is not right. You cannot assume that this object has dimension of mass.

So this one is not safe to start with, because it has phi whose dimension we want to determine. But it also has another parameter m whose mass dimension we do not know. This one is the place where we should start because it has phi, which we do not know. But del mu this mass dimension we know. So this will, the kinetic term will always have only the fields, the fields whose mass dimension we do not know, everything else we know.

So always start from here, fix the mass dimension of phi, then move to the next term. And then to the next, okay. So that is how you should proceed. So once you have determined the mass dimension of phi here, then you can go here, there is no problem. You can and then you can find the mass dimension of lambda is zero. Instead, we can check here itself. You already know that mass dimension of phi is 1.

That we have either done or either or I gave as an exercise. Okay, so this brings to and this one is -4, right? So that one is -4. This one is 2 and the derivatives also give you 2, okay. Your lengths have mass dimension -1, so the derivatives will have mass dimension +1. So this entire operator that you have here, the first this entire operator that has dimension 4, the entire thing, okay?

And actually all of them will have dimension 4 because this has to work or this is -4, so each of them has to be 4, so that S comes out to be dimensionless. Okay, action will be dimensionless in these units. So once you know that phi is dimension 1, so this is dimension 4 and that is -4. So it already makes 0. So lambda has to be dimensionless and that is why you have dimension 0, okay.

Symmetries present in this theory

- Translation invariance
- Lorentz Invariance
- $\phi(x) \to \tilde{\phi}(x) = -\phi(x)$

We are working in natural units, $\hbar = c = 1$, the mass dimensions of λ is $[\lambda] = M^0$

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}\partial_0\phi\partial^0\phi - \mu\,\partial_i\phi\partial^i\phi \tag{1}$$

Due to Lorentz symmetry $\mu = \frac{1}{2}$

It is a dimensionless parameter. Okay, so that is one thing. Another thing suppose you want to consider not this theory, but something else. Suppose you want to have one more term and you want to keep this symmetry intact and you do not want any derivatives in the term then you would write a term which will be something of this sort. And this is one of the terms which is allowed.

So you can add this in the Lagrangian, okay. So whenever you are adding a new term in the action, you have to introduce a new parameter, okay. This is lambda, this is g. So you have to introduce new parameters because there is no reason why the coefficient of this operator should be the same or a factor of 2 or 3 whatever for this operator, okay.

So we have to keep new parameters and those parameters get determined by the experiments okay, that you do not fix a priori unless you have some symmetry reasons to fix them. Let me

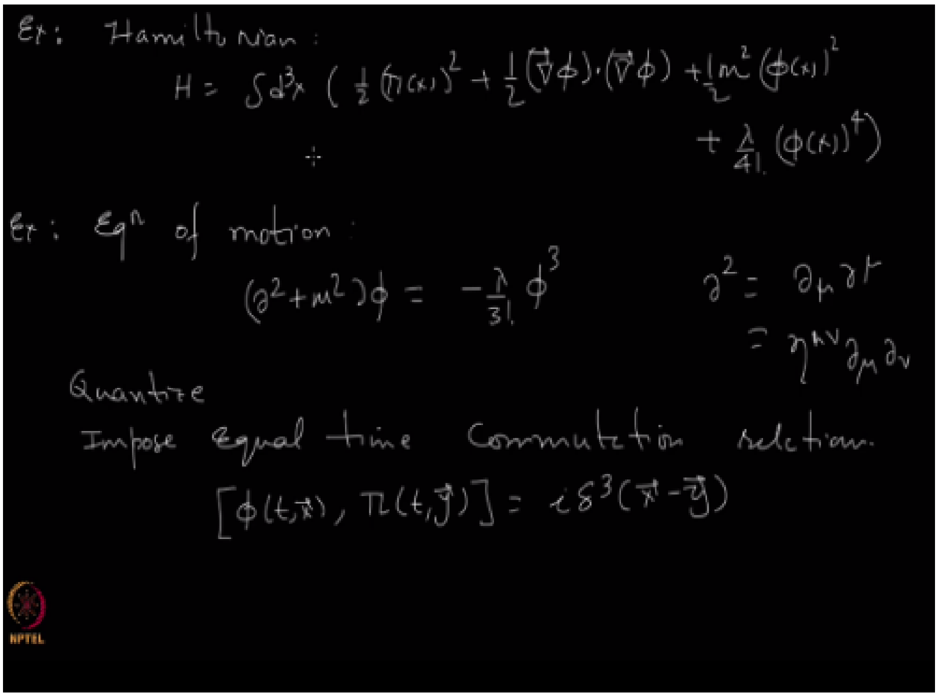


Figure 2: Refer Slide Time: 08:36

make this little bit more clearer. Let us look at the first term, okay. You could say okay because you are saying every time I should introduce a term in the theory, in the action, I should introduce a new parameter.

So instead of writing it like this, maybe I should have written the following. Okay, so that is the phi dot square minus let us say okay, and I introduced a new parameter let us say mu. So you could do that and actually you should do that. But then you want Lorentz symmetry, okay. So in principle one should start like this okay, because this is a different term. So you should have a different parameter.

But then you want to impose Lorentz symmetry and Lorentz symmetry you cannot have if you have something else here okay, some mu here. Lorentz symmetry forces this to be half, okay. So it is because of Lorentz symmetry that your coefficients get fixed like this, okay. So similarly, you will have situations where you have different terms and their coefficients are related due to some symmetry that you have imposed.

But if such, if any symmetries are not present then in principle the term should come with their own parameters which are free okay and they get determined by the experiments. Okay. So I hope that part is fine, so an exercise.

Exercise: Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi(x))^2 + \frac{1}{2}(\vec{\nabla}\phi(x))(\vec{\nabla}\phi(x)) + \frac{1}{2}m^2(\phi(x))^2 + \frac{\lambda}{4!}(\phi(x))^4 \right) \tag{2}$$

Exercise : Equations of motion

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \tag{3}$$

$$\partial^2 = \partial_\mu \partial^\nu \tag{4}$$

$$= \eta^{\mu\nu}\partial_\mu \partial_\nu \tag{5}$$

So find the Hamiltonian of the above system and show that Hamiltonian is this. I should put x in the argument but I am omitting it. I shall put half m square phi x square. And then you have plus lambda over 4 factorial phi x to the 4. And this is something you expect. So here this is you should read it like T - U okay. So this is these are the potential energy terms with the minus sign, overall minus sign.

So T - U okay. So not only this. This one has this piece this also goes to the potential. So only the kinetic term is really this one. And when you are looking at the Hamiltonian, you know that it will be T + U and that is why you have plus U here. So all the signs are reversed and this is what you will get. But do the calculation explicitly and show that this is true, okay. And also another exercise.

Find the equation of motion for this theory, for the theory described by the section, okay. So using principle of least action you will find the equation of motion and show that the following is, following equation of motion you get. Okay, so where del square is just del mu del mu, which is same as eta mu nu. Or maybe let me put it this way. Eta mu nu del mu del nu, okay.

So when you had free theory, Klein-Gordon theory this the right hand side was 0. There was no source term here. And that is why any two solutions, you could add them up, make a linear combination of any two solutions of the equation motion and the sum would also be a solution. But that is not going to be true here because of this right hand side not vanishing, okay.

This is a inhomogeneous equation and superposition principle will not hold. And that is what is going to reflect into the fact that this theory is not going to be free theory, okay. Now how do you quantize this theory? The procedure is same as always. To quantize you should construct the conjugate momentum that we have, the fields which we already have okay, and impose equal time commutation relations.

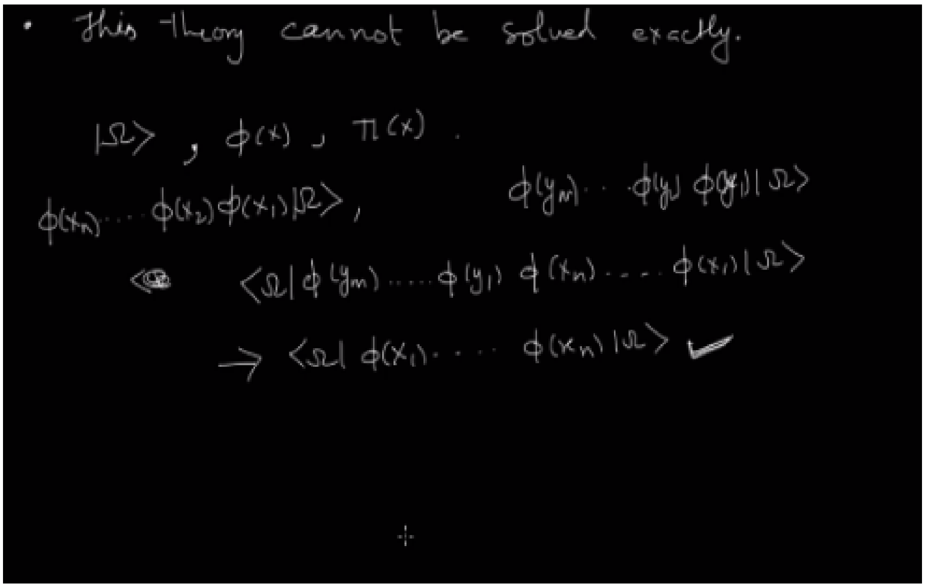
So same as always. So you take phi t, x look at the momentum density t, y and impose this commutation relation okay, i h bar I am putting h bar equal to 1. So this is just this. i delta cube x - y, okay. So if you do so you will quantize the theory. Now unlike the case of free Klein-Gordon theory that we were looking at earlier, this theory cannot be solved.

When it cannot be solved exactly you can of course do something but you cannot solve exactly like we could do before. So let me spend a minute or so in explaining what I mean by being able to solve it exactly, okay. So let us see. So let me first make this point. This theory cannot be solved, okay or you cannot integrate this theory, can be solved exactly is what I mean, okay. In fact, almost all the theories that you will see cannot be solved exactly because of interactions. There are few theories which can be solved, but none of them are theories which describe the nature, okay. So we will have to live with that.

But let me first try to make precise what I mean by we cannot solve them exactly, okay. So the answer will be a little longer. So we have to wait a bit. So let us see what we have got in the theory. So what you have got is the action okay, the field phi which you turn into operators and the conjugate momenta which you turn into the operators, right. So what the things that you have got to work with are the following.

You have the vacuum state, the ground state of the theory that you will have. Then you have the operators phi. Remember we are doing quantum mechanics, so this is not just a field. It is a quantum mechanical operator, we are going to quantize it or right now we have quantized already and the pi's. Okay, so these are all the things which are at your disposal, disposal to work with.

Okay, and phi is not so, it is not something different from, sorry the pi is not different from phi because pi you can I mean by being different okay maybe I will I should wait to make the statement, otherwise it is a bit confusing. So these are the three objects you have, okay. So you can take your operator phi or pi and hit on the vacuum to create some new state, okay.



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Impose equal time commutation relation

$$[\phi(t,\vec{x}),\Pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \tag{6}$$

Or you can hit twice to create another state and so forth. You can hit this n times, okay. So that is, those are the states that you can create in this theory. Now the questions that you will ask in quantum theory is what is the probability amplitude that a state like this has an inner product with some other state, right. These are the questions that you ask.

So you would end up asking things like so you have created one state like this, you have another state like this one. Let us say, so you hit on vacuum. You have phi y 1, phi y 2, phi y m. So you have another state and you want to know the inner product between these two states. What is the amplitude that this state okay can be found in this state.

And that you find by taking the conjugate of this which we will just get into bras and put the daggers on phi's. But we are working in real theory. So the phi daggers are same as phi. And then when you construct the inner product, you will find objects like this. Sorry. Okay, or there is no real meaning to having these kind of labels when they are all phi's.

So you write it more simply as just phi x 1 to phi x n. So you have n number of fields. That is the more general thing, okay. So these are the objects you would like to calculate in the theory, okay. And the thing is that you will not be able to calculate these objects exactly. See all your questions that you can ask out of quantum field theory can be construct can be formulated in terms of these objects.

And if you cannot calculate these objects exactly in the interacting theory, then we are saying that we cannot solve exactly our interacting theory. So that is what I meant by not being able to solve them exactly. So what do we do then? To understand what we do, and in fact that the other thing you have done before. So let us remind ourselves what we have been doing in classical mechanics when we study, okay. So what we do is let us forget about that we are doing quantum field theory. Let us forget about the fact that we are doing quantum mechanics, okay. And let us go back and think about harmonic oscillator, okay.

$$|\Omega\rangle\,, \quad \phi(x), \quad \Pi(x) \tag{7}$$

$$\phi(x_n) \cdots \phi(x_2)\phi(x_1) |\Omega\rangle \tag{8}$$
$$\phi(y_m) \cdots \phi(y_2)\phi(y_1) |\Omega\rangle \tag{9}$$

$$\langle\Omega| \phi(y_m) \cdots \phi(y_2)\phi(y_1)\phi(x_n) \cdots \phi(x_2)\phi(x_1) |\Omega\rangle \tag{10}$$

So that is the Lagrangian of the harmonic oscillator. Or in fact, I will keep it more general. No let us keep it harmonic oscillator. So where U is some alpha times q square and q square so the potential is quadratic, and that is why it is harmonic okay, and alpha is greater than 0, okay. And why have we studied the system all our, in all our education is because we can solve this system exactly, okay.

Because it is a simple system that you can solve exactly, okay. You know what the solutions are, okay. Now it does not really describe the system, it does not really describe anything physical. I mean, there is no there are no real simple harmonic oscillators in nature, okay. But the good thing about this system is you can solve exactly. And another important thing why we study this is because many systems can be approximately close, can be approximated closely to this system, okay?

So that is the advantage of studying a harmonic oscillator. Because if you wish to study a system, which is different, let us say it looks like this, okay. Let us put a minus sign here which is of course not quadratic in potential and this we will not be able to solve exactly. But nevertheless what we can do is if lambda is small, if the parameter is small okay, if parameter is very small, let us say.

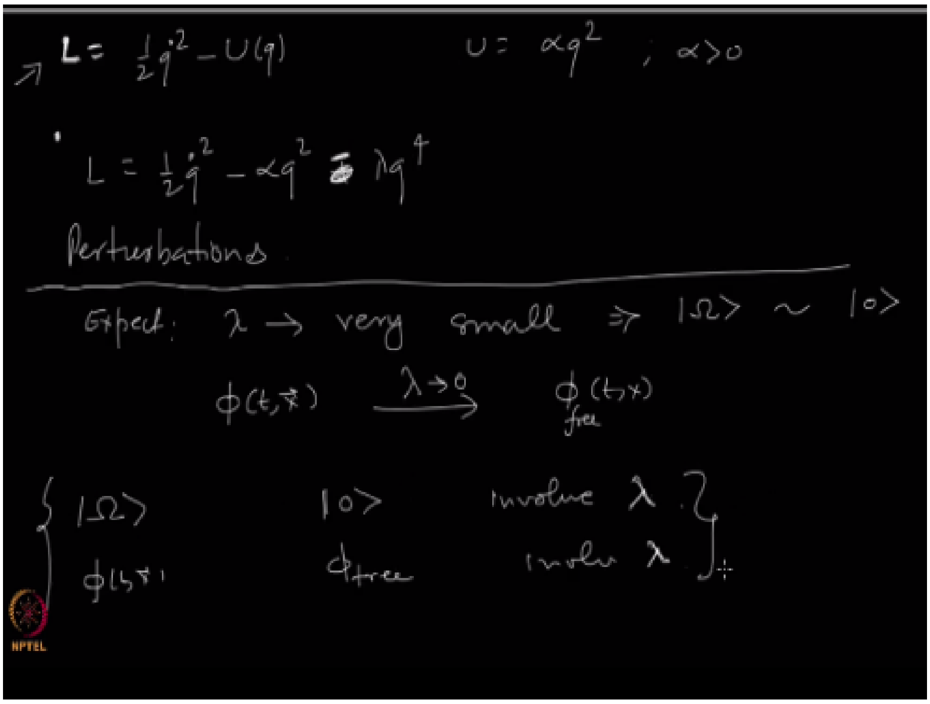
Then of course, the system behaves very closely to that of a harmonic oscillator. So the features, the broad features that this system will have will be, we will be able to describe by a harmonic oscillator. And if you want to know more, a little more about the system and include the effects of the small parameter lambda, we will try to seek solutions which are small deviations away from the solutions of the harmonic oscillator.

Okay, that is what we do in perturbation theory, okay when we are studying perturbations. So that is the same strategy we will adopt to study our phi-4 theory. So what we want to do? We would want to know that if this parameter lambda, the coupling constant, it is also called coupling constant the lambda, if this is small okay, then how things look like? And I know that if this is small, let us say this is very small, okay.

If this is very small, then my system is going to behave almost like a free theory, free Klein-Gordon theory right. So I expect that in that limit, so I expect that when lambda is very small, this should imply or when lambda is very small I should expect that the vacuum of my theory will be close to the free vacuum, right? And in the limit lambda is 0, omega should become ket 0, it should really become the free vacuum, okay.

That is good and how about our operators? Our operators phi t, x okay they should also behave like the operators in free theory. So let me for the moment put a free here okay as lambda goes, if you take lambda to be very small. So as lambda goes towards 0, the operators in this theory should also behave like free operators, okay. So that is the expectation.

So what we want to do is we want to figure out how omega and ket 0 the free vacuum are related, okay. And of course, this relation will involve, this parameter lambda. Similarly, these okay and this will also involve free parameter lambda and you could depending on how much accurate description you want you could go to different orders in lambda.



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$$\langle\Omega|\,\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle \tag{11}$$

Langrangian of harmonic oscillator

$$L = \frac{1}{2}\dot{q}^2 - U(q) \tag{12}$$

$$U = \alpha q^2, \quad \alpha > 0 \tag{13}$$

$$L = \frac{1}{2}\dot{q}^2 - \alpha q^2 - \lambda q^4 \tag{14}$$

Perturbation

Expectation $\lambda \to$ very small

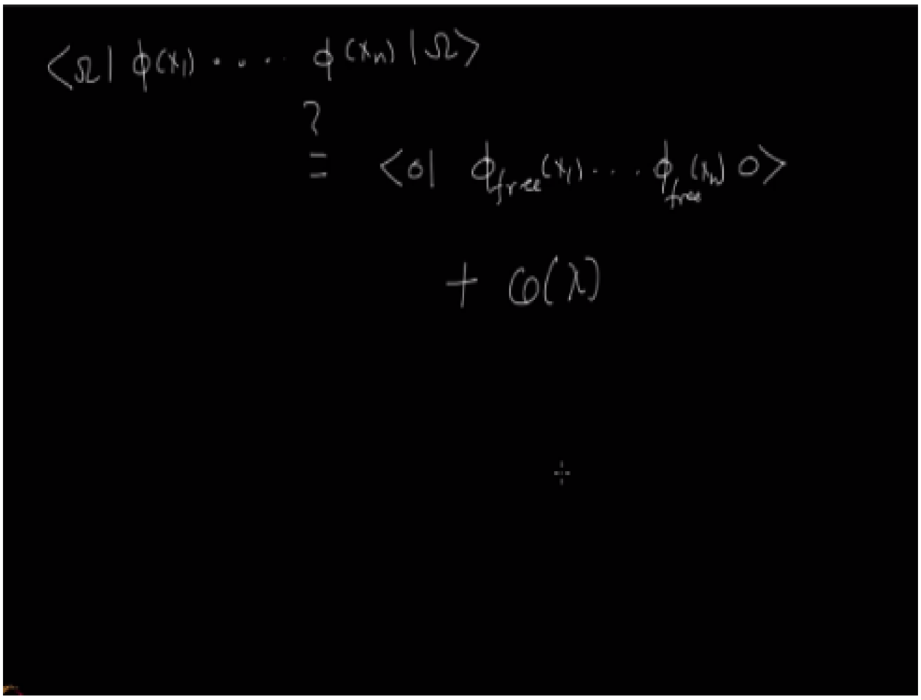
$$|\Omega\rangle \sim |0\rangle\,, \quad \phi(t,\vec{x}) \xrightarrow{\lambda\to 0} \phi_{free}(t,\vec{x}) \tag{15}$$

$$|\Omega\rangle \;\&\; |0\rangle \qquad \text{involve}\,\lambda \tag{16}$$

$$\phi(t,\vec{x})\,\&\,\phi_{free}(t,\vec{x}) \qquad \text{involve}\,\lambda \tag{17}$$

Then we will find out

$$\langle\Omega|\,\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle \stackrel{?}{=} \langle 0|\,\phi_{free}(x_1)\cdots\phi_{free}(x_n)\,|0\rangle \tag{18}$$



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But that is what basically it will, finding solutions in this theory would amount to really searching for first this relation. And once you have done so then we can go ahead and try to find out these objects okay, and try to express them as, of course this is not going to happen because you have to have lambda appearing, okay. And but because you know that if lambda you were to take to 0, this should be exactly true. So whatever changes you are going to observe are going to appear at order lambda, okay.

So this would indeed be true. So this will, this expansion, this perturbation theory would start like this. So this you would have, the left hand side would be equal to the first term which will look like really a free theory thing and then corrections to order lambda. Okay, so this will be our goal in the next video, to find relations of this kind, okay. So I think now we are clear what we are after. So we will do the computation explicitly in the next video.