# Introduction to Quantum Field Theory

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## Lecture 32:Conserved Currents and Charges due to Lorentz Symmetry

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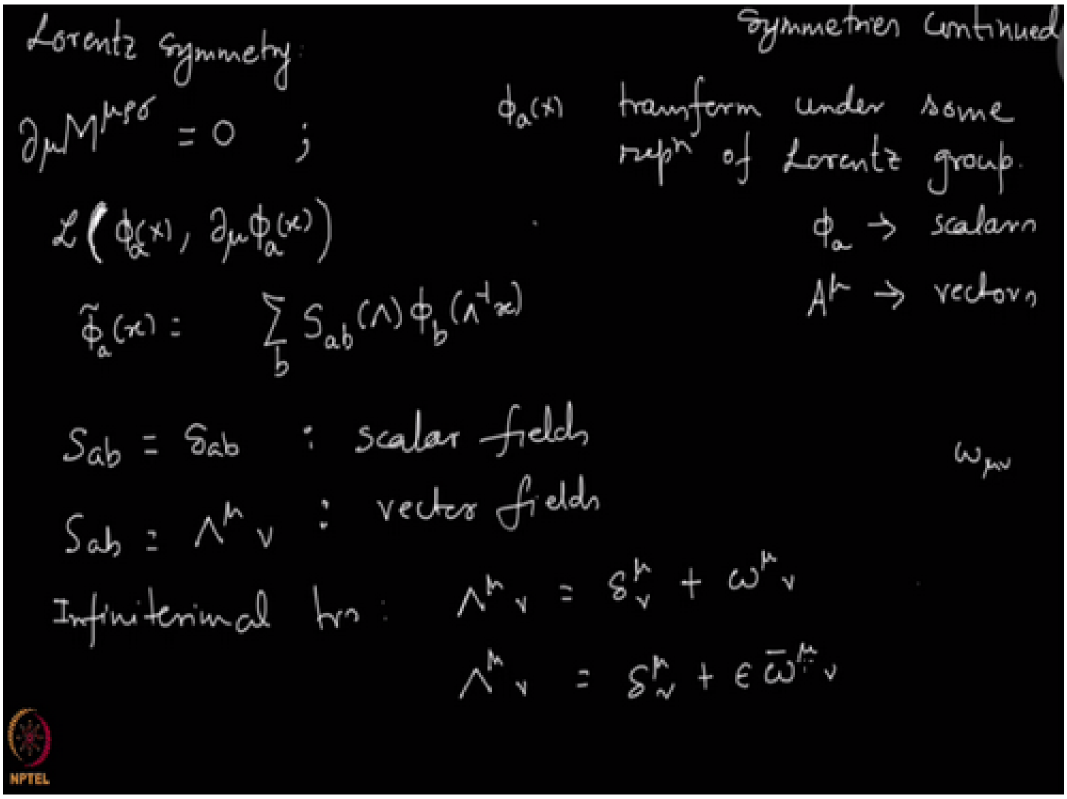


Figure 1: Refer Slide Time: 00:13

**Lorentz symmtery**

Okay let us continue our discussion on symmetries and today we plan to talk about conserved charges and conserved currents arising from Lorentz symmetry and in fact last time we have built up our expectation on what we should get from Lorentz symmetry and we saw that we are expecting a conserved current which we denoted last time by M mu rho sigma where this mu index is the one which contracts with del mu to give you 0.

So, expecting a conserved current M mu rho sigma an object with three indices and in fact we further argued that it should have the following form. In fact, I can go back and show you here this one. So, we expect something like that so let us proceed and see what our actual calculation gives. So, even though this is a course on scalar fields I will keep the discussion general in this video.

And talk about theory which is not necessarily scalar and could contain fields which transform under some representation of the Lorentz group and it will be useful because you will see that you get pieces which are present only if the fields are not scalar and it will be good to see the full results instead of having just seeing the special case of scalar fields. So, anyhow let us begin.

So, I will start with a theory which is described by Lagrangian density this curly L and there are several fields which are labelled by phi a and I am interested in those Lagrangians which depend on the fields and their first derivatives and I am interested in this because you saw earlier that in this case I could write an explicit expression for general expression for conserved current.

The formula which we wrote earlier I will write again, but that is valid when this is true. So, you may have several fields so a runs from 1 to whatever and here the field phi a of x they transform under some representation of Lorentz group. So, phi could be a scalar or phi a could let me write phi a they could be scalars so you may one scalar so in that case index a is irrelevant.

Did I check whether sound is getting recorded yeah it is getting recorded and if you have n number of scalars that is also fine. So, each field will be a scalar and they will not mix under Lorentz transformations or you could have the phi's could represent A mu's which are vectors and so forth that is what phi a's are and so I should specify the transformation property of phi's. So, if you have a particular representation.

Then the transformation property is already specified. So, let us say the fields phi a transform like this. So, it will transform under some representation and that representation matrix are denoted by S we have already talked about this in one of the lectures earlier and there is a summation over b. This part is not so important here all we are saying is if I take the fields which are located at lambda inverse x.

Then under Lorentz transformation they give you fields which are located at x. This part S is telling you how the different components mix under the Lorentz transformation for example how A 0, A 1, A 2 and A 3 mix under Lorentz transformation. So, that piece of information is encoded in S the matrix S whose elements are S ab and if you were to have this differently like phi tilde of x is S times phi b of lambda x.

$$\partial_\mu M^{\mu\rho\sigma} = 0 \tag{1}$$

$$\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) \tag{2}$$

$\phi_a(x)$ transforms under some representation of Lorentz group

$$\phi_a \to \text{Scalar} \tag{3}$$
$$A^\mu \to \text{Vector} \tag{4}$$

Then your S would be related to my S by simple transformation that which you can figure out easily, but beyond that there is no mystery about this. So, what is S ab if your phi's are scalars so let us say if you have n scalars phi 1, phi 2, phi 3 so and so forth to phi n what would S ab. So, in that case S ab would be just delta ab right because there is no mixing between different phi's in that case.

And this is true for scalar field and if you are looking at vector fields then your S ab would be lambda mu nu because in that case your fields will be labeled as a nu's so your S ab would also carry the indices mu and nu corresponding to these indices and you know that four vectors transformed with this matrix mu nu that is the definition of a four vector. So, in that case your S ab would be lambda mu nu so these 4 cross 4 matrices.

And similarly if it was some other representation you will have something else here. For example we have already discussed about a field in very brief which is a two dimensional representation and you saw that here you will have two dimensional matrices constructed out of Pauli matrices so that is another example.

Anyhow and as always we are interested in infinitesimal transformation because we are going to do this Noether's theorem. So, I am interested in infinitesimal transformation. So, in that case I can write lambda mu nu okay I am interested in mu nu not so much because of this reason because of this argument. So, lambda mu nu as we have been writing before for infinitesimal transformation it is identity meaning no transformation plus an infinitesimal amount of transformation given by this omega mu nu where you recall omega mu nu is an anti-symmetric matrix.

So, this is an anti-symmetric matrix, but what I want to do will help me later in the derivation. I want to slightly modify the notation and write this as from this omega mu nu I want to pull out an epsilon to make it explicit that I am looking at small transformations. Earlier I used to call omega small, but this time I want to pull out an omega epsilon explicitly and define omega as epsilon times omega bar. So, epsilon is infinitesimal this is not infinitesimal this is finite and this is of course it has to be anti-symmetric and remember that omega mu nu and omega bar mu nu they are constant matrices. So, the entries in omega mu nu do not depend on space time parameters these are some constants because if you choose a Lorentz transformation that is a rotation by some amount then omega mu nu tells you I mean it encodes that rotation.

So, because you are talking about a fix rotation omega mu nu will contain some fix numbers similarly omega bar mu nu or if you are doing a boost by some amount omega mu nu elements will correspond to that boost which will be some fixed number and that does not depend on x. So, these are constants. So, that is good and maybe on the next page probably.

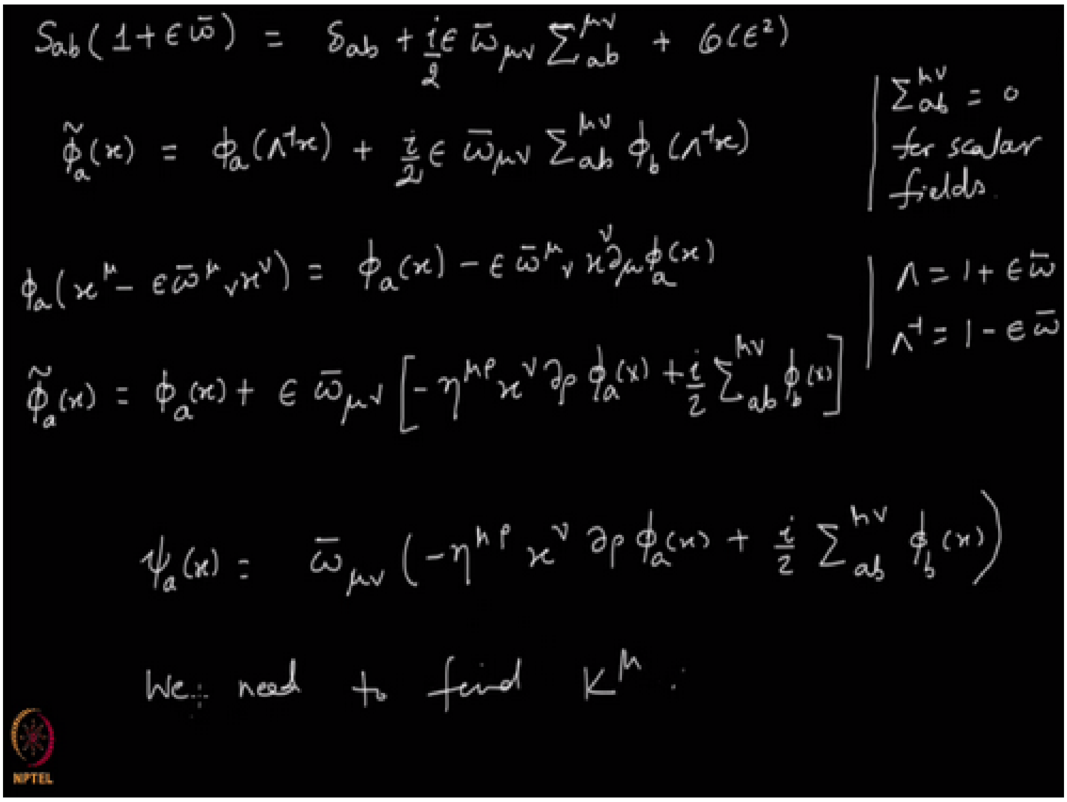


Figure 2: Refer Slide Time: 11:18

$$\tilde{\phi}_a(x) = \sum_b S_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \tag{5}$$

$$\begin{aligned} S_{ab} &= \delta_{ab}, \quad \text{Scalar fields} && (6)\\ S_{ab} &= \Lambda^\mu{}_\nu, \quad \text{Vector fields} && (7) \end{aligned}$$

Infinitesimal transformation: $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, where $\omega^\mu{}_\nu$ is anti-symmteric matrix

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu_\nu + \epsilon\,\bar{\omega}^\mu{}_\nu && (8)\\ S_{ab}(1+\epsilon\,\bar{\omega}) &= \delta_{ab} + \frac{i}{2}\epsilon\,\bar{\omega}^\mu{}_\nu\Sigma^{\mu\nu}_{ab} + \mathcal{O}(\omega^2) && (9)\\ \tilde{\phi}_a(x) &= \phi_a(\Lambda^{-1}x) + \frac{i}{2}\epsilon\,\bar{\omega}^\mu{}_\nu\,\Sigma^{\mu\nu}_{ab}\,\phi_b(\Lambda^{-1}x) && (10)\\ \Sigma^{\mu\nu}_{ab} &= 0 \quad \text{for scalar fields} && (11) \end{aligned}$$

So I want to write the infinitesimal version of S ab matrix. So, S ab of lambda now I am writing as 1 + epsilon omega bar so this would be of course it will have delta ab because if you are not doing any transformation if epsilon is 0 there is the components are not mixing at all they remain whatever they are. So, it has to be delta ab the first term and the changes will be proportional to epsilon.

And because epsilon and omega bar come together as a product this will also have omega bar multiplying it which comes with these indices and then whatever mixing happens for that I define this matrix sigma. This is not very standard, but several authors use sigma and some authors use S, but nevertheless so mu nu I have to contract here of course and then the ab index on the left and this one that also has to be carried by the sigma.

Now see all the indices are fine, but I will also multiply a factor of i over 2. If you do not multiply then your definition of sigma is different from mine, but because I have used i over 2 earlier in one of the lectures I will keep i over 2 here and of course there will be order epsilon square terms. So, I have written down the matrix S ab then I should be able to write down how fields will transform under Lorentz transformations given this matrix and that is easy to write.

So, I am asking how a field will transform under Lorentz transformation remember the tilde was I always used to refer the symmetry transformation. So, what would that be? That is phi a of lambda inverse x that is the first term coming from here. So, what I am doing is I am substituting the expression of S that I have written the infinitesimal version of S in this expression.

So, first term is delta ab so that multiplies phi b and makes it phi a lambda inverse x that is the first term and the second term is coming from here and this is what you have seen earlier maybe I can try to show you. I think it was on page number 152. See here this is what we had written earlier the same discussion we have had earlier and that is what I am using now. Now, if you had a scalar field this sigma would be 0 because there will be no mixing there is nothing to mix or no mixing between different fields.

So, this will just be this piece so let me write for scalar fields, but here I have lambda inverse x not lambda x. So, I should find out what lambda inverse is and put in here so that I have to expand around x and that is easy because if your lambda is 1 + epsilon omega then your lambda inverse would be 1 – epsilon omega because when you multiply these two this when you multiply lambda times lambda inverse you expect 1.

And that you see here because all order epsilon term will cancel and what you get as order epsilon square, but we have ignored order epsilon square in both of these. So, we cannot say about order epsilon square here and we ensure that order epsilon term is 0. So, up to order epsilon terms lambda inverse would be 1 – epsilon omega bar. So, that is what lambda inverse would be.

So, I can write phi of let us look at this phi a lambda inverse x would be x mu – epsilon omega bar mu nu x nu and that is the transformed field. So, if you put epsilon = 0 you are at x mu. So, it is easy I do a Taylor expansion so phi a of x here I have just put the indices I could have dropped the indices also, but to make it explicit I did that so minus the change in the derivative of the field at point x times the change and plus times the change and the change is minus epsilon omega bar mu nu x nu so that is what I put. So minus I bring back epsilon omega bar mu nu x nu.

Let us see whether everything is correct so nu is contracted, mu is contracted so there is no known index here so which is a same here, but I have missed the index a so I should have an a here. So, that is how the field will transform phi a will I mean only this part will transform. Sorry bad choice of words what I mean to say is this phi a for infinitesimal transformation can be written like this.

I am sorry for not using the right words now I will find out what phi tilde of x is. So, phi tilde of x is this piece combining together with this one and that is easy let me see let me write down the answer which you can easily derive there is not much work in that and then tell you one of the steps which will be needed. So, I am just collecting the order epsilon terms so this piece gives it this and this so phi a of x and these two order epsilon terms I collect.

So, plus epsilon omega bar mu nu and because here your mu and nu are one up one down to bring this one down you will use the matrix tensor and that gives you minus eta mu rho I have taken the minus sign inside x nu del rho of phi of x plus i over 2 sigma mu nu ab phi b of x. This is let me check whether expression is correct x nu del rho phi of x + i over 2 sigma mu nu ab ph b of x.

So, that is how my field transforms. So, what are the things which I need to find the conserved current let us go back and check. So,, to find the conserved current I need to know psi i of x which is just the change in the fields that I have already figured out and second thing which I need to find out is k mu and what is k mu? K mu let me remind you I think it was on a previous one.

$$\Lambda = 1 + \epsilon\,\omega \tag{12}$$
$$\Lambda^{-1} = 1 - \epsilon\,\omega \tag{13}$$

$$\phi_a(x^\mu - \epsilon\,\bar{\omega}^\mu{}_\nu\, x^\nu) = \phi_a(x) - \epsilon\,\bar{\omega}^\mu{}_\nu\, x^\nu \partial_\mu \phi_a(x) \tag{14}$$
$$\tilde{\phi}_a(x) = \phi_a(x) + \epsilon\,\bar{\omega}^\mu{}_\nu \left( -\eta^{\mu\rho} x^\nu \partial_\rho \phi_a(x) + \frac{i}{2}\Sigma^{\mu\nu}_{ab}\,\phi_b(x)\right) \tag{15}$$
$$\psi_a(x) = \bar{\omega}^\mu{}_\nu \left( -\eta^{\mu\rho} x^\nu \partial_\rho \phi_a(x) + \frac{i}{2}\Sigma^{\mu\nu}_{ab}\,\phi_b(x)\right) \tag{16}$$

We need to find $K^\mu$

We are only interested in those theories for which $L$ is a Lorentz scalar

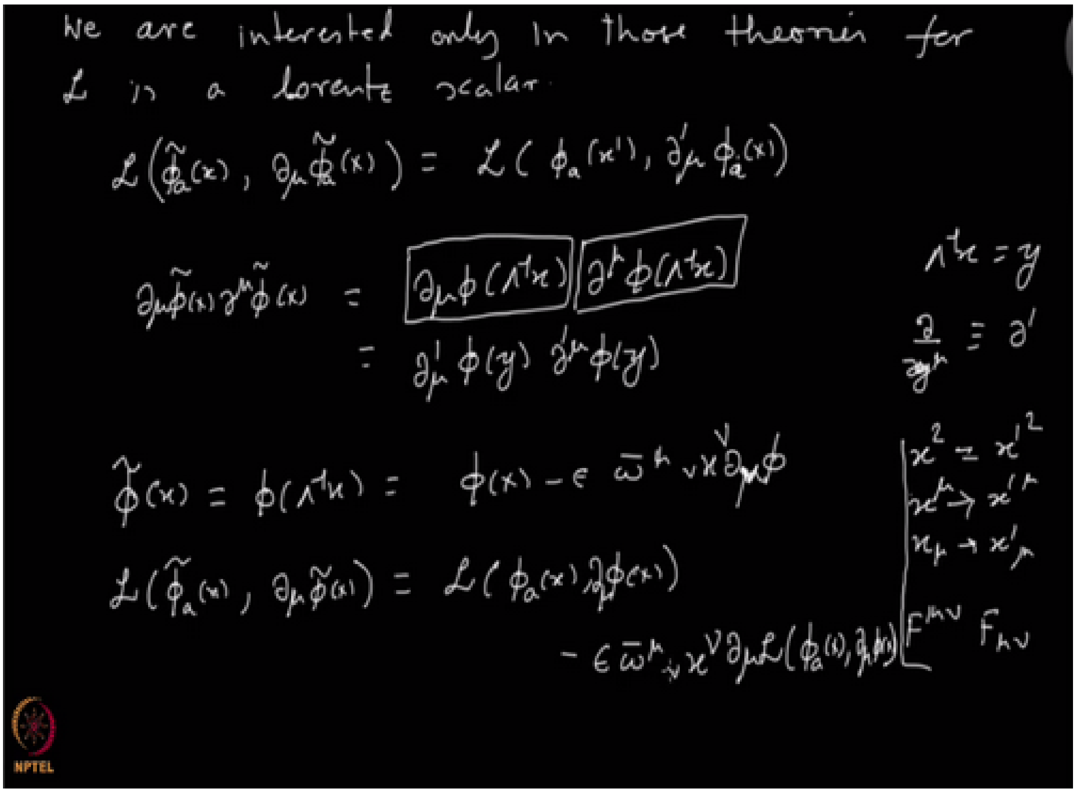


Figure 3: Refer Slide Time: 24:06

$$\begin{aligned}
\mathcal{L}\Big(\tilde{\phi}_a(x), \partial_\mu \tilde{\phi}_a(x)\Big) &= \mathcal{L}\Big(\phi_a(x'), \partial'_\mu \phi_a(x)\Big) && (17)\\
\partial_\mu \tilde{\phi}(x) \partial^\mu \tilde{\phi}(x) &= \partial_\mu \phi(\Lambda^{-1}x) \partial^\mu \phi(\Lambda^{-1}x) && (18)
\end{aligned}$$

$$\begin{aligned}
\Lambda^{-1}x &= y && (19)\\
\frac{\partial}{\partial y^\mu} &= \partial' && (20)
\end{aligned}$$

$$\partial_\mu \tilde{\phi}(x) \partial^\mu \tilde{\phi}(x) = \partial'_\mu \phi(x) \partial'^\mu \phi(\Lambda^{-1}x) \quad (21)$$

$$x^2 = x'^2, \quad x^\mu \to x'^\mu, \quad x_\mu \to x'_\mu \quad (22)$$

$$\begin{aligned}
\tilde{\phi}(x) &= \phi(\Lambda^{-1}x) = \phi(x) - \epsilon\, \bar{\omega}^\mu{}_\nu\, \partial_\nu \phi && (23)\\
\mathcal{L}\Big(\tilde{\phi}_a(x), \partial_\mu \tilde{\phi}_a(x)\Big) &= \mathcal{L}\Big(\phi_a(x), \partial_\mu \phi_a(x)\Big) - \epsilon\, \bar{\omega}^\mu{}_\nu\, x^\nu \partial_\mu \mathcal{L}\Big(\phi_a(x), \partial_\mu \phi_a(x)\Big) && (24)
\end{aligned}$$

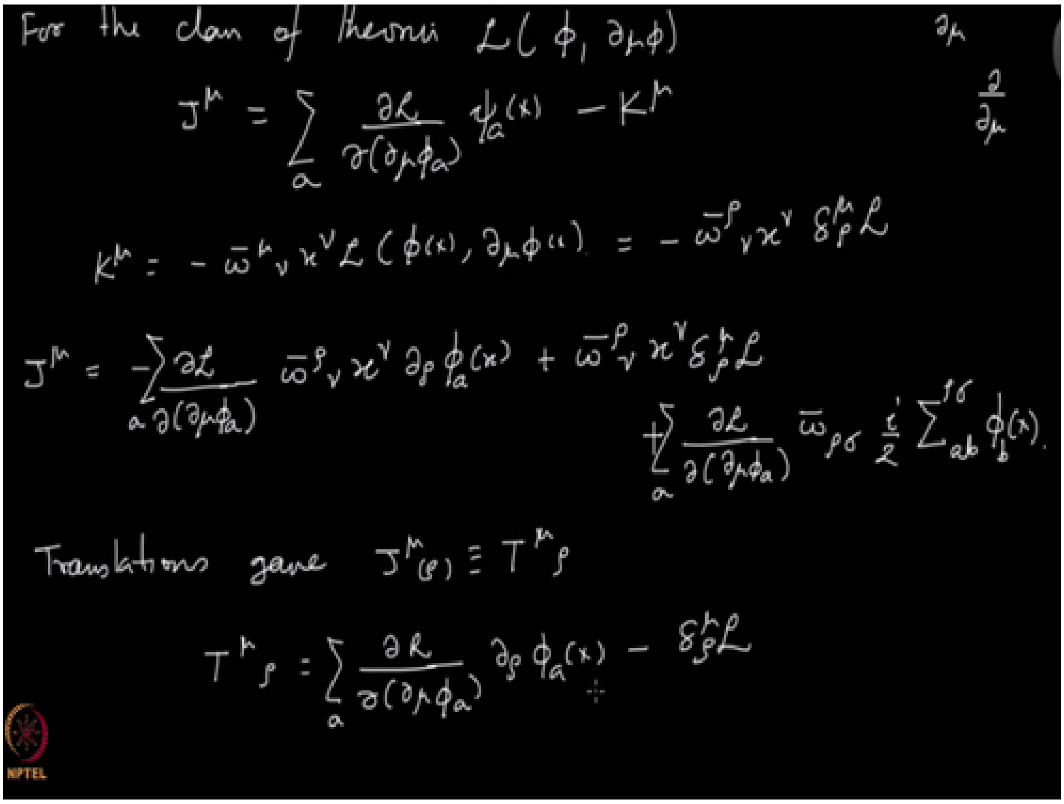


Figure 4: Refer Slide Time: 34:37

For the class of theories, $\mathcal{L}\bigg(\phi(x), \partial_\mu\phi(x)\bigg)$

K mu is this piece so Lagrangian density with the transformed fields will be equal to the original Lagrangian density plus some total derivative term and this k mu which enters here. So, that is the k mu which I want to find out for my present case and our J mu is this the signs you remember we have chosen these sign so that energy comes out to be correct. So, let us go.

So, from here I can write down the psi the change in the field which is this piece this entire piece here, but now let us find out the change in the Lagrangian density. So, we will come back to this page so let me write here itself so psi a of x is omega bar mu nu – eta mu rho x nu del rho phi a of x + i over 2 sigma mu nu ab phi b of x. Now we need to find K mu okay let us do that.

Now all the theories that we have constructed or looked at and that we will construct or we will construct later in your academic career they would all be Lorentz invariant meaning I mean if you are writing down a Lagrangian density you will construct the Lagrangian density such that it is a Lorentz scalar that is the requirement you have. Let me write this down so we are interested in only those theories invest for which a Lagrangian density is a Lorentz scalar and by that I mean that if you take the Lagrangian density and transform the fields by taking the fields to be Lorentz transform fields the way we did earlier which is psi tilde a of x would be this one. If you do this then what you get should be expressible in this form where the derivative with respect to the primary coordinate.

So, maybe I will give you example and then it will be clear and also I will tell you what exactly it means. So, remember the Lagrangian density for real scalar fields where you had apart from some factors of half and other things you had del mu phi del mu phi and let us look at the transformed version of this. So, the transformed version is del mu phi lambda inverse x and del mu phi lambda inverse x.

Sorry this mu should be up that is what it means, but now if you take lambda inverse x and define it to be y and you define the derivative with respect to y as delta let me write del over del y mu let us say we call it del prime and then you can check that this is phi of x del mu prime sorry phi of y I have put lambda inverse x as y and del mu prime phi of y and this is happening because it is a scalar.

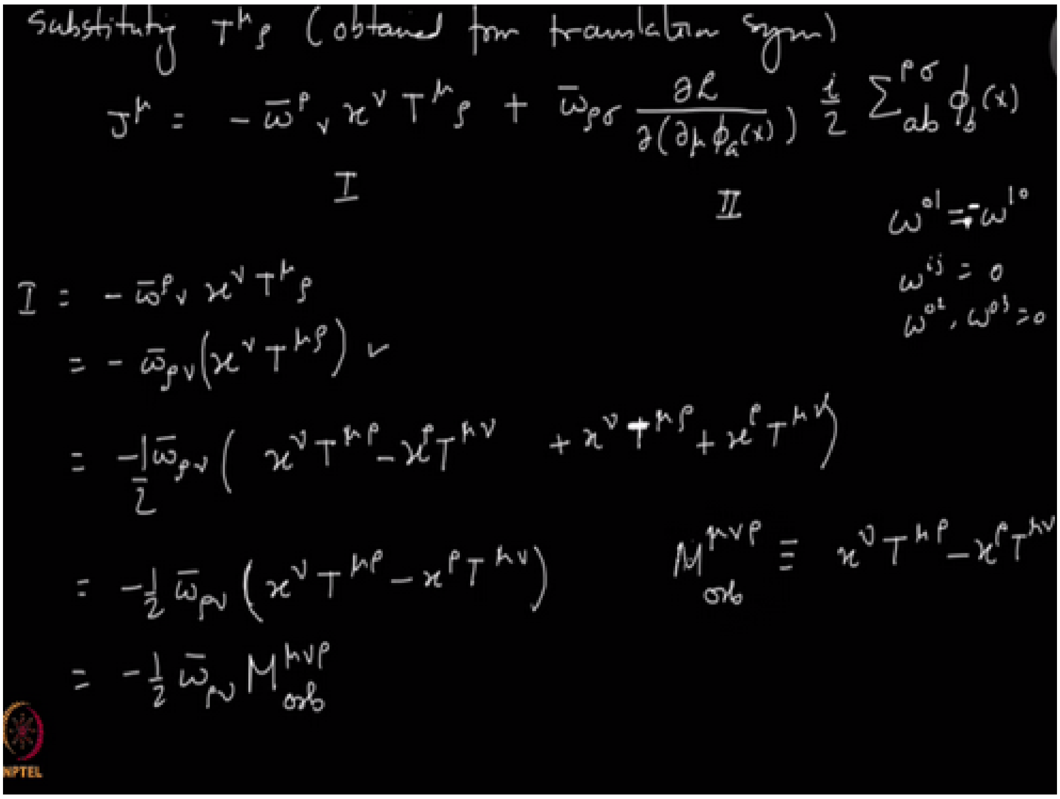


Figure 5: Refer Slide Time: 46:14

You see it is easy to understand why this will happen in all the theories which you are interested in. So, if you have written down your Lagrangian such that all the indices are contracted no matter what the Lagrangian contains what fields it contains and how they transform and how many derivatives there are if all the indices are properly contracted then this will happen this will always happen that you will be able to write it as the same Lagrangian at y with derivatives with respect to y.

And this is going to happen because look at this piece there is an index mu downwards so it says that this object for the timing let us forget about lambda inverse x they are the trivial part of the argument they do not matter so much. So, this index mu tells you that how del mu phi transforms and it tells you that it transforms like a covariant vector. Now this one as a mu index which says that del mu phi transforms like a object which is contravariant vector.

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \psi_i(x) - K^\mu \tag{25}$$

Using

$$\bar{\omega}^\mu{}_\nu \, \delta^\nu_\mu = \bar{\omega}^{\mu\nu} \eta_{\nu\mu} \tag{26}$$

$$K^{\mu} = -\bar{\omega}^{\mu}{}_{\nu}\, x^{\nu} \mathcal{L}\Big(\phi(x), \partial_{\mu}\phi(x)\Big) \tag{27}$$

$$K^{\mu} = -\bar{\omega}^{\rho}{}_{\nu}\, x^{\nu}\, \delta^{\mu}_{\rho} \tag{28}$$

$$J^{\mu} = -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_a)}\partial_{\rho}\phi_a(x) - \delta^{\mu}_{\rho} L \tag{29}$$

Substituting $T^{\mu}{}_{\rho}$ (obtained from translation symmetry)

$$J^{\mu} = -\bar{\omega}^{\rho}{}_{\nu}\, x^{\nu}\, T^{\mu}{}_{\rho} + \bar{\omega}_{\rho\sigma}\, \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_a)}\, \frac{i}{2}\, \Sigma^{\rho\sigma}_{ab}\, \phi_b(x) \tag{30}$$

And these transformation laws for this one and that one are such that they are inverses of each other and that it why this turns out to be a scalar because the way this guy transforms cancels with the way this guy transforms and that is why it forms a Lorentz scalar. Let me remind you like if you have x square this x square is invariant under Lorentz transformations why because when you do a Lorentz transformation a boost or whatever x goes to x prime.

X mu goes to x prime mu, x mu goes to x prime mu, but x square remains unchanged meaning x square still is x prime square and this is only possible if the transformation property of mu with up cancels against the transformation property of mu down and now you can understand if you are writing a Lagrangian which has fields which themselves carry some indices.

Even then it is going to work because if you have arranged for the contraction of all the indices then you have really arranged for the cancellation of effects of Lorentz transformation just like here. Let me give you another example if you have let us say F mu nu F mu nu where these are F mu nu is del mu a nu – del nu a mu. You see that this will also be a Lorentz scale because everything is contracted.

This object will transform under Lorentz transformation in some manner, but this object is going to transform just in the opposite manner so that the effect of transformation is cancelled and that is why it is a Lorentz scalar. So, this is always going to work out or try to play around with some Lagrangians and try to see if you agree with what I am saying. So, if this is the case then all the theories which I am going to construct would be of this form.

So, this will satisfy this property and I am going to arrange this because I want my theories to contain Lorentz symmetry. So, this is the precise statement of saying that my Lagrangians will be Lorentz scalars. Good then something becomes very easy to understand now then this object L the Lagrangian density because it is a Lorentz scalar in the sense then it is going to transform exactly in the manner in which phi transforms a scalar field transforms.

And how does a scalar field transforms? A scalar field transforms like this phi tilde of x which is the transform field which is phi lambda of inverse x is just phi of x – epsilon omega bar mu nu del nu phi that is what we wrote previously also then if this is how a scalar transforms and if I am arguing that Lagrangian is a scalar then of course Lagrangian density is a scalar then this Lagrangian density is also going to transform in the same manner.

So, I will look at the Lagrangian density at the transform fields and how should it transform it should transform like this. So, instead of phi I can put L, but now L should be a function of here phi x and the change in the Lagrangian would be something everything is fine so it should be sorry there is a mistake here that is why I was thinking something is going wrong. So, I have missed the factor of x I can show you here omega this mu contracts here and you have x nu this x nu I was I had missed and instead I have written (()) (33:45) by mistake.

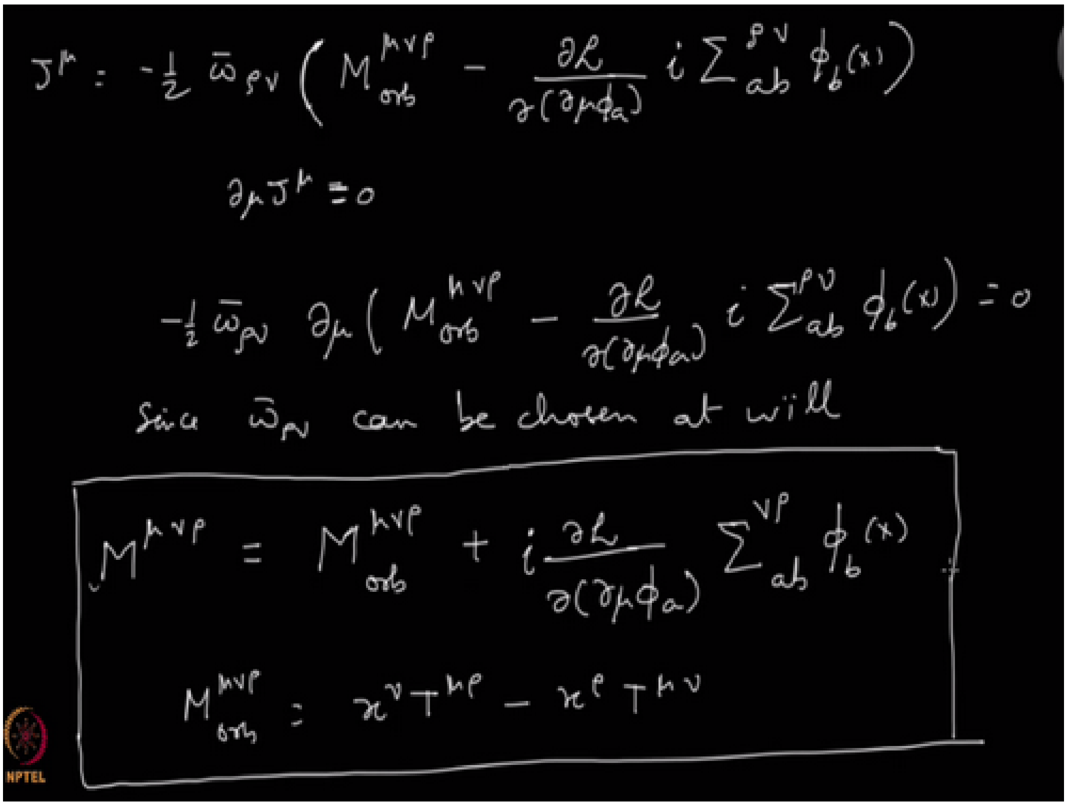


Figure 6: Refer Slide Time: 56:14

Now it is all correct. So, what should I get here I should get del mu of L with again the fields evaluated at x and the derivatives evaluated at x and here minus epsilon the same thing here omega bar mu nu and x nu. Nu is contracted mu is contracted there is no index all good. So, that is how the Lagrangian density would change which means I have the k mu now that is the K mu for us. So, let us carefully extract the K mu let me write it down for ease. So, for the class of theories where Lagrangian density is a function of phi and first derivative we have written down explicit form for the conserved current which was this. I can make the summation explicit here minus K mu and so let me write down psi a I have written down already and k mu is the following. So, we can see that psi I wrote down K mu is minus omega bar mu nu x nu.

Let us go back so you have del mu of L so actually I can take the del mu outside I can pull it out and put it before x nu and I can do so because this will generate a term which will give you 0, 1 multiplied with the omega bar mu nu. All you have to use it omega mu nu this is anti-symmetric under mu nu interchange and this will generate a piece which will disappear when you contract with something anti-symmetric.

And that you can see easily why it will happen. Let me show you what I can do is take this piece and write as omega bar mu I think this not even a need to show. So, here this is a derivative with respect to x mu. And when you pull it out here it will generate a derivative term acting on x nu so it will give you delta nu mu and this will be contracted against omega bar mu nu which is same as omega bar mu nu and eta nu mu.

This is a symmetric tensor this is completely anti-symmetric and that is why that extra contribution will get kill. So, I can pull it out here or in front of this because omega is a constant. So, then you get a minus epsilon del mu times K mu and whatever remains is the K mu. So, then you can see that K mu would be this. So, now you understand why this is K mu because the derivative has been pulled out which I can also write as - omega bar rho nu x nu del mu rho L.

I have written it this way because I wanted the mu index not to be sitting on omega bar because omega bars are the parameters and I am going to pull them out eventually so that is the reason I want to free up this omega bar from the rho index. So, this is fine so what is our conserved current then? I just have to plug the expression of psi from here and K from here into this expression and then I get the conserved current.

So, let me write down the result so J mu = - and the minus sign is I think you can check I will not explain del L over del del mu phi a and omega bar rho nu x nu del rho phi a of x perfect then what + omega bar rho nu x nu del mu rho times Lagrangian density plus del L over del del mu phi a omega bar rho maybe I should write it down in the next line plus derivative of Lagrangian density with respect to del mu phi a.

$$\omega^{01} = -\omega^{10} \tag{31}$$
$$\omega^{ij} = 0 \tag{32}$$
$$\omega^{02}, \omega^{03} = 0 \tag{33}$$

First term in the expression

$$I = -\bar{\omega}^{\rho}{}_{\nu}\, x^{\nu}\, T^{\mu}{}_{\rho} \tag{34}$$
$$I = -\bar{\omega}_{\rho\nu}\, x^{\nu}\, T^{\mu\rho} \tag{35}$$
$$= -\frac{1}{2}\bar{\omega}_{\rho\nu}(x^{\nu}\, T^{\mu\rho} - x^{\rho}\, T^{\mu\nu} + x^{\rho}\, T^{\mu\nu}) \tag{36}$$
$$= -\frac{1}{2}\bar{\omega}_{\rho\nu}(x^{\nu}\, T^{\mu\rho} - x^{\rho}\, T^{\mu\nu}) \tag{37}$$

Then you have omega bar rho sigma i over 2 sigma rho sigma ab phi b of x. Let me check if I have written correctly let us check first in one check we can always do is look at whether all the indices are contracted properly and if they are it is highly likely that things are correct. Of course, factors of 2 can be wrong and the signs can be wrong, but one level of check would be there.

Left hand side has only mu free so I should have a mu up on the right hand side also so rho is contracted nu is contracted a is contracted because there is a sum over a I should write here maybe and then you have a del mu sitting in the denominator which is equivalent to having a mu index up. So, if you have a del mu this means it transforms like a covariant vector, but if you have del over del mu this object transforms like a contra variant vector.

So, this mu behaves as if it was on something as an up index so that is fine that matches with what you have here. Here mu is up which is good rho is contracted, nu is contracted so all other indices are contracted which is fine. Again the mu here behaves as a up index transforms like a up index that is good rho sigma all gone ab b so you have only a as free which is not good, but it is all okay because a is contracted here.

So, there is a summation over a implied here as well so I can leave that or I can write down. So, this is the current which we get. Before I do something more let us see whether we are happy with this and of course you will verify the expression it is easy just plug in, but we said that we expect 6 currents 3 for rotations and 3 for boost and what I have here is only 1. So, have I made any mistake or is it all fine.

In fact there is only one J mu I will put mu = 0 I get one charge it appears like I do not get 6 charges, but only 1. I will answer it maybe after a little while. So, now this these two terms that you see they appear very familiar to us and if you recall when we were looking at translations.

See right now we are looking at Lorentz transformations which most because we are connected to identity, we are doing infinitesimal transformations.

$$M^{\mu\nu\rho}_{orb} = x^{\nu}T^{\mu\rho} - x^{\rho}T^{\mu\nu} \tag{38}$$

We are basically looking at rotations and boost because that is the part which belongs to proper orthochronous transformations, but earlier we looked at translations which has nothing to do with Lorentz transformations and in that case we had found that let me write down translations gave us the following. So, we found that the conserved current which we were writing as J mu rho and the standard notation for which is T and we found that T mu rho was del L over del mu phi a del rho phi a of x.

So, a is contracted again there is a sum over a and mu is up and rho is down which matches here and also we had this term delta mu rho times L and this piece you are able to see here also that it is exactly the same except for this omega bar times x. So, if you pull this omega bar times x what you have here is exactly this piece of course the sign is interchanged that is what you have.

So, I can write J mu using T mu rho so let me do that then so I substitute this expression in here the expression of J mu so I get the following.

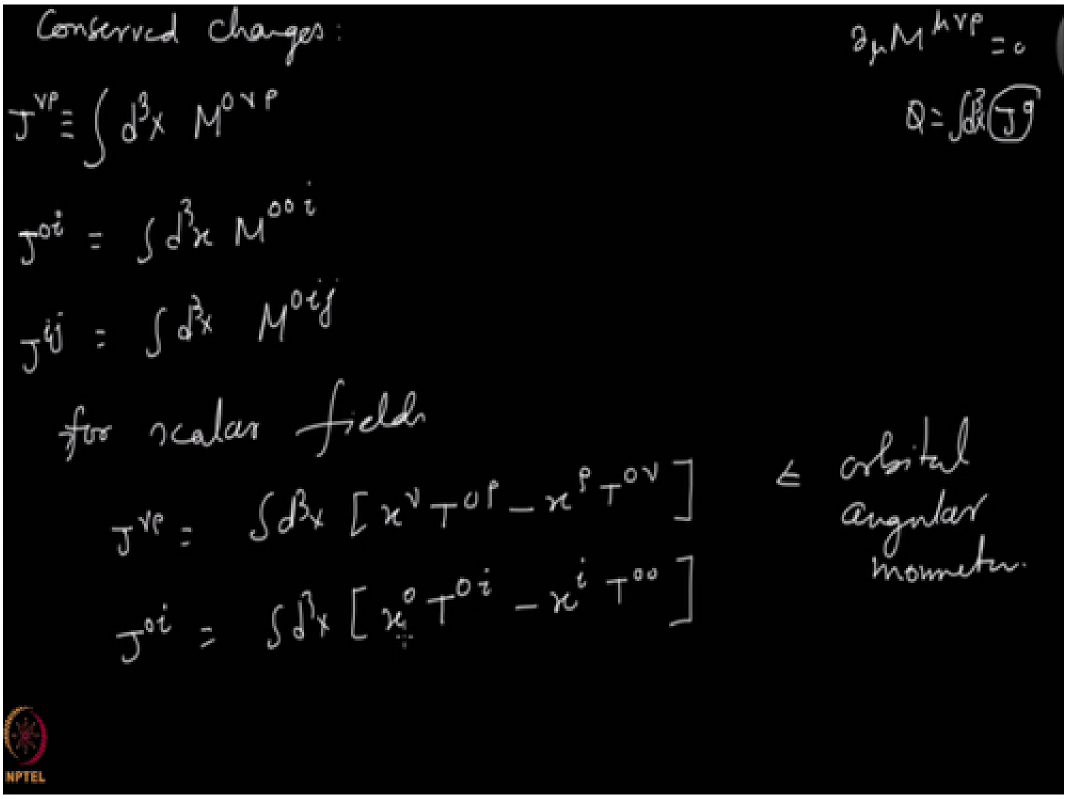


Figure 7: Refer Slide Time: 01:03:37

$$I = -\frac{1}{2}\bar{\omega}_{\rho\nu}M^{\mu\nu\sigma}_{orb} \tag{39}$$

$$J^{\mu} \quad = \quad -\frac{1}{2}\bar{\omega}_{\rho\nu}\left(M^{\mu\nu\rho}_{orb} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_a)}\, i\,\Sigma^{\rho\nu}_{ab}\,\phi_b(x)\right) \tag{40}$$

$$\partial_\mu J^\mu \;=\; 0 \tag{41}$$

So, substituting T mu rho translation symmetry what do I get? I get J mu = minus omega bar rho nu x nu T mu rho and then you had the other term starting this one. Remember this one comes because of your fields transforming under representations different from scalar representation. If it was just scalar representation this term would be 0 and you get only this piece.

So, equivalently only this piece, but because we are looking at general representations we get this term as well. I will leave summation over a implicit all the things which are contracted are summed over. So, let us write back to that question whether it is one current or 6 currents. You see the omega bars these are the constants which parameterized how much rotation or how much boost you have done.

Now, you may choose to do only rotation about one of the axis and do not do boost at all so you put all the other omegas to be 0 or you may do a boost along one axis and put all the parameters corresponding to rotations and boost along other directions to be 0 then you will get one J mu corresponding to that one. So, for example, if you choose to put omega 01 as non zero and of course omega 10 because they are related by symmetry.

So, this is actually equal to – of this if you put this as non-zero and all others to be 0. So, all omega i j to be 0 and all omega 02 and omega 03 to be 0 then you get a conserved current corresponding to this boost. Choosing a different let us say omega 02 as non zero and all other remaining to be 0 you get another conserved current corresponding to the boost along y direction and similarly for rotation.

So, you indeed have 6 conserved currents here, but they all are summed here everything has been summed together because of this summation. So, we are going to pull them out explicitly in a short while, but there is no issue with this. Let us do a little bit more on the first term. So, let me write the first term and the second term this entire thing is first term, this entire thing is second term.

So, in the first term minus omega bar rho nu x nu T mu rho – omega bar I have pulled down the index rho downwards then I will have to pull this upwards and if you are not already familiar with this pulling down or pulling up you can do it by putting the eta matrices explicitly. So, you lower it down using a eta you raise it up using another eta and then you discover that this is what you will get eventually.

$$J^\mu \;=\; -\frac{1}{2}\bar{\omega}_{\rho\nu}\partial_\mu\left(M^{\mu\nu\rho}_{orb} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\, i\,\Sigma^{\rho\nu}_{ab}\,\phi_b(x)\right) \tag{42}$$

Let us go back and check here. We were expecting a conserved current with three indices and which was x times T you see the rho and sigma rho is the first index second index on the left it is a first index here and if you go back and check here it is the same expression. So, indeed we have gotten what we were expecting at least one of the terms is the same which is this piece.

And I am calling it orbital because you explicitly have x here so angular momentum is x times p and if I put mu = 0 here that gives you the momentum density and you see because it is x times p it is orbital angular momentum basically and that is why I have put the label orbital in here, but I see that my conserved current J mu is not just having an M orbital which I was expecting in the last video, but an additional piece coming because of this sigma.

So, if it was a scalar field then my expectations coincides and I get exactly only this term. So, let me go further and write the J mu eta I have found now

Since $\bar{\omega}_{\rho\nu}$ can be chosen at will

$$M^{\mu\nu\rho} \quad = \quad M^{\mu\nu\rho}_{orb} + \frac{i\,\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\Sigma^{\nu\rho}_{ab}\,\phi_b(x) \tag{43}$$

$$M^{\mu\nu\rho}_{orb} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{44}$$

Six conserved currents!
Conserved charges

$$\partial_\mu M^{\mu\nu\rho} = 0, \quad Q = \int d^3x J^0 \tag{45}$$

So, the J mu is – 1 /2 omega bar rho nu M mu nu rho orbital – another term del over del mu phi a i times sigma rho nu ab phi b of x. Now, if I take del mu J mu I should get 0 because J mu is conserved current that is my construction we are doing Noether's theorem. So, I get – 1 / 2 omega bar rho nu del mu of this object this should be 0, but now because my omega they are arbitrary right I can choose whatever I wish them to be.

I can put many of them to be 0 or do rotations by whatever amount and no matter what I choose the omega bars to be this has to come out to be 0. So, this contraction of omega bar with this object has come out to be 0 independent of my choice is of omega bar which means that this can be true only if this object here it vanishes on its own. And otherwise you cannot arrange that no matter what you choose here you get a 0.

So, since omega bar rho nu can be chosen at will this has to be identically 0 meaning the conserved current now is this object J mu is instead of J mu of course you know this object has three indices now mu nu rho so the conserved current is this and it carries this indices nu rho. Here it is fine the reason I was choosing an anti-symmetric part was I wanted to drop this piece because if I do not that and if I continue with this piece.

Let us say I do not symmetrize and anti-symmetrize and drop the symmetric part and I continue with this then I would be concluding that here the piece correspondingly will be x nu T mu rho and I would be concluding that x mu T mu nu T mu rho – this piece is J mu, but that may not be correct because the symmetric part which is contained in this. Our derivation cannot even touch that symmetric part because this kills that symmetric part.

So, I should be careful and I should include only those parts about which omega rho nu is aware of it should not be touching those parts which omega rho nu is not aware of so this part is fine and this because it is anti-symmetric in rho nu the sigma also we should have anti-symmetric in rho nu. So, you take it to be anti-symmetric under rho nu interchange. So, I will interchange the signs and put a plus sign here to match with literature.

Sorry not so much to match with literature, but here nu is second index and rho follows it so nu comes first and rho later and that is what I want to have nu first and rho later and there is no big deal about this and there is a contraction over a contraction over b and nu is up, nu rho up so indices here all are which means all good and let me here write M mu nu rho orbital is what x nu T mu rho – x rho T mu nu.

So, we have found all the 6 conserved currents corresponding to Lorentz transformations 6 because you have nu rho and nu rho is anti-symmetric under these two indices are anti-symmetric I am going to interchange. So, there are only 6 independent entries so that is why you have 6 conserved currents and correspondingly 6 charges and interesting thing is the first piece is written in terms of translation current.

So, if you are looking at a scalar theory all you have is this piece and this term is wrong and then you can write everything in terms of energy–momentum tensor. That is good and maybe I should remark is the del mu of this entire thing which is vanishing. So, del mu of only this piece is not is going to be 0 on its own unless it is a scalar field. In general if you have to take the del mu of the entire thing and then only put equal to 0. Let us see if I have more to say.

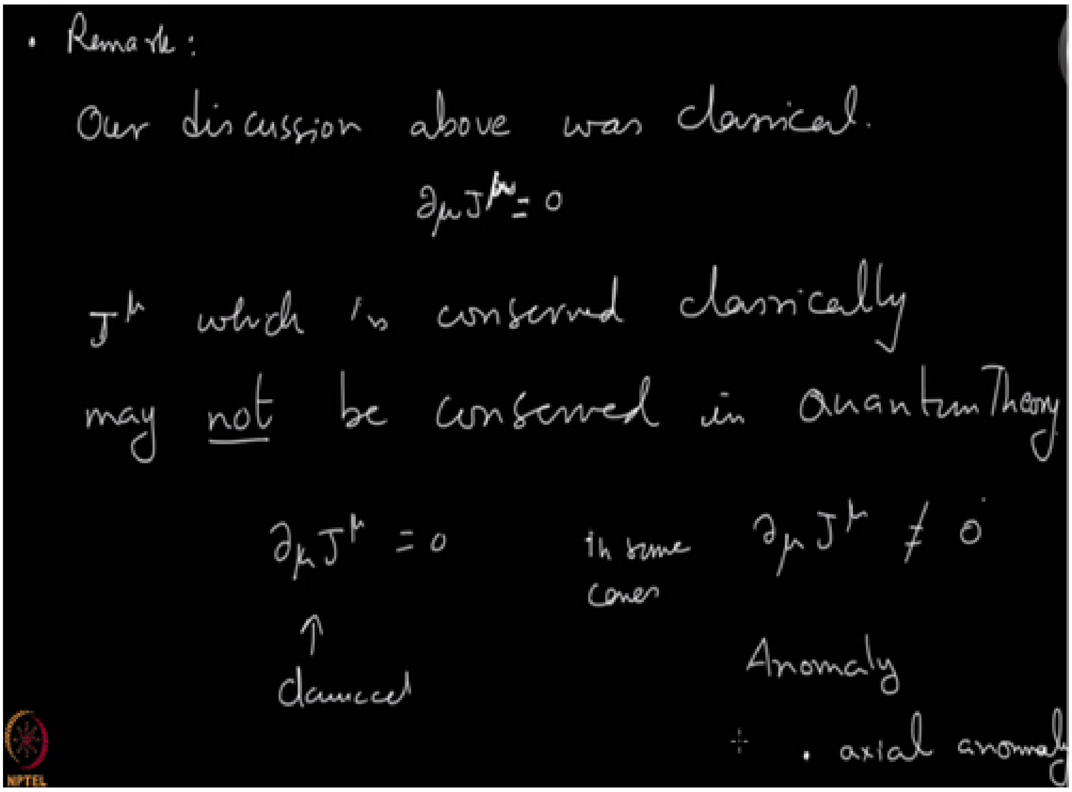


Figure 8: Refer Slide Time: 01:11:03

$$J^{\nu\rho} = \int d^3x M^{0\nu\rho} \tag{46}$$

$$J^{0\rho} = \int d^3x M^{00\rho} \tag{47}$$

$$J^{ij} = \int d^3x M^{0ij} \tag{48}$$

For scalar fields

So, we have found the conserved current and I should write conserved charge now. So, we have M mu nu rho and this is the thing and you remember you have to put J 0 that integral of this gives you the charge. So, we should put mu to 0 and then I get 6 charges because of this. So, what are the conserved charges? They are given by this expression and of course d over d t of this entire thing will be 0.

So, let us define this object to be J nu rho so this J is an object with two indices so this is different from whatever we have been writing. So, there is no reason for any confusion so I will use this notation. So, if I look at what is J 0 i these are the ones corresponding to boost this is d cube x M 0, 0, i and if I look at what are J i, j then this will be M, 0, i J and if the field is scalar field for scalar fields your J nu rho is integral d cube x x nu T 0 rho – x rho T 0 nu.

So, this gives you the angular momentum this one gives you the orbital angular momentum actually because we are looking at only the scalar part. Here it is not just the spatial, but also the time nu can take 0 values 0 and you can put rho = i. So, this is a generalized version of angular momentum, but if you look at only the special indices then this is what is coinciding with angular momentum so that is the relativistic version of our vital angular momentum.

So, that is one object which is one set of conserved charges you get from here if you put one of the indices to be 0 you see that this conserved charge which corresponds to boost now so I am putting nu to 0 I should put i rho is i T nu 0, 0. You see the time appear explicitly in this. So, the conserved charge corresponding to boost has explicit time dependence here say you have an integral over d cube x but x 0 is still there.

And that is why you will not use it so much because it does not commute with the Hamiltonian, but this one is of course going to be very useful with the three indices. So, what we have seen is that Lorentz symmetry implies 6 conserved currents and corresponding 6 conserved charges and these conserved charges are for a field that transforms under some representation of Lorentz group.

These charges correspond to angular momentum, but the angular momentum has two parts. One is orbital angular momentum which you are seeing here, but it also has another piece which comes from this term and this is remember why it is coming it is coming because of this matrix sigma if it is a scalar this term is 0 it is gone, but if sigma is non zero this term is here.

And what sigma does sigma is the matrix which mixes the different components of the field psi b if these are the components. It is they all are mixed so this is intrinsic spin angular momentum. So, this piece of the angular momentum is coming because of something that is intrinsic to the field itself it is not coming because of orbital angular momentum. So, if you have for example fields which let us say electromagnetic fields. If you are electromagnetic fields then of course your sigma is not 0 you know what it is you can calculate.

And then you see in addition to your electromagnetic fields carrying an orbital angular momentum they will also carry a spin angular momentum something which is intrinsic to the fields and this you might have already heard of or if you are looking at for example field theory of let us say a field theory we describes electrons and if you are willing to believe me that a field theory of electrons is not described by scalar fields.

Then you can expect that there also you will get in addition to the orbital angular momentum a term which will be giving you a angular momentum corresponding to something intrinsic to the fields corresponding to electrons and jumping a little ahead without any proofs or without any concrete arguments I will just say that if you were to quantize that field which corresponds to electrons you would get something which will correspond to the spin of electron because spin is what is an angular momentum.

So, in addition to your electron having an orbital angular momentum because it is moving in certain way with respect to the origin it will also have an angular momentum because of its intrinsic spin and the reason will be this term, but at this stage we do not have any particles I have not quantized these fields and I have just done a discussion at the classical level. So, all these current J mu and all these things are at classical level. Maybe it is a good point to make one more remark. So, our discussion in the context of Noether's theorem has been purely classical means that the current I have obtained the J mu. They have been obtained using classical arguments and we have found that they are conserved. Now whether these conserved currents will still remain conserved after we start doing quantum theory is not guaranteed.

It works in almost all the cases except for some anomalies, but this we should remember. So, this J mu which is conserved classically may not be conserved in quantum theory. So, meaning classically del mu J mu is equal to 0 so you found to conserve current and in some cases it is not that it is going to break in all the cases, but in some cases the same J mu will not be conserved so this will not happen.

But your translations and Lorentz transformations whatever conserved current they give here they are fine they are going to be good symmetries and when this happens we say it is anomaly and if you are more interested you can look into books for example axial anomaly, but anyway that is not what we are interested in further. I think this is all that I wanted to say. Okay then let us continue this discussion on quantum fields in the next video.

$$J^{0\rho} \;=\; \int d^3x x^\nu T^{0\rho} - x^\rho T^{0\nu} \tag{49}$$

Orbital angular momentum

$$J^{0i} = \int d^3x \left[x^0 T^{0i} - x^i T^{00}\right] \tag{50}$$

Our discussion above was classical

$$\partial_\mu J^\mu = 0 \tag{51}$$

$J^\mu$ which is conserved classically may not be conserved in Quantum theory, in some cases

$$\partial_\mu J^\mu \neq 0$$

then that is called **anomaly**, axial anomaly.