Introduction to Quantum Field Theory

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Lecture 35 : A Boring World: Scattering in a Free Theory

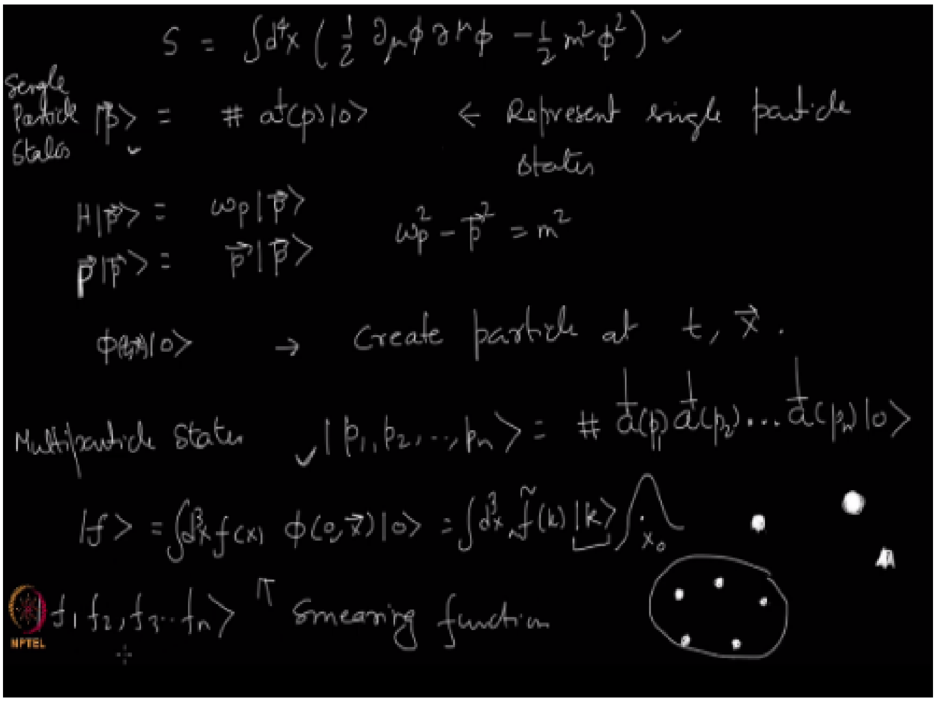

Figure 1: Refer Slide Time: 00:28

Okay, let us return to our discussion that we were having on free scalar field theory before we started discussing about symmetries. And remember that we were looking at this section at that time. In fact, this is the one we have been looking most of the times, okay. And we had learned that if we quantize this theory I find particles or states that are represented by this. So if I find that there are states which I can label by some momentum p, which you obtain by acting with a daggers on the vacuum and we had put some normalization factor here.

And that normalization is not so important right now for us, so I will not bother to write it, okay. And we saw that these states represent single particle states. And what is the basis of saying that these represent single particle states? The basis was that if you take the Hamiltonian of the theory and act on the state you get the energy or you get something, okay. And if you take momentum operator and act on the state you get this.

So this is the operator P capital P, this is small p okay. And the relation between omega P and p was the following, okay which is the dispersion relation for relativistic particle okay; that is the relation between energy and momentum of relativistic particle. And that is why together with this and these two statements we conclude that the state represents a particle of momentum p, okay. So that is what we had learned.

Also we had learned that you can take the vacuum, the ground state of this theory. And if you act on it with the operator phi and you create a particle at position t, x. Create particle at, you create a single particle at time t at position x okay. That is what the interpretation was. And similarly, if you look at the following states, so here let me write single particle state.

And similarly, you could we could construct multi-particle states which we labelled as, this is a state representing n particles. And apart from the overall factors you get this state or this state is defined by the following. I am putting p like this so. Okay, so this is what a multiple particle state is.

And again the way to conclude that this represents a state with many particles of momentum having momentum p 1, p 2 and so forth is again by doing this analysis, right? You look at the Hamiltonian, look at the momentum and then you conclude that this represents a multi-particle state, okay. Now of course, if you are looking at a single particle okay, this one gives you a eigenstate of momentum p.

So this state or this particle has a very precise momentum, which is exactly p, okay. And if you construct this one, then this guy, this state represents a particle which is located exactly at position x at time t okay. But you know that when you are thinking of particles, you are not thinking of particles with a precise momentum, because having a precise momentum tells you based on Heisenberg's uncertainty principle that the particle is completely de-localized, okay. But we know that you can have particles which are localized somewhere, roughly in space and also in momentum. See we can prepare states, which are localized somewhat and have somewhat well-defined momentum, okay. By somewhat I mean, it is not exactly having a precise value, but there is some spread around the momentum value you are looking at. So there will be some spread, okay.

And similarly, in the position space it will not be exactly at some point, but there will be some spread around it okay. For example, if you are thinking of firing an electron from an electron gun, okay. You know that that electron gun will fire electrons with some momentum, okay.

So, when it comes out of that electron gun, it will not have precisely that momentum, one momentum, but it will be spread around that momentum. But you also know it is coming out of the gun. So it is not that it is not localized. It is localized when it is coming out. So you have a notion of some momentum and some position and things are smeared around those values, okay.

Okay, but these states of fixed momentum they serve as a good basis for describing particles, single particles. So for example, if you want to describe a particle which is let us say localized, okay let us look at momentum or even yeah, let us look at position first. So suppose I want to describe a single particle state which is labelled by this. And you will know why I am labelling it like this.

What I do is first I create a particle, okay. Let us say we are creating that particle at time t equal to 0. You could choose t non-zero but let me write it anyway. So let us say you are creating a particle at position x at time 0 by doing this, okay. Now as we said we want this not to be fully localized at one point, because that would mean that the momentum is completely uncertain.

What I would do is rather than having this localized at point x, I would smear it around this, okay. So which means I want to sum it up, not at all these points, I want to create a sum okay. And I want to have a weight factor which I will call f. This is

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right) \tag{1}$$

Single particle state

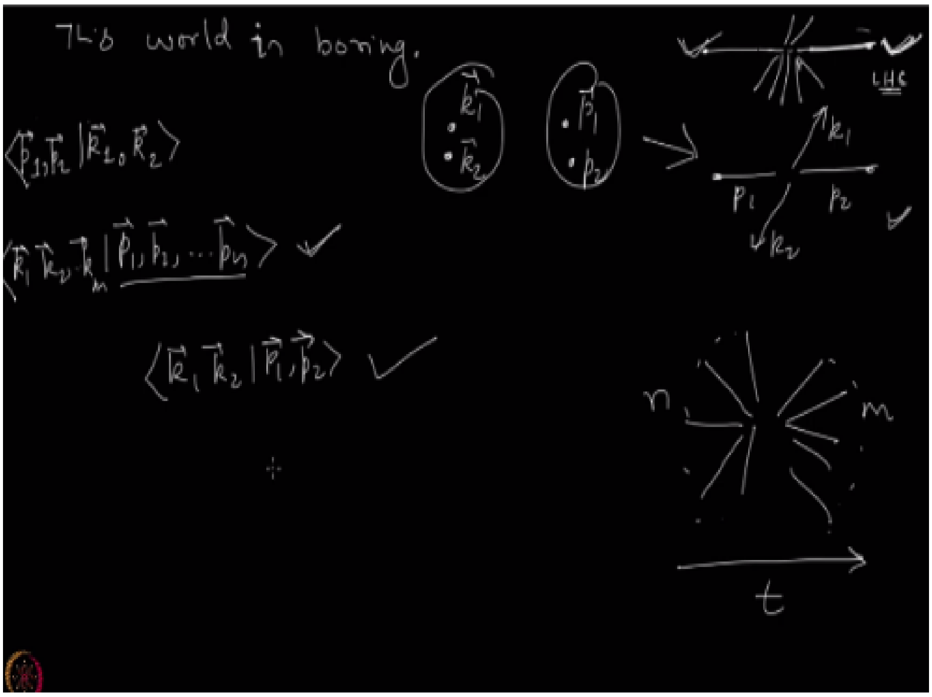

Figure 2: Refer Slide Time: 12:29

$$|\vec{P}\rangle = \#a^{\dagger}(p)\,|0\rangle \tag{2}$$

Represents single particle state $\omega^2 - \vec{p}^2 = m^2$

$$H\,|\vec{p}\rangle = \omega_p\,|\vec{p}\rangle \tag{3}$$
$$\vec{P}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle \tag{4}$$

So if you choose a function which is a Gaussian centred around this point x, some point x okay. So you choose some point x naught and take a Gaussian which is peaking around that point x naught okay in x in the physical space. Then you will get a state which is representing one particle state and localized around that point x naught, okay.

And if you write down the expression of phi in terms of a and a daggers okay, then of course a acting on the vacuum will give you zero and only a dagger will remain. And that a dagger acting on vacuum gives you momentum, momentum eigenstate, okay. And you can convert this f into the Fourier transform and this same state you will be able to write as let me write it down.

Okay, I am not being very careful with the normalizations right now. But it will be of this sort, okay. So there will be some normalization here depending on how we normalize the states okay, depending on what these factors were you will have something here okay.

But the point is this is again a sum of single particle states smeared according to this so that this also represents a particle localized around some momentum value depending on what Gaussian you took. So because Gaussian would give you another Gaussian for a Fourier transform. This will accordingly give you a particle which is localized in momentum around some peak value of that momentum, okay.

So that is how you construct single particle states, which represent particles in the real sense. And similarly, you can do for multi-particle states, okay. So you can fold each of these p 1, p 2 and p n just like in this manner. So you introduce several f's, okay? And then you can localize several particles at different locations and assign some momentum to them, okay. So that is how you will define single particle states and multi-particle states. But, the point is, the point that I wanted to make was that these states these get k for, these form a good basis for writing down single particle states in general, okay. This precise momentum eigenstates form a good basis.

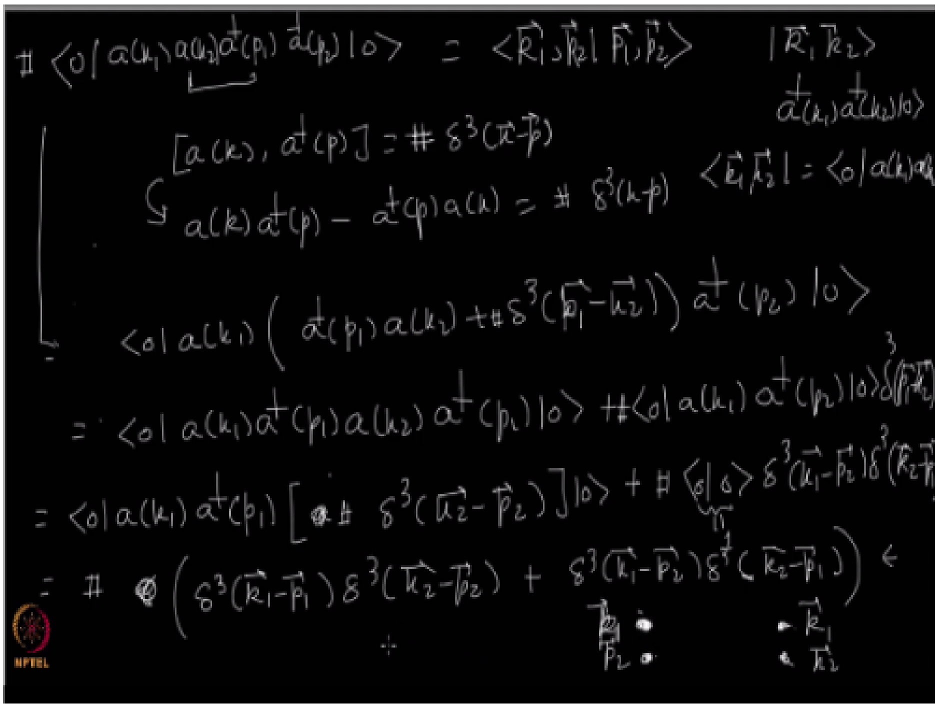

Figure 3: Refer Slide Time: 19:29

$\phi(t,\vec{x}) \rightarrow$ creates particle at $t, \vec{x}$ Multi-particle state

$$|\vec{p_1}, \vec{p_2} \cdots \vec{p_n}\rangle = \# a^\dagger(p_1) a^\dagger(p_2) \cdots a^\dagger(p_n) |0\rangle \tag{5}$$

$$|f\rangle = \int d^3x\, f(x) \phi(0,\vec{x}) |0\rangle = \int d^3x \tilde{f}(k) |k\rangle \quad f(x) \rightarrow \text{Smearing function} \tag{6}$$

And similarly, these states, these multiple particle states okay, they form a good basis for writing down a multiple particle state which represent particles like this. Okay, so they form a good basis for describing things. Okay, so good then. Then I know in this free theory that I have single particle states, I have multi-particle states. Now let us see what is happening in this world, okay. Now we are going to discover that this world is really very boring, okay.

$$\langle \vec{p_1}, \vec{p_2} | \vec{k_1}, \vec{k_2} \rangle \tag{7}$$

$$|\vec{k_1}, \vec{k_2}\rangle = a^\dagger(k_1) a^\dagger(k_2) |0\rangle \tag{8}$$
$$\langle \vec{k_1}, \vec{k_2}| = \langle 0| a(k_1) a(k_2) \tag{9}$$

$$\langle \vec{k_1}, \vec{k_2} | \vec{p_1}, \vec{p_2} \rangle = \langle 0 | a(k_1) a(k_2) a^\dagger(p_1) a^\dagger(p_2) | 0 \rangle \tag{10}$$

here we will use the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{p})] = \#\delta^3(\vec{k} - \vec{p}) \tag{11}$$
$$a(\vec{k}) a^\dagger(\vec{p}) - a^\dagger(\vec{p}) a(\vec{k}) = \#\delta^3(\vec{k} - \vec{p}) \tag{12}$$

By this world I do not mean the world we are living in. I mean the world described by the action that I wrote before, here this one, okay. Let us see why it is boring. It is boring, because nothing happens here. And by nothing happening here, I mean that if you were to start with a state with some particles, okay. And you were to look at what happens later to these states to this, these particles.

You will find that nothing happens. As time changes, they all just move around, and nothing happens. But in real world where we live in if you were to take two particles and collide them, let us say you are at LHC and you are colliding some particles, then what you expect is some things will happen at the very least okay.

Even in ordinary case, forget LHC, even in the ordinary case, when you are colliding things even billiard balls, okay some balls are colliding. If you throw with some momentum p 1, some momentum p 2, they will scatter off and go in different directions, right. So something has happened, some things scattered around, okay. Or if you are at some very high energies like at the machine LHC, then if you collide things, many things come out eventually in the final state, okay.

So even though if you start with a collision of two particles, several particles, different kind of things will be produced, okay. So these are things which happen in real world and that is why the things are more interesting in the real world. But in our world, which is getting described by this action, nothing is happening and that is what I want to show you, okay.

So let us see whether you get something of this sort, if you were to collide two particles, okay. So what I do is instead of looking at the states like this, which involves smearing function f, I work with the basis states okay, these states, okay. Because any such state get f single particle state or any other one with different smearing functions for each a okay.

I can construct a multi-particle state like this which I was saying, okay. I started with such a state and see what happens, okay. So let us take the basis states and for now let me take a state with only two particles and ask what is the amplitude to find that state into this state, okay. Now what I am, see this what will this give?

This will give you the probability amplitude that this state which is described by momentum k 1 and k 2 okay which means that you are starting with two particles one with momentum k 1, one with k 2. Of course, you have which one is which is not something you can tell because you have the both symmetry here. So you can in the a daggers commute, so you can interchange them, okay. So that symmetry is there. But within that, you know that you have two particles. And you are asking what is the probability amplitude that this state okay, is found in this one, that it has different value of momentum p 1 and different value of momentum p 2. Okay, what is the probability amplitude of finding a state like this into another state like this, which is equivalent of asking this, right?

You start with p 1, p 2. So you start with that state or it is a probability amplitude that you find it in this state, okay. And if you could answer these questions for the basis states, you would have answered all the questions about scattering in this theory, because everything will be constructed out of these objects like these ones, p1, p 2. What is the quality amplitude of this guy, this state appearing as this one k m.

So n and m are different, okay. I am saying different because in general something of this sort can happen. Okay even this is not the most general situation. Here I am saying that you are starting with two particles and several particles come out. The most general situation is that you arrange for several particles coming together. So you have n number of particles coming together.

They collide and they produce different number of particles in the final state, okay. So time is flowing like this. So meaning this is what is. So these are the particles which are coming in, and these are the ones which are going out after the scattering. So this is the more general situation. This is a special case, which is often what we arrange in the experiments. But this is the most general situation.

Okay, so we want to know about these objects in our free theory, okay. Let us see, we can easily calculate because we can solve the theory fully. In fact, we have. So what are these objects? So how do I construct? Let me take a simple example by having only two rather than arbitrary number and look at only the question where I am looking at this going to sorry. So right now I am interested in this one. But whatever I do here, will generalize and the same algebra you can repeat and get the similar conclusion, same conclusion actually. So that is what we want to look at, okay. And that is easy. So let us see.

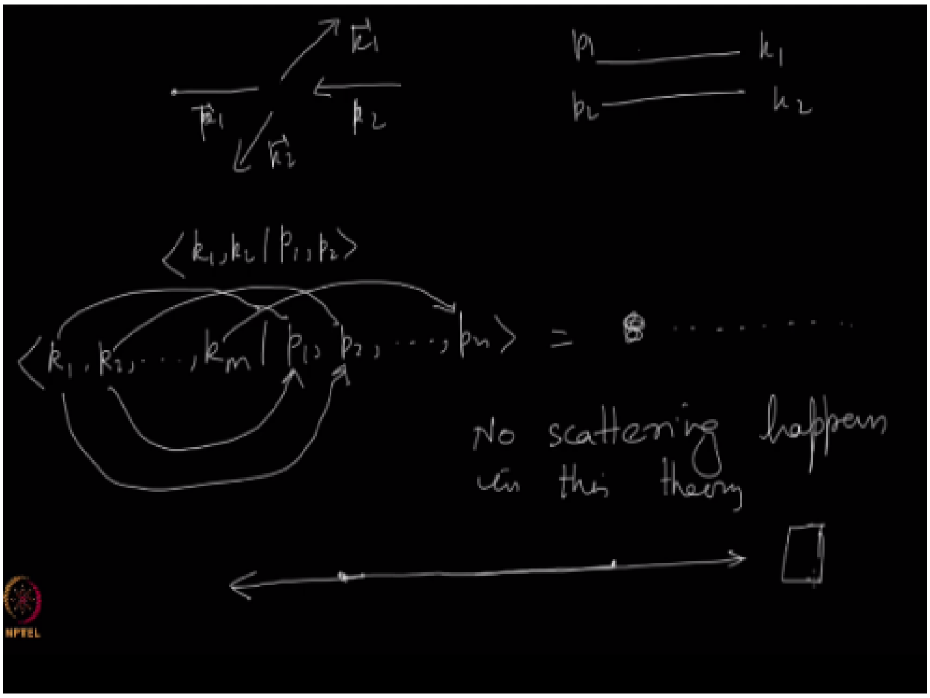

Figure 4: Refer Slide Time: 27:46

Upon using these commutation relations Eq.(10) will be,

$$
\begin{aligned}
&= \langle 0|a(k_1)a^\dagger(\vec{p_1})a(\vec{k_2}) + \#\delta^3(\vec{k_2}-\vec{p_1})a^\dagger(\vec{p_2})|0\rangle && (13)\\
&= \langle 0|a(k_1)a^\dagger(\vec{p_1})a(\vec{k_2})a^\dagger(\vec{p_1})|0\rangle + \#\,\langle 0|a(k_1)a^\dagger(\vec{p_2})|0\rangle\,\delta^3(\vec{k_2}-\vec{p_1}) && (14)\\
&= \langle 0|a(k_1)a^\dagger(\vec{p_1})[\#\delta^3(\vec{k_2}-\vec{p_2})]|0\rangle + \#\,\langle 0|0\rangle\,\delta^3(\vec{k_1}-\vec{p_2})\delta^3(\vec{k_2}-\vec{p_1}) && (15)\\
&= \delta^3(\vec{k_1}-\vec{p_1})\delta^3(\vec{k_2}-\vec{p_2}) + \delta^3(\vec{k_1}-\vec{p_2})\delta^3(\vec{k_2}-\vec{p_1}) && (16)
\end{aligned}
$$

So you had p 1, p 2, and that you create by having a p 1, a p 2 acting on vacuum, and of course they are some factors of omega sitting here. I am not worried about them right now, okay.

And then you have the state, what was that k 1, k 2, right? It is given by a (k 1) dagger a (k 2) dagger acting on vacuum. And when you look at the brow k 1, k 2, this would just give you vacuum with a (k 1), a (k 2), okay?

So I have here, a (k 1), a (k 2). You can interchange the order it does not matter because they commute. Okay, so this is basically this thing apart from some overall factors. Let me write it down, okay. So let us compute and computing this is easy, because we have this relation a (k), a dagger p is delta q (k – p), okay. I think there were, probably we, I do not remember the factors we chose here.

But that is besides the point. I mean we will not need this factor. So I am not worried. I am not careful in writing this, okay. You can find in the previous notes, but let us not worry about the factors here. Okay, so what I will do is, I want to take this a and pull it through these factors of a daggers and bring it here. Because when I do so that a will kill the vacuum, go away.

And while I am pulling it through, it will pick up the commutators and they are just numbers, the commutators are just C numbers, complex numbers. So that is how I am going to eventually get rid of all the operators, a's and a daggers and be left with only complex numbers, okay. So that is what we are going to do. So let us look at this one. You get this one, let me show you.

So here it is. If you look at this commutator, this is a (k) a dagger p minus a dagger p a (k) is delta q k minus p, okay. So if I interchange the order of a and a dagger like this, I get an additional delta from this. So that is what I am putting in here. So you have zero vacuum, a (k 1) and this gives you a, so I have this; a dagger p 1, a (k 2) plus something here, delta cube p 1 minus k 2, okay.

And then of course, you have a dagger p 2 outside, okay. So you have now the following. a (k 1), I am just writing the two terms, a dagger p 1, a (k 2), these things. Then a dagger p 2 plus a (k 1) whatever that factor was, okay. And then you have a dagger p 2 and this delta function delta cube p 1 minus k 2. Okay, so let us see. Then I bring this one here. And that will kill the vacuum.

That order will be reversed and a delta function will be picked up. So let us write that; a (k 1) a dagger p 1. And as I said, this will give you the reverse order and that will kill the vacuum. So that is gone. And you have something here times the delta cube of k 2 minus p 2 plus here again I do the same thing. I interchange the order and I get this. From here I will get delta k 1 one minus p 2.

And here you have delta 1. Let me write it differently. Let me write as k 2 minus p 1, okay. And this is 1 of course. This you can normalize the vacuum to 1. So you get this delta function and this is 1, okay. So here again this is a complex number. The delta function I will take it out and again interchange the order. This will kill the vacuum and you will pick up another delta function which will be delta of delta cube of k 1 minus p 1.

$$\langle \vec{k_1}, \vec{k_2} | \vec{p_1}, \vec{p_2} \rangle \tag{17}$$

$$\langle \vec{k_1}, \vec{k_2} \cdots \vec{k_m} | \vec{p_1}, \vec{p_2} \cdots \vec{p_n} \rangle \tag{18}$$

And again you will have this factor and you will have a 1, okay. So you will get the following result. You do the algebra, put all the factors and you will see that the following will come out. This is all gone. You will get delta cube. These two are going to give you k 1 minus p 1 delta cube k 2 minus p 2 plus delta cube k 1 minus p 2 delta cube k 2 minus p 1.

Okay, so that is the result that you are going to get and let us see what it means. It means that this right hand side, this expression is zero. It vanishes unless k 1 is equal to p 1 and k 2 is equal to p 2. Either this is true or k 1 is equal to p 2 and k 2 is equal to p 1. So it is saying this is k 1 this is k 2. Let us say you, no we started with p 1, p 2 and we said we find k 1 k 2. So these are the momenta.

So it is saying only if either this guy's moment, I mean this momentum is same as that momentum and this momentum is same as that momentum. Or if this momentum is same as this momentum or this momentum is and this momentum is same that is that same as this momentum, unless one of these is true the above expression vanishes, right? So this is zero unless one of these two are satisfied, okay.

So it says that your momentum k 1 cannot be different from p 1 and p 2. It has to be either p 1 or p 2. And similarly k 2 has to be, depending on what you have chosen for k 1 the other one, okay. So there is no chance of something of this sort happening So you cannot have k 1 coming in, k 2 coming in, and sorry p 1 coming in, p 2 coming in and the momentum that you get in the final state k 1 and k 2, they are different from p 1 and p 2, okay. So what is happening is the things were going like this and they continue going like this, okay. There is no exchange of momentum, it just goes okay. So there is no scattering happening.

And similarly, if you can try to convince yourself that this is not only true for two going to two, I am inner product of states having only two particles, but this is true in this theory for any such inner product with any number of particles here, any number of particles.

So if you were to calculate p 1, p 2, p n and what is the probability amplitude of this guy being found in this state k m where n and m could be different in principle, you will again find that this would be zero, sorry not zero, those set of delta functions, okay. So you will have all the terms here which will enforce that either k 1 is equal to p 1.

So one possibility of having non-zero result is that k 1 is equal to p 1, k 2 is equal to p 2, k 3 is equal to p 3 and k m is equal to p n okay where m and n would be same, okay. And similarly, another possibility could be that k 1 is equal to p 2 and k 2 is equal to p 1. And you can shuffle other things also around, okay. So those things will give you non-zero results. But other than those, everything will be zero, okay.

So again there is no change in the momentum of the particles in the final state, okay. So whatever particles momenta was available for the particles in the initial state, the same momentum, same momenta are available for the particles in the final state. So again no scattering is happening. So no scattering happens ever in this theory.

Which means, if you were to take a particle and fire at another particle in the same direction, they will just not see each other, they will just go through. There is nothing like they will collide or deflect, they will just go through, okay. They will go through each other. So that is the situation in this Klein-Gordon theory. And that is why I said it is boring because nothing happens here.

What can be worse that two particles can just go through each other, okay. So this is what is true in free theory. And let me tell you why this has happened okay. The reason that this is happening is because the system, the modes which you have in your theory, they are completely uncoupled, there is no coupling between different modes. See your Hamiltonian was just you could write it just as a sum of harmonic oscillators, right. But these harmonic oscillators they never talk to each other. There was no term which coupled the coordinates of one harmonic oscillators, one harmonic oscillator to the other one, okay. So there was no coupling. This Hamiltonian was completely diagonal in the a's and a daggers which you wrote, right?

It was just and you had the integral over k but it was basically this. Instead of sigma you had integral, but that is the Hamiltonian that we had, right? So these oscillators are not talking. So what we have to arrange to have something interesting happening in the theory is an interaction between different oscillators, okay. And once we do so then our scattering results will have features in them rather than being so featureless.

So the first thing we could do is to have a term in the action in addition to what we have already taken. So till now we had half del mu phi del mu phi minus half m square phi square and this gave us a boring world. Now let us allow for some terms. So I will choose only one term for now, you could choose other terms also. But before I write something in here, I should decide upon the symmetries that I want to preserve, okay.

Do I want to spoil the symmetries which were present in this theory or do I want to keep them, that choice I have to make. So I choose that I should not spoil translation invariance. I choose so because I know that this is something which leads to momentum conservation and we know that momentum is conserved and energy is conserved in nature. So I know the translation symmetry is a good symmetry of nature and I want to retain it. So I will keep. I also know that Lorentz symmetry is a good symmetry of nature. We know that things do not change in going from one frame to another frame. And things, physics does not change in doing, if you were to rotate your system, okay. So boosts and rotations are good symmetries and I want to make sure that my action also reflects that symmetry.

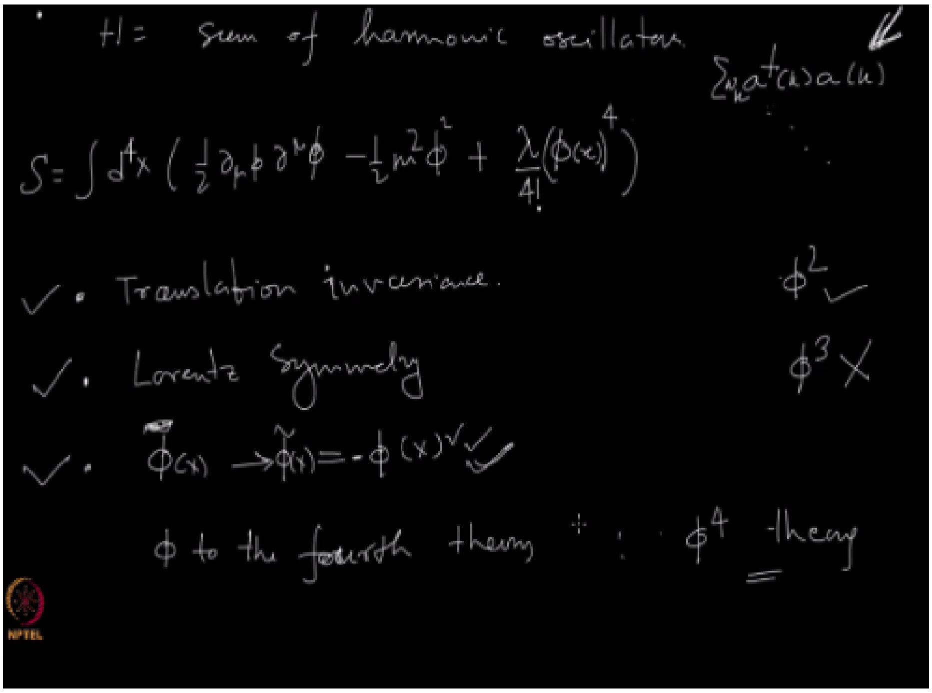

Figure 5: Refer Slide Time: 31:30

No scattering happens in this case, H = sum of harmonic oscillator

$$\sum \omega_k a^\dagger(k)a(k) \tag{19}$$

So these two I will for sure keep. One symmetry that was present in this theory was an internal symmetry which was phi of x going to minus phi of x okay, which we used to write as not this, phi tilde x is equal to minus phi (x). So this symmetry was there, right. So if you change all the phi's to minus phi nothing changes. Now we may choose to keep it or not keep it, okay.

But one of the models which we will construct, which we will write here is the one which will keep this symmetry also, okay. So phi into minus phi symmetry I want to keep. So what are the possibilities, simplest possibilities? The simplest possibility is having a phi square term, which is going to respect, but that is same as here. So if I were to add another term with the phi square with some constant here, I can club it with m square.

So it will just change the value of m square. So that is not something new, not something very useful. I could keep a phi cube term. So right now I am looking only for the polynomials in phi.

But this one will violate this symmetry. Because if you were to put a phi cube and if you were to change phi to minus phi, that phi cube term will pick up a sign. So your action will not remain unchanged.

So I say okay, I do not want this one purely because I want to have this symmetry. Then the next option is phi to the 4. So let me write it with an x, so it will be phi x to the 4, okay. So that is something you can add here. But now when you add a term like this in the theory, it will come with some coupling here, some constant here, right? And that constant is not determined by the theory, okay.

Nothing tells you what the value of that coupling should be, that term which you can write here should be, okay. So that is a free parameter okay, just like m is a free parameter here, you could change it. Similarly that coupling parameter, the coupling constant is a free parameter in the theory, okay.

So we will put a lambda here and for avoiding certain factors which appear later in the calculations, I will just divide it by 4 factorial. You may choose not to divide by 4 factorial, it is up to you, okay. But useful to have this 4 factorial here because factors of 4 factorial will appear at certain other places and they will just cancel with this one, okay. But there is nothing wrong if you do not keep it here.

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{20}$$

Symmetries we want to conserve

- Translation invariance
- Lorentz Invariance
- $\phi(x) \to \tilde{\phi}(x) = -\phi(x)$

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} (\phi(x))^4 \right) \tag{21}$$

Then you will just start seeing those 4 factorials at other places, okay. So that is action that you can write and of course you can write more terms here. No one stops you from writing other terms. But this is one simple theory, one simple world okay, and we will study this one and this one has a name it is called phi fourth, phi to the fourth, phi to the fourth theory.

Or I will sometimes just call phi-4 theory, okay. By the way, this is the to some extent, this is the one which is this theory is one which is used to describe Higgs physics, okay. Yeah, but there is some spontaneous symmetry breaking involved there. But other than that, that is a phi-4 theory. Okay, so we have decided upon looking at a world which has interactions okay, which is not a free word.

So if you have only quadratic terms, the action is going, the theory is going to be free. But now we have a quadratic term. And this will spoil this feature, and which is what we want. We do not want a theory where modes are not talking to each other, okay? And this is precisely what this piece does, okay? And that is why this theory provides interactions.

Even at the classical level, you can see that because if you are looking only at the classical solutions, because this is free okay, any two solutions, which you find, the superposition is also a solution, okay. But now if you have a non-linear term like this okay, you will see that if you excite one mode, it also disturbs other modes because things are coupled with each other. Okay, so that is the theory which we will start looking at next.