Introduction to Quantum Field Theory

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Lecture 33 : Conserved Charges as Symmetry Generators

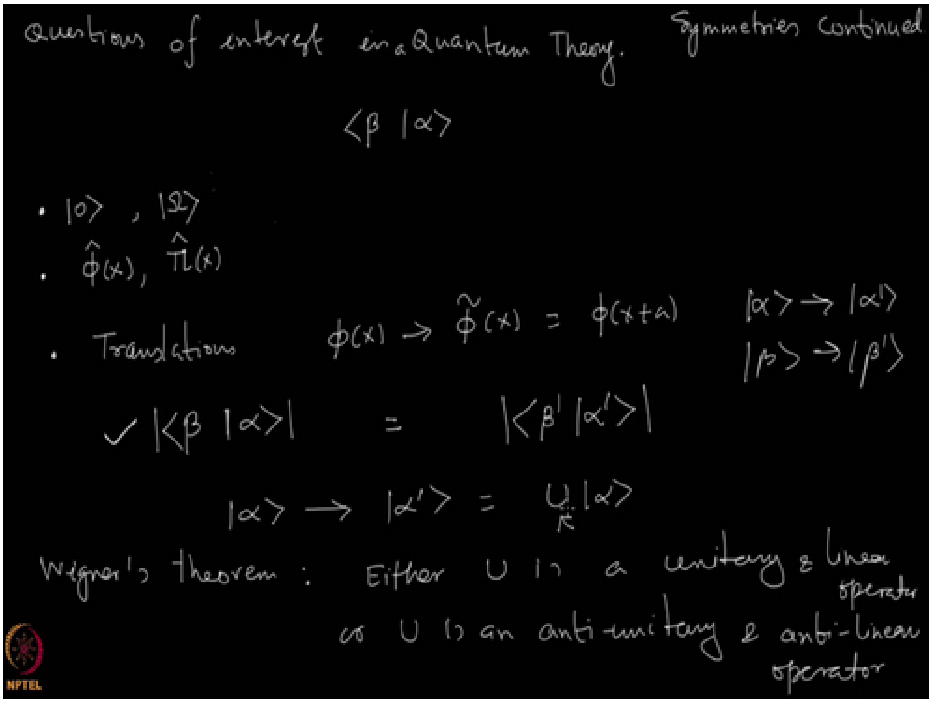


Figure 1: Refer Slide Time: 00:14

Let us continue our discussion on symmetries and today we begin by asking what are the questions that we ask in a quantum theory? So, the things that we ask are what is the probability amplitude of one state evolving into another state or what is the probability amplitude of a state alpha that we find it in some state beta. So, these are the things that we are interested in when we are working in a quantum theory.

Now, how are we going to construct states alpha and states beta. So, let us list down the things that we have got. We are given the ground state of the theory for the free case we denote the ground state by this we also call it vacuum and in general if the theory is interacting we represent the ground state by ket omega. So, that is the ground state how do you create other states?

Well, you do so by acting on the ground state by operators and the operators that you have; so the states that you create in quantum theory is by acting with operators and what operators do you have? You have the field operators and the operators which you obtained from the momentum densities. So, if you wish I can put a hat for a while. So, these are the operators that you have in your theory.

So, everything is going to be constructed out of these objects. So, you act with phi's and pi's on the vacuum to create different states. So, all that we want can be formulated in terms of these operators and these states. So, how about our discussion on symmetry that we had, what does that imply for these states that we construct. See one consequence that we found from symmetry was the existence of Noether's charges conserved current in Noether's charges that is one thing that we found.

But we can expect that those symmetries under which the action is going to remain unchanged they will also affect or they will have consequences on these objects on these matrix elements and that is what we plan to explore in this video. So, for example, if you are looking at translations symmetry where you take the field operator phi and you transform it to the operator phi tilde of x.

Remember that now we are talking about quantum theory. Now we are not looking at field as a classical field, but rather as a field operator where phi tilde of x is phi of x + a. If this leaves the action invariant then we expect that if you are looking at the probability amplitude that this state alpha is in state beta this will reflect upon transformation this fact. So, if alpha goes to some other state under this transformation beta goes to some other state under the transformation.

The inner product of these let me call it alpha prime so alpha goes to alpha prime under this translation and beta goes to beta prime under that. So, we expect that the probability amplitudes of finding alpha and beta and alpha prime and beta prime they would be the same. So, that is what we expect we can put a model is here because it is a statement about probability, but the amplitudes may change by a phase.

So, where I have said that under the transformation ket alpha goes to ket alpha prime whatever the transformation is for example translation in this case beta goes to beta prime and we expect this kind of a relation. Now when you go from alpha I mean you are taken from alpha to alpha prime by some operator U. So, let us call it U so U is the one which takes you from ket alpha to ket alpha prime so that is the operator we does get.

Questions of interest in quantum theory

$$\langle \beta | \alpha \rangle \tag{1}$$

- $|0\rangle, |\Omega\rangle \rightarrow$ ground state
- $\hat{\phi}(x), \hat{\Pi}(x)$
- Translation $\phi(x) \rightarrow \tilde{\phi}(x) = \phi(x+a)$

So, let us look at a state alpha in our Hilbert space and let us say that we do a transformation by so the transformation operator is T 1 so that is the transformation we are doing and in the Hilbert space this transformation is done by the unitary operator T 1. Now if we look at another transformation T 2 which acts on this so I am doing one transformation after the other. So, first I do T 1 and then the T 2 then this state so let me write it like this.

So, alpha goes to U T 1 of alpha and then this state if I do further transformation we will go to this one U T 2, but we could have done this in one go. So, instead of going doing first T 1 and then T 2 I could have started from alpha ended and done T 1 T 2. So, T 1 on the right means first this is operation so this product I can do in one go and then it should give me the unitary operator corresponding to this transformation which means that I should get U of T 1, U of T 2 this acting on alpha this one same as, but there could be a phase because if there is a phase the probability amplitudes will not change.

But I am not going to include the phases here and discussion on that you can find in Weinberg volume 1 for example. So, this is what we get and which means that these unitary matrices should satisfy the following property that U T 2, U T 1 = U T 1 I should have written it other way. So let me do it and that is what the property of a representation is. So, this is saying that if you are looking at these matrices these operators T 1, T 2.

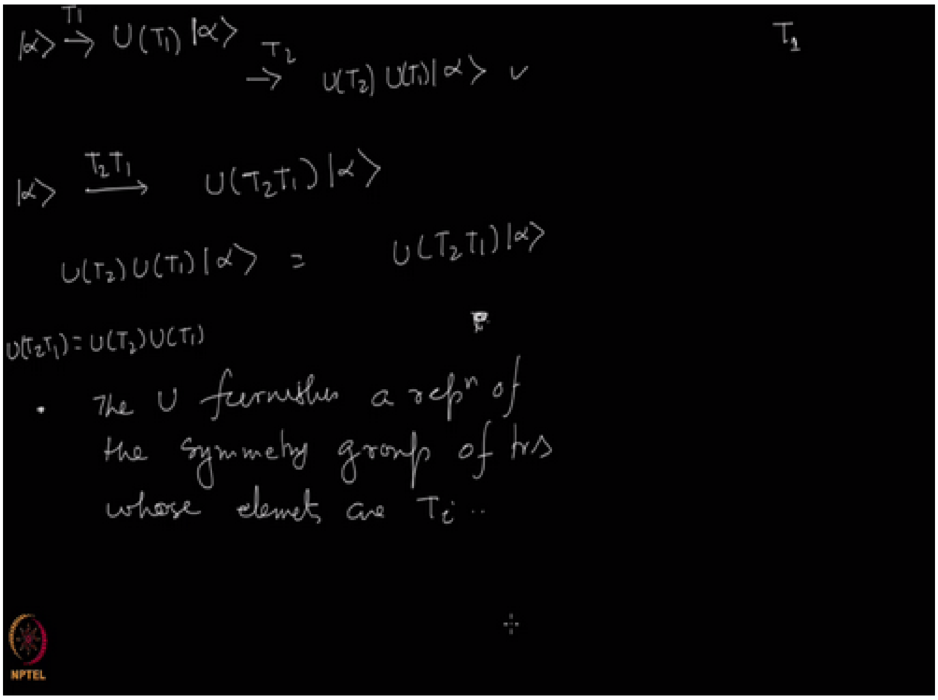


Figure 2: Refer Slide Time: 08:42

Then they will form a group as we have seen and the group will have property that T 1 I mean so this is the group multiplication for the representation generated by the unitary matrices. So, this is one thing I wanted to tell the U furnishes a representation of the symmetry group of transformations whose elements are T 1, T 2 and T 3. I will try to put these things in a context that is familiar to us which we already knew.

And then after we have talked about that familiar setting in which we are already aware of these results I will move on to a general discussion. So, for that what I do is I start looking at quantum theory in the Heisenberg picture. So, here as you know that the operators evolve with time and the states are independent of time. So, if you are looking at an operator O then it is time evolution is given by this for now I am putting i h bar, but later I will drop so this equation you are familiar with it is a Hamiltonian of the system and you are taking a commutator with the operator O again the states do not change.

They do not evolve with time so we have some states i or alpha whatever and that does not evolve with time because the entire evolution time dependence has been transferred to the operators away from the states. So, that is an equation we know so what is the solution of this equation this differential equation the solution is this e to the i over h bar H t - t 0 we can immediately verify that this is indeed true.

That this is a solution so let us check that so if you calculate d over d t this is anyway constant so it does not matter. This one when I take the derivative it will give me i over h bar times H, but that H I can put this side or that side does not matter because it will commute because this object is made up of only H so I can write it as i over h bar times H times the same thing again.

Then the derivative x on this one it again produces the same thing with a minus sign so you get − i over h bar and this entire thing is again h O t sorry this entire thing is again O of t and then derivative has pulled out an H which will be on the right side and this is what we have. You get your differential equation.

$$|\langle\beta|\alpha\rangle| = |\langle\beta'|\alpha'\rangle| \tag{2}$$

Where

$$|\alpha\rangle \rightarrow |\alpha'\rangle \tag{3}$$
$$|\beta\rangle \rightarrow |\beta'\rangle \tag{4}$$
$$|\alpha\rangle \rightarrow |\alpha'\rangle = U\,|\alpha\rangle \tag{5}$$

Wigner's theorem: Either U is a unitary and linear operator or anti-unitary and anti-linear operator.

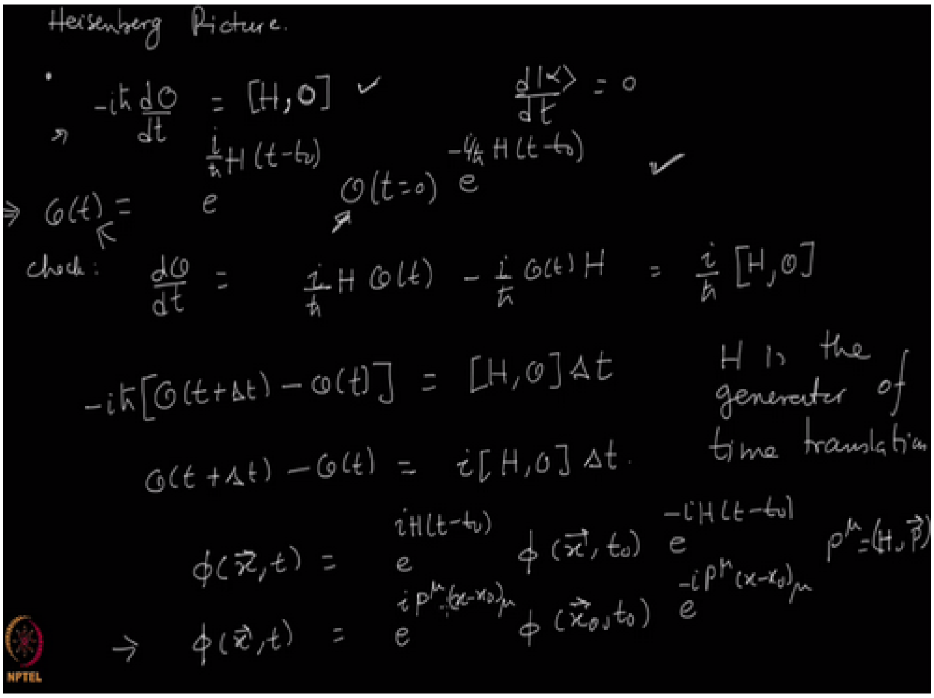


Figure 3: Refer Slide Time: 12:59

$$|\alpha\rangle \xrightarrow{T_1} U(T_1)\,|\alpha\rangle \xrightarrow{T_2} U(T_2)\,U(T_1)\,|\alpha\rangle \tag{6}$$
$$|\alpha\rangle \xrightarrow{T_2T_1} U(T_2T_1)\,|\alpha\rangle \tag{7}$$

$$U(T_2)\,U(T_1)\,|\alpha\rangle = U(T_2T_1)\,|\alpha\rangle \tag{8}$$
$$U(T_2T_1) = U(T_2)\,U(T_1) \tag{9}$$

The U furnishes a representation of the symmetry group of transformation whose elements are T_i

So, this is indeed a solution to our equation here. So, this equation here let me write it slightly differently this one let me write it as – i h bar O at time t + delta t – O at time t. So, you see that your operator evolves from O of t to O of t + delta of t and that difference is given by this commutator on the right hand side. So, we say that h is the generator of time translation. See you have translated in time you were at time t then the operator evolve to time delta t.

And this translation from t to delta t has been carried out by the operator H. So, we say that H is the generator of time translation. I will drop h bar and put i there on the right hand side which makes it following little bit cleaner so I multiply plus i on both sides I get this. So, you see this is the infinitesimal version of this statement. Here you are starting at time t 0 and these two unitary operators take you from time t.

So, evolve from time t = 0 through these unitary operators and reach at time t and this is the infinitesimal version of it. It is saying you go from time t to time t + delta t through generator H. So, these are two same statements. One is infinitesimal version of the other one. So, let us take this operator O to be phi so let us take that operator to be phi and I want to write this statement for phi.

Phi of x t I can write as e to the i H I am dropping the H bar t – t 0 phi of x t 0 so I am starting from time t 0 and evolving to time t – i H t – t 0. So, this is what you get from here and clearly if you find the commutation relation you will get phi of I mean this relation where O gets replaced by phi. So, this is good this is for the time translation. Then you realize that when we were calculating the conserved charges Hamiltonian was you got Hamiltonian where we were looking at time translations.

But you got 4 generators corresponding to space time translation the p mu which was H and P so we have 4 generators. So, if I were to look at not just time translation from t 0 to t, but look at a space time translation so I will start from x 0 and t 0 and evolve to x t I should expect that I will get something of this form not something, but I should expect the following. So p mu and then you get x – x 0 mu and here I should get – i p mu x – x 0 mu that is our expectation.

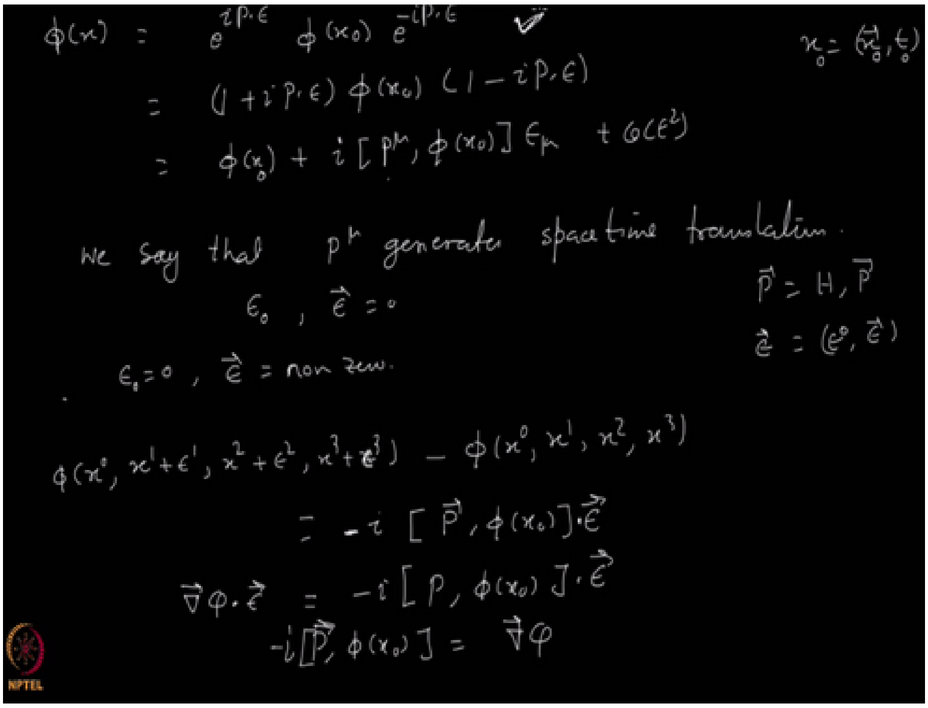


Figure 4: Refer Slide Time: 21:20

Heisenberg picture

$$-i\hbar \frac{\partial O}{\partial t} = [H, O] \quad ; \quad \frac{d}{dt}|\alpha\rangle = 0 \tag{10}$$

Purely from relativistic ground because I know that p mu is 4 vector and I know what one component is doing. So, I can almost guess what other component should do. So, let us look at the infinitesimal version of our expectation. So, what I have to do is just take x – x 0 to be infinitesimal and expand this.

$$O(t) = e^{\frac{i}{\hbar}H(t-t_0)}O(t_0)e^{\frac{-i}{\hbar}H(t-t_0)} \tag{11}$$

Check

$$\frac{dO}{dt} = \frac{i}{\hbar}H\,O(t) - \frac{i}{\hbar}O(t)H \tag{12}$$

$$= \frac{i}{\hbar}[H, O] \tag{13}$$

$$-i\hbar[O(t+\Delta t) - O(t)] = [H.O]\Delta t \tag{14}$$

H is the generator of time translation

$$O(t+\Delta t) - O(t) = i[H, O]\Delta t \tag{15}$$

$$\tag{16}$$

And let us see what we get. So, I get phi, I can work with so x is this so you have looking at above thing you have e to the i p dot epsilon is the same thing which I wrote, but now x – x 0 I have written epsilon and instead of mu's I have put a dot and you have phi x 0 where x 0 is this e to the – i p dot epsilon. When I expand so I get 1 – i p dot epsilon phi x 0 1 – i p dot epsilon so this is phi of x 0 + i and you get the commutator.

An order epsilon square term; which we will not worry about. So, we say that phi x – phi x 0 that is the translation that has happened is generated by the operator p mu. So p mu this momentum operator is the generator of space time translations. So, we say that we say that p mu generates space time translations or it is the generator of space time translations. Now of course if you put mu = so you put epsilon 0 to be non zero and all others epsilon to be 0.

Then you get the p 0 which is Hamiltonian and you recover this equation this one and similarly now we can put epsilon 0 to be 0 and look only at space translations then you will get the following. So, if you are looking at epsilon 0 to be 0 and all other epsilons to be non zero then you get phi of x 0 is fixed and here I have x 1 + epsilon I can put epsilon 1 x 2 + epsilon 2 x 3 + epsilon 3 – time is again kept at the same value what that would be?

That would be so I have just taken to the left taken the difference and I have i P phi x 0 and so I can put a dot here. This is correct except for a sign. So, here you have P mu epsilon mu so you have P is this your epsilon is so P dot epsilon will have a negative sign for the space bar so just a minus sign here. So, if you look at this; this is the difference and if you divide by the change you get the following.

You get divergence of phi dot epsilon I am doing a Taylor expansion here = - i P phi x 0 dot epsilon and which tells you that P phi x 0 this commutator – i is gradient of phi. So, you see that this the P generates space translation so we say that this momentum generates space translations. Now this was based on guess work that the transformation of the field under space time translation would be of this form which was a good guess, but now we are going to prove it. So, we will appeal to the Noether's theorem which we have used and which we have studied. So, recall that for infinitesimal transformations for whatever symmetry we are looking at your transform field be wrote like this. So, epsilon is the small parameter and the change we wrote as this. So, this is the part which is the change in the field and we are considering these actions for which we can write down an explicit expression we were looking at the theories for which we have only single derivatives of fields.

And we had found that the conserved current J mu we can write in the following manner. The Lagrangian density and you have to take the derivative with respect to del mu phi a and I should sum over all the fields and this has psi a and there was a – K mu term and Q the charge I get by looking integral d cube x of J 0 which is the following del phi dots I have put mu = 0 so that becomes a del phi dot psi a – k 0.

That is the conserved charge Q corresponding to this symmetry. Now recall that del L over del phi dot a is this is pi of x where pi is the momentum density which says that I can write Q as d cube x summation over a pi a psi a – K 0 and my experience shows that I should be constructing quantities like this. So, I take the conserved charge and look at the commutation relation with phi.

So, this is what I am going to do so I will take Q and look at the commutation relation with respect to phi. Let me write it like this so Q that is the commutation relation I want to make, but then I remember that I know see the Q is made out of psi and pi and fields. So, I have to use commutation relations, but I know commutation relations at equal ranks. So, it is useful to have Q at time t here because here I am writing at time t.

And it does not matter because Q is time independent we have shown that d Q over d t. So, you can choose whatever time you like and this is useful because then I can use equal time commutation relations. So, I have integral d cube x prime so I use a dummy variable here summation over a pi a of t x prime. So, this time is same as here then I have psi a of t x and – K 0 of t x prime.

So, that is what I get. I forgot to make the commutator sorry just a second so I have to take a commutator phi x t here was a prime there was a prime here and you have this commutator and you have this additional term with K 0 so you have – K 0 and t x prim e phi x t. You have one more commutator here. So, the summation is for this one this is the commutator and this closes here.

$$\phi(\vec{x},t) = e^{iH(t-t_0)}\,\phi(\vec{x},t_0)\,e^{-iH(t-t_0)} \tag{17}$$
$$P^\mu = (H,\vec{P}) \quad , \quad \epsilon=(\epsilon^0,\vec{\epsilon}) \tag{18}$$
$$\phi(\vec{x},t) = e^{ip^\mu(x-x_0)_\mu}\,\phi(\vec{x}_0,t)\,e^{-ip^\mu(x-x_0)_\mu} \tag{19}$$

$$\phi(x) = e^{ip\cdot\epsilon}\phi(x_0)e^{-ip\cdot\epsilon} \tag{20}$$
$$= (1+ip\cdot\epsilon)\,\phi(x_0)\,(1-ip\cdot\epsilon) \tag{21}$$
$$= \phi(x_0)\,+i[P^\mu,\phi(x_0)]\epsilon_\mu+\mathcal{O}(\epsilon^2) \tag{22}$$

So, that is what we have in general, but if I am looking at transformation such as space translations rotations and other internal symmetries in those cases K 0 will be 0 as you can verify

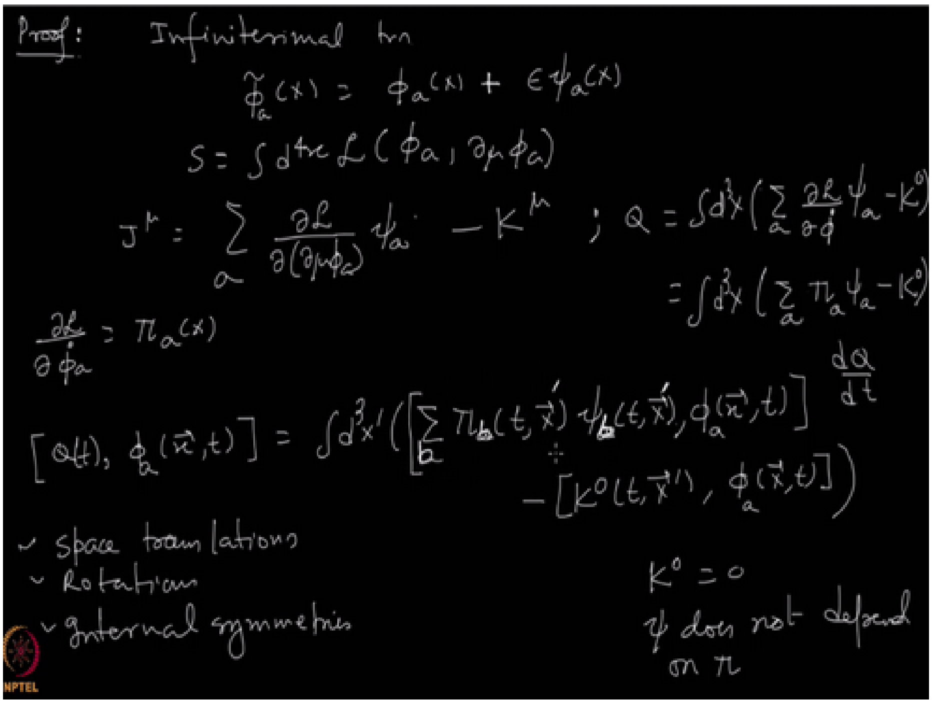


Figure 5: Refer Slide Time: 26:58

and the psi a which is the change in the field will not involve pi's. So, psi a will be independent of pi's. So, in these cases the result that I am going to proof I am trying to proof becomes easier because then this term is not there.

And here I do not have to worry that psi may have pi and I just have one commutation relation between pi and this phi and that makes our life easy, but the result which we are going to derive is true in general, but we are going to work in fact in a easier context. So, what do I get so I will leave this term out because I am saying that I am looking at these ones and then all I have to do is use this work out this commutation relation.

So, the psi because I am saying it is made up of only phi's that one will get out and sit on the right and I will have a commutation relation between pi and phi and that you know it gives delta cube and that I can integrate easily. So, let me do that. So, Q of t what did I write here I should have written phi of a and here you can make the variable the dummy variable as b it will be less.

We say that P^μ generates space time translation

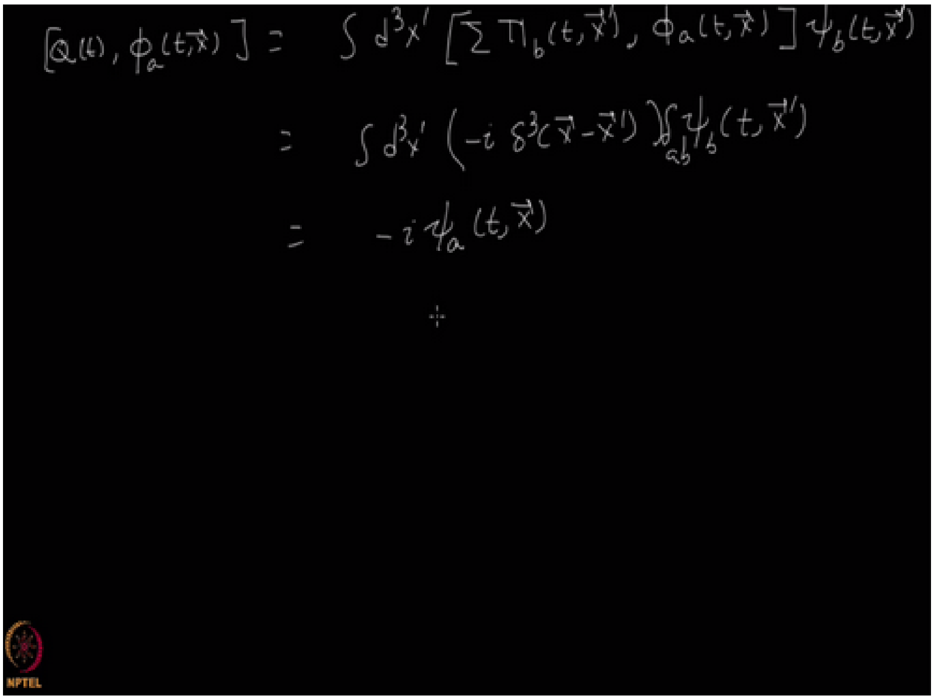
$$\epsilon = (\epsilon^0, \vec{\epsilon}) \tag{23}$$

$$\phi(x^0, x^1+\epsilon^1, x^2+\epsilon^2, x^3+\epsilon^3) - \phi(x^0, x^1, x^2, x^3) = -i[\vec{P}, \phi(x_0)]\cdot\vec{\epsilon} \tag{24}$$

$$\vec{\nabla}\phi\cdot\vec{\epsilon} = -i[\vec{P}, \phi(x_0)]\cdot\vec{\epsilon} \tag{25}$$

$$-i[\vec{P}, \phi(x_0)]\cdot\vec{\epsilon} = \vec{\nabla}\dot{\phi} \tag{26}$$

Momentum is the generator of space translation



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Proof: Infinitesimal transformation

$$\tilde{\phi}_a(x) = \phi_a(x) + \epsilon\psi_a(x) \tag{27}$$
$$S = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu\phi_a) \tag{28}$$

$$J^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\psi_a - K^\mu \tag{29}$$
$$Q = \int d^3x \left(\sum_a \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}\psi_a - K^0\right) \tag{30}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a} = \Pi_a(x) \tag{31}$$

$$Q = \int d^3x \left(\sum_a \Pi_a\psi_a - K^0\right) \tag{32}$$

$$[Q(t), \phi_a(\vec{x}, t)] = \int d^3x' \left[\sum_b \Pi_b\psi_b, \phi_a(\vec{x'}, t)\right] - \left[K_i(\vec{x}, t), \phi_a(\vec{x'}, t))\right] \tag{33}$$

Space translation, rotation, internal symmetry, $k^0 = 0$, ψ does not depend on Π

So, we have Q t phi a t x commutator I am just writing what I had in the previous slide d cube x prime and then have summation over pi of b t x prime and then you have phi a t x and as

I said the psi will come out this will commutate with a phi we have assumed the psi depends on phi only and this one I used the commutation relation and the commutation relation is that this object is following d cube x prime – i delta cube x – x prime time psi b of t x prime I am missing a prime every time.

And this gives you when you do the integral over x prime it gives you – i psi there is a delta ab I should put because this is b, this is a so I have a delta ab so psi a t of t and x. So, this is the general relation that I have proved and you see the change in phi was psi. The change in phi was psi of course there is an epsilon, but the coefficient of epsilon is psi and we see that psi or the change in field is being generated by the generator Q or the charge Q.

$$[Q(t), \phi_a(\vec{x}, t)] = \int d^3x' \Big[\sum_b \Pi_b(\vec{x}', t), \phi_a(\vec{x}', t) \Big] \psi_b(\vec{x}', t) \quad (34)$$

$$= \int d^3x' \Big(-i\delta^3(\vec{x} - \vec{x}')\delta_{ab}\psi_b(\vec{x}', t) \Big) \quad (35)$$

$$= -i\psi_a(\vec{x}', t) \quad (36)$$

So, Q is called the generator of symmetry transformations and this is exactly what you have seen for time and space translations even though the proof I have given does not apply for time translations, but the result is general as you see that this result coincides with what you have here this one. So, this is a nice result that we have proved I will stop the video here and we will continue discussion in the next video.