Introduction to Quantum Field Theory

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Lecture 34 : Consequences of Symmetry

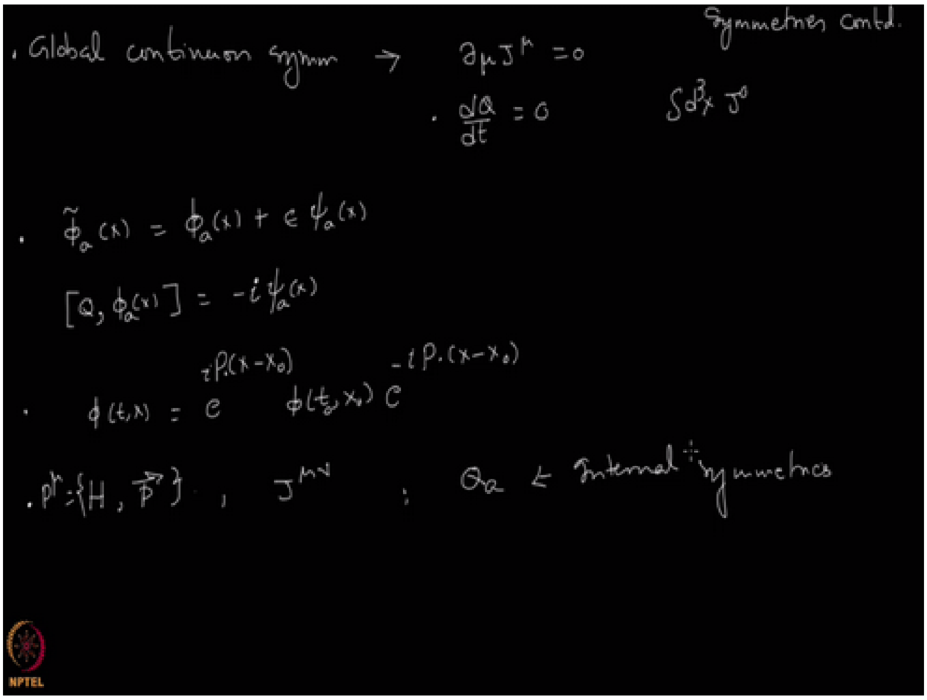


Figure 1: Refer Slide Time: 00:14

Global continuous symmetry

So we have seen till now that if you have global continuous symmetry then it leads to conserved currents and you can write down a conserved charge from this current and d Q over d t is 0 for their conserved charge and you form the conserve charge from J 0 and we have also seen so that these conserved charges are the generators of symmetry transformation.

So, what it precisely means is the following. So, if you have a symmetry transformation which is this so epsilon is a small parameter and this is the transformational field then if you take the commutator of the conserved charge Q with the field then it gives you a factor of − i this piece the psi of a. We also learn how to go from a field configuration at t 0 x 0 to a field configuration phi of t of x using the conserved charges which for the case of translations are the momenta or the momentum operator.

So, it is as you recall and we have seen several conserved charges till now so we have seen Hamiltonian and momentum which together form p mu. We have seen angular momentum J mu nu this corresponds to rotations and boost these are from translations and if you have some internal symmetries then you have corresponding charges for them. So, this comes from internal geometries.

So, I will give you one very simple exercise it is not difficult at all so you can take the action of complex scalar field so you take d 4 x del mu phi i del mu phi i so the fields phi are complex there is a summation over i. So, here by phi i phi i I mean phi i phi i the summation runs from 1 to N so there are total N number of fields and I should write this and phi i and phi i starts and then you have – m square phi i star phi i and some potential term which is also function of this.

So as you know this is invariant under U n and you have a s u N subgroup. So, if you look at infinitesimal transformations of this form where phi i tilde x is delta i j + i times epsilon k t k i j and you have k summation over all the k's and other terms so these ones you can ignore let us keep only order epsilon terms and then phi j of x. So, t k's are the generators of your s u N.

So, all we are saying is that the phi i tilde is same as phi i if you are looking at the order 0 terms so you have phi i here and then order epsilon term is proportional to this so these are the generators of s u N. Now, just to remind you that the generator satisfy the following algebra so if you take two generators t k and t l and the commutation relation is and there generators are Hermitian and these are real numbers this structure constants are real number.

$$\partial_\mu J^\mu = 0 \tag{1}$$

$$\frac{dQ}{dt} = 0 \tag{2}$$

$$\tilde{\phi}_a(x) = \phi_a(x) + \epsilon\psi_a(x) \tag{3}$$

$$\Big[Q, \phi_a(x)\Big] = -i\psi_a(x) \tag{4}$$

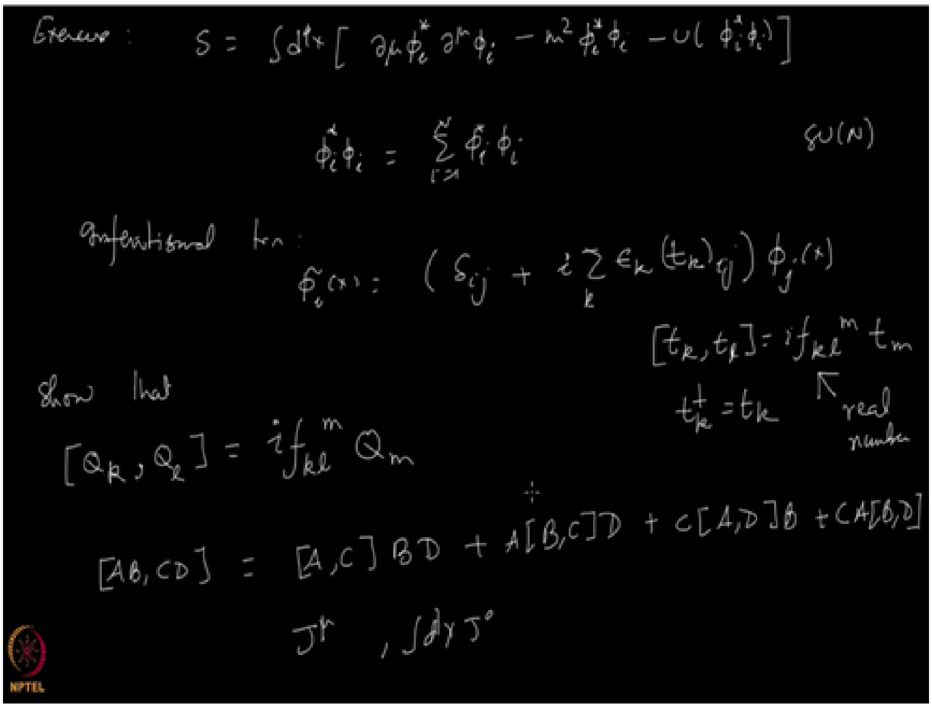
$$\phi(\vec{x}, t) = e^{ip\cdot(x-x_0)}\,\phi(\vec{x}_0, t_0)\,e^{-ip\cdot(x-x_0)} \tag{5}$$

$$P^\mu = \{H, \vec{P}\}, \tag{6}$$

So, this is what you will need to use and show that if you construct a conserved charge Q k so if you take epsilon k so right now there is a summation, but let us say you take one of the epsilons to be non zero and all other epsilons to be 0 and find out the charge corresponding to their symmetry transformation then you will get some charge and I am saying let us call that epsilon to be epsilon k.

So, the corresponding charge I am writing as Q k for that symmetry transformation. Similarly, do it for take some other epsilon to be non zero and all others to be 0 and similarly find the conserved charge Q l and find out the commutator and show that this commutation relation gives you the following. So, what you see is that the charge is here this satisfies the same algebra as these generators of SU(N).

So, if you look at the generators of SU(N) they satisfy this algebra and if you calculate in this field theory the conserved charges and the conserved charges corresponding to these internal symmetry you will find always exactly the same commutation relations and the algebra is same. So, this is simple exercise that you can do it is fairly straightforward and one thing which you would need to use is the following.



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So, when you are doing this you will need to use commutators like this. So, if you have these four then you can use that following relation that is one thing you would require and you will proceed by first constructing J mu's and then looking at d cube x J 0 and also you will need to use that these are real numbers and then these are Hermitian with that you will be able to show this and this is a nice interesting reason that the charge is the Q satisfies the same algebra is the generators of the group. So, please do that exercise.

Exercise:

$$S = \int d^4x \,[\partial_\mu \phi_i^* \partial^\mu \phi_i - m^2 \phi_i^* \phi_i - U(\phi_i^*, \phi_i)] \tag{7}$$

$$\text{where,} \quad \phi_i^* \phi_i = \sum_{i=1}^{N} \phi_i^* \phi_i \tag{8}$$

Infinitesimal transformation

Another exercise which also would not be difficult is that show that if you take the generators of translation that is the momentum and take the charge corresponding to internal symmetry transformation the Q k they commute. Similarly, if you take J mu nu angular momentum and take charge Q k which corresponds to internal symmetry again they commute and anyway we have seen that two charges corresponding to internal symmetry do not necessarily commute.

So, this is something you should show and just to remind you what J mu nu is? J mu nu is d cube x M 0 mu nu this is what we had seen earlier. So, these are some exercises that you have to do. Now we want to further explore the consequences of symmetry in a theory. So, once we have done these exercises and you also use this one that the momentum operator P mu and P nu they commute.

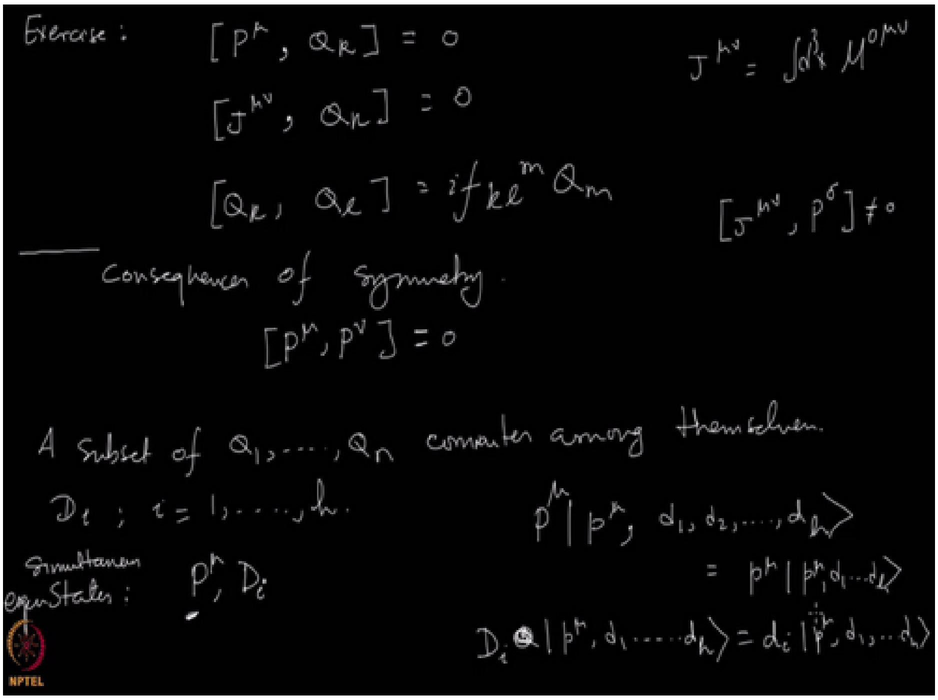
So, together with this we see the following that the P is commute amongst themselves the P is commute with Q and the Q is commute with J mu nu, but Q do not commute among themselves. So, if you take a subset see not all Q commute, but some of the Q's may commute. So, if you take a subset of whatever number of Q's you have in your theory corresponding to that symmetry.

Let us say a subset which is a subset of this commutes among themselves. Now, if a subset of this commutes among themselves let us call them D i first so D i are the generators of symmetry transformation which are commuting so we can diagonalize them simultaneously so let us say they are h of them. Now these D i together with the P mu's we can diagonalize them together.

So, if we can simultaneously diagonalize them then you see that the states in your theory the quantum states that you have in your theory you will be able to find a basis in which the states are labelled by the eigenvalues of P mu and the eigenvalues of D i's. So, we can find states or eigenstates or simultaneous eigenstates of P and Q not Q the subset of Q's which we can diagonalize.

I think I should mention one more thing which is important so I forget to write, but if you look at the commutation relation of J mu with P mu you will not get 0 because P mu is a vector so it will transform like a vector. So, maybe you should work out as an exercise so find out the commutation relation for this and you will see this is not 0 it is just because this is vector so it has to transform like a vector.

So, because of this you can only the P's and the Q's to label the states because these are the only operators that you will be able to simultaneously diagonalize so simultaneous eigenstates here. So, how will we label the states? We will label the states like this. So, small p mu is the eigenvalue of this operator P and d 1, d 2 these are the eigenvalues of the operators capital D i.



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And remember capital D i are just the charges some of the charges corresponding to this internal symmetry. So, if you take a P mu and act on this you get P mu and the same state back and if you take Q and act on the state not Q let us call it D i you will get d i p mu. So, you see that the states in your theory are not just labelled by the momenta they carry, but also they may carry other charges which are these d 1, d 2 and d h up to d h. So, you may states which have charges in them and in addition to the momentum.

$$\tilde{\phi}_i(x) = \delta_{ij} + i\sum_k \epsilon_K (t_k)_{ij}\phi_j(x) \tag{9}$$

$$\tag{10}$$

$$[t_k, t_l] = i f_{kl}{}^m t_m \tag{11}$$

$$t_k^\dagger = t_k \tag{12}$$

Shwo that

$$[Q_k, Q_l] = i f_{kl}{}^m Q_m \tag{13}$$

Hint

$$[AB, CD] = [A, C]BD + A[B, C]D + C[A, D]B + CA[B, D] \tag{14}$$

Then construct $J^\mu, \int d^3x J^0$

Exercise:

$$[P^\mu, Q_k] = 0 \tag{15}$$

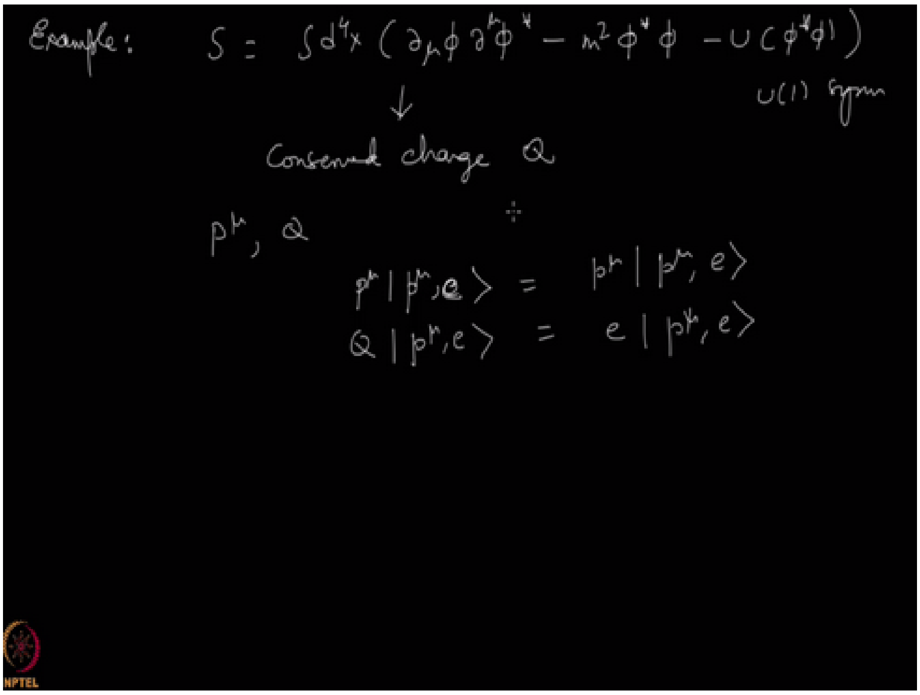
So, you can do another exercise which will help to pressure this point further. So, if you take the theory of a complex scalar field with only U(1) symmetry. So, there is no for the label i it is only phi and phi star only two fields here and look at this theory and this has a u 1 symmetry. Now with this symmetry you will able to construct the conserved charge I am sure I give this certainly as an exercise, but again you can do if you have not.

So, get the conserved charge now we can find states in this theory which will be simultaneous eigenstates of the momentum operator P mu and this conserved charge Q. So, let us write the states with the following label so if I take P mu and act on this eigenstate it gives me P mu small p mu and when I act with the charge operator Q this eigenstates gives me e times p mu e.

So, e is the eigenvalue that you have for this charge so when this operator acts you get an eigenvalue e so e is the charge you have for the state and now you see that you can have states in the theory which will carry momentum and some charge and here it was u 1 so you had one charge, but if you have more general symmetry s u N you will have more than one charges that will label the states in the theory. Now let us derive further consequences of symmetry. So as I argued sometime back that the objects of interest in a quantum field theory are this because all the things that you can construct you can construct out of the field operators and their derivatives so this is what you need to calculate in the quantum field theory. Also you will have if we have written down the theory which is invariant under space and translations you will have p mu.

If there is some internal symmetry you will have a set of charges Q a let us for the moment just write 1 because whatever I say will apply for more than 1 also and then you will have a ground state in your theory and you build everything else out of these kind of objects and we can use this to analyse some of the properties of these states which are due to symmetry. So, I will assume that the ground state in the theory has zero energy and momentum.

So, it has annihilated. So, the eigenvalues are 0 so p mu annihilate the vacuum. I also assume that the ground state has no charge. So, Q when it acts on this it has eigenvalue 0 so it also annihilates the vacuum. So, the ground state has no charge and if it does if this is not true then your theory has broken symmetry spontaneously broken symmetry. It means that you have other ground states also with momentum 0 and energy 0.



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But when you do a internal symmetry transformation on the ground state it takes you from one ground state to another ground state. So, we will not deal with those in this course and the theories which you are going to deal with are the ones which satisfy these conditions.

$$[J^{\mu\nu}, Q_k] = 0 \tag{16}$$

$$[Q_k, Q_l] = i f_{kl}{}^m Q_m \tag{17}$$

Consequences of symmetry

$$[P^\mu, P^\nu] = 0 \tag{18}$$

A subset of $\phi_1, \cdots \phi_n$ commutes among themselves
$D_i; i = 1 \cdots h$

Simultaneous eigenstates: Now with these assumptions we can obtain relations between certain correlation functions. So, you would naively expect this and it is going to turn out to be true that if you take the following.

If you take a correlation function or this matrix element in the vacuum state of phi tilde x 1 so phi is the field and this is the transform field under whatever symmetry you are looking at phi tilde x 2 phi tilde x n and since we are saying that the ground state does not transform under the symmetry transformation this statement and these are the transform fields, but my action remains unchanged under phi going to phi tilde meaning theory remains unchanged you would expect that the correlation functions also remain unchanged.

So, that is expectation we will have based on physical grounds and let us see whether whatever we have learned so far does give that result. So, let us start from the left hand side and this one

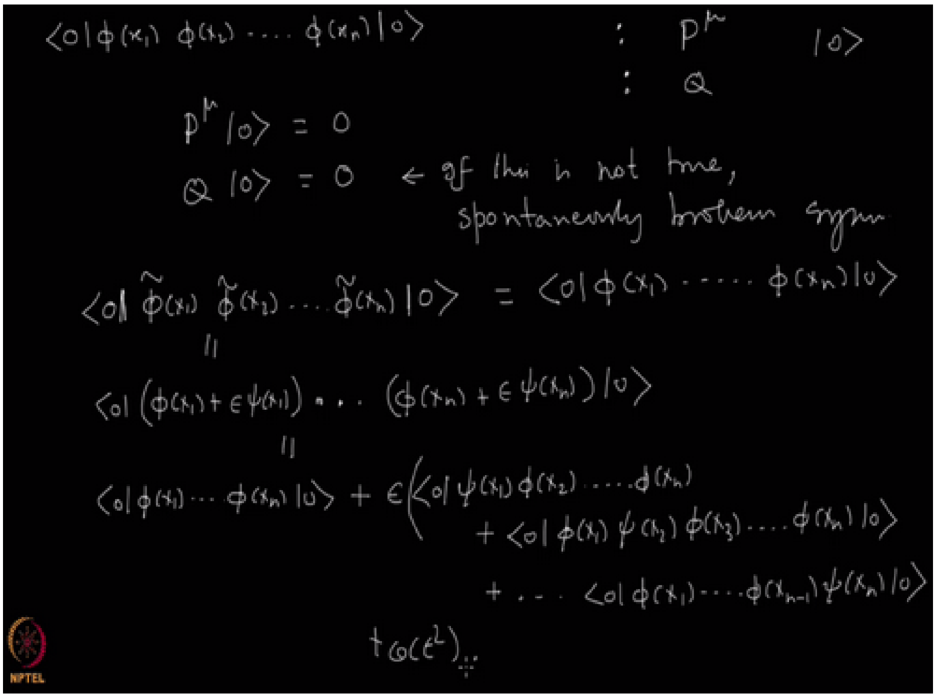
I will write as 0 vacuum phi of x 1 + epsilon psi x 1 so that is what phi tilde of x 1 is. Similarly, for phi tilde x 2 and phi tilde x n. Now this object if you look at order epsilon to the 0 terms it is simply this x n.

Then if you collect all the order epsilon terms you get the following. So, you take epsilon psi x 1 from here phi x 2 from here, phi x 3 from here and so forth phi x 1 from here so you get one order epsilon term which will be psi x 1 phi x 2 phi x n plus now you take epsilon psi terms from the second factor and you will get from the first one phi x 1 you should take here you should take psi x 2 and you have phi x 3 phi x n.

And then similarly you continue and you get phi x n – 1 psi x n and plus other order epsilon square terms. So, you see that these terms have in each such term you have one factor of psi and all the remaining are phi. So, that is the object we get. Now, I will use following result that if I take the generator Q okay remember all the phi Q all these are operators right now we are working in quantum theory. So, the Q is an operator and if you look at this commutation relation just trying to do something let us now do that. So, phi x 1 phi x n this is just a commutator and you can see that it gives you the following. So, first is you take a commutator with phi x 1 and other fields are sitting outside.

Then you have another term which is phi x 1 and then you have Q commutator with phi x 2 other fields are sitting here. Others terms and you have phi x 1, phi x n – 1 and you have Q phi x n this commutator. So, this is something you can verify. Now let us sandwich this on both sides by vacuum. So, I will apply a vacuum here and vacuum there. So, when I do that so sandwich with this and this and use the following.

Use the fact that our vacuum is annihilated by the conserved charge. So, this is 0 and our conserved charge we assume to be Hermitian so that Q dagger is Q which also implies that if you are working on the bra vector you get this to be 0. So, now let us see what we get so when I apply on both sides with the vacuum and I write this as Q times this entire product minus this entire product times Q.



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$$\hat{P}_i^{\mu}\hat{D}_i\left|p^{\mu}, d_1, d_2\cdots d_n\right\rangle \tag{19}$$

Exercise :

$$[J^{\mu\nu}, P^{\sigma}] \neq 0 \tag{20}$$

$$P^{\mu}\left|p^{\mu}, d_1, d_2\cdots d_n\right\rangle = p^{\mu}\left|p_1^{\mu}, d_1, d_2\cdots d_n\right\rangle \tag{21}$$

$$D_i\left|p^{\mu}, d_1, d_2\cdots d_n\right\rangle = d_i\left|p^{\mu}, d_1, d_2\cdots d_n\right\rangle \tag{22}$$

So, you will have Q times all these operator let me write phi 1, phi 2, phi n instead of spending time in writing the arguments I will just phi 1 to phi n. So, we will get one term this and another term will be this one and because our vacuum is annihilated by the charges both the terms vanish. So, when I sandwich this entire equation with the vacuum on both sides the left hand side will give me 0 and let us see what we get for the right hand side. So, we have a 0 and here you will have 0, but Q with phi is the generator of the symmetry and as I wrote earlier let us say where was it this one. So, apart from a factor of minus i what you get is psi. So, Q phi is psi and because I have a 0 on the left I will not worry about the factor of i that you can remove. So, I will get psi of x 1 operators psi of x 1 times phi x 2 so and so forth plus phi of x 1.

So, this term here then this one give you psi x 2 then you have phi x 3, phi x n and similarly other terms. So, you will have one psi here next time psi here next time psi here and so forth and this is fine. Now let us go back here. So you see this piece this order epsilon term is exactly what we have written down on the next slide. So, if the psi is switching places so psi is here and next time it is here and last time it is here.

So, this piece this entire order epsilon term is we can so let me see how I can write that there is no need to write I will just declare that it is 0. So the order of epsilon term the coefficient of epsilon is this piece. Let me show you again this plan in the round brackets. So, that is 0 that is what we have shown which means that under the transformation this object is the same as this object which is constructed out of untransformed fields.

And the order epsilon term vanishes and the changes appear at order epsilon square. So, that is what we have shown.

Example:

$$S = \int d^4x\, \partial_{\mu}\phi\, \partial^{\mu}\phi^* - m^2\phi^*\phi - U(\phi^*\phi) \quad ; \quad \text{U(1) symmetry} \tag{23}$$

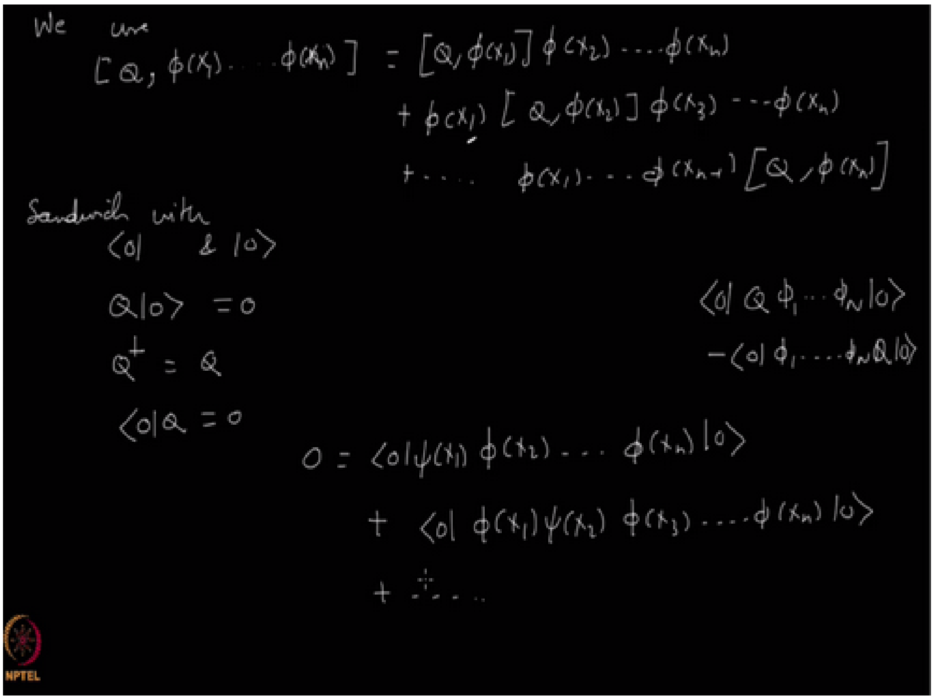
Conserved charge Q

$$P^{\mu}\left|p^{\mu}, e\right\rangle = p^{\mu}\left|p^{\mu}, e\right\rangle \tag{24}$$

$$Q\left|p^{\mu}, e\right\rangle = e\left|p^{\mu}, e\right\rangle \tag{25}$$

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{26}$$

We have



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$$P^{\mu}, \quad Q, \quad |0\rangle \tag{27}$$

$$P^{\mu}|0\rangle = 0 \tag{28}$$
$$Q|0\rangle = 0 \tag{29}$$

If Eq.(29) is not true, then we have spontaneous broken symmtery

So what we have shown is this 0 vacuum phi tilde x 1, phi tilde x n this is the same as phi tilde x 1, phi tilde x 2, phi tilde x n and the changes are of order epsilon square. So, this is good if you are looking at infinitesimal transformations it is not changing to order epsilon and that is consistent work with what expectation we had on physical grounds, but how about finite transformations?

Finite transformations is easy to build because you can construct any finite transformation in n steps and you can take those N steps to be very small and in the limit you take N to be large, your steps becomes infinitesimal. So, we can build in finite steps so let us we have total capital N steps. So, the entire thing I divided in steps so that each step is of order epsilon and when I multiply order epsilon with k I get order 1.

So, this is finite so we have one finite step which is build out of case N steps where e steps is of order epsilon which implies that where epsilon is order 1 over N. So, the different between this (()) (33:25) and this without transformed fields this is order epsilon square. So, when you do successive transformations to build up a finite transformation at each step you make the difference that you have between this and that is of order epsilon square.

$$\langle 0|\tilde{\phi}(x_1)\tilde{\phi}(x_2)\cdots\tilde{\phi}(x_n)|0\rangle = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{30}$$

$$\langle 0|(\phi(x_1)+\epsilon\psi(x_1))\cdots(\phi(x_n)+\epsilon\psi(x_n))|0\rangle \tag{31}$$

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle &+\epsilon\,\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle\\ &+\langle 0|\phi(x_1)\psi(x_2)\cdot\phi(x_2)\cdots\phi(x_n)|0\rangle\\ &+\cdots\langle 0|\phi(x_1)\cdots\psi(x_{n-1})\phi(x_n)|0\rangle+\mathcal{O}(\epsilon^2)\end{aligned} \tag{32}$$

We use

$$\begin{aligned}[Q,\phi(x_1)\cdots\phi(x_n)] = [Q,\phi(x_1)]\,\phi(x_2)\cdots\phi(x_n)\\ +\phi(x_1)\,[Q,\phi(x_2)]\phi(x_3)\cdots\phi(x_n)\\ +\cdots\phi(x_1)\cdots\phi(x_{n-1})\,[Q,\phi(x_n)]\end{aligned} \tag{33}$$

Sandwich with

$$\langle 0| \quad \text{and} \quad |0\rangle \tag{34}$$

$$Q\,|0\rangle = 0 \tag{35}$$

$$Q^\dagger = Q \tag{36}$$

$$\langle 0|\,Q = 0 \tag{37}$$

$$= \langle 0|Q\Phi(x_1)\cdots\phi(x_n)|0\rangle - \langle 0|\Phi(x_1)\cdots\phi(x_n)Q|0\rangle \tag{38}$$

$$\begin{aligned}0 = &\langle 0|\psi(x_1)\Phi(x_1)\cdots\phi(x_n)|0\rangle\\ &+\langle 0|Q\Phi(x_1)\psi(x_2)\phi(x_3)\cdots\phi(x_n)|0\rangle \quad (39)\\ &+\cdots \quad (40)\end{aligned}$$

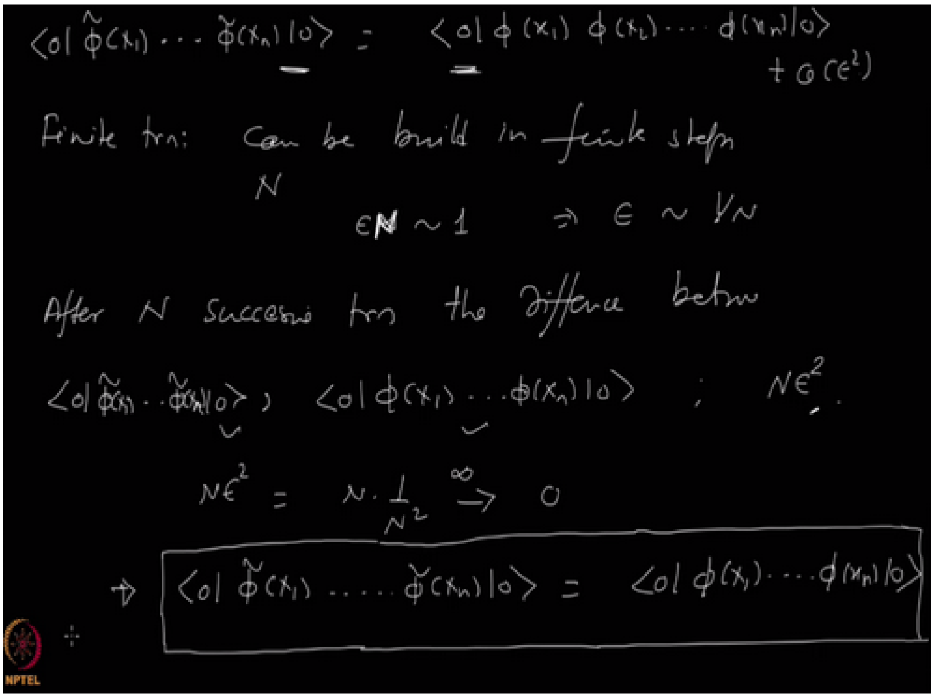
$$\langle 0|\tilde{\phi}(x_1)\tilde{\phi}(x_2)\cdots\tilde{\phi}(x_n)|0\rangle = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle + \mathcal{O}(\epsilon^2) \tag{41}$$

Finite transformation can be build in finite steps

$$\epsilon N\sim 1, \quad \epsilon\sim\frac{1}{N} \tag{42}$$

So they add up and give you N times epsilon square, but then you realize that epsilon times N is of order 1 which means if you look at N epsilon square this is N times epsilon is 1 over N so 1 over N square and as you take N going to infinity this goes to 0. So, you see that the difference between this and this vanishes when you are taking N to be very large. So, when you build up the finite transformations those difference disappear.

And you get the exact result this and make sure that you also have a intuition why this should happen. So, imagine your field theory you have some field configuration and then you are doing transformation on that field so you have this transformed field configurations in ground state is



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still the same. And when you construct that quantity that should turn out to be this because your action does not change and now we have an explicit proof of that also.

After N succesive transformation, the difference between
$\langle 0|\tilde{\phi}(x_1)\tilde{\phi}(x_2)\cdots\tilde{\phi}(x_n)|0\rangle, \quad \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \,:\, N\epsilon^2$

$$N\epsilon^2 = N \cdot \frac{1}{N} \to 0 \tag{43}$$

$$\langle 0|\tilde{\phi}(x_1)\tilde{\phi}(x_2)\cdots\tilde{\phi}(x_n)|0\rangle = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{44}$$

So, we have talked quite a lot about symmetries and their consequences and mostly I have talked about continuous symmetries and similarly the discrete symmetries will have similar consequences of course there will be no conserved charges, but relations of this form will also be true there. So, we will stop our discussion on symmetries here and we will continue further lectures in the next video.