Introduction to Quantum Field Theory

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Lecture 31 : Noether Currents corresponding to Lorentz symmetry

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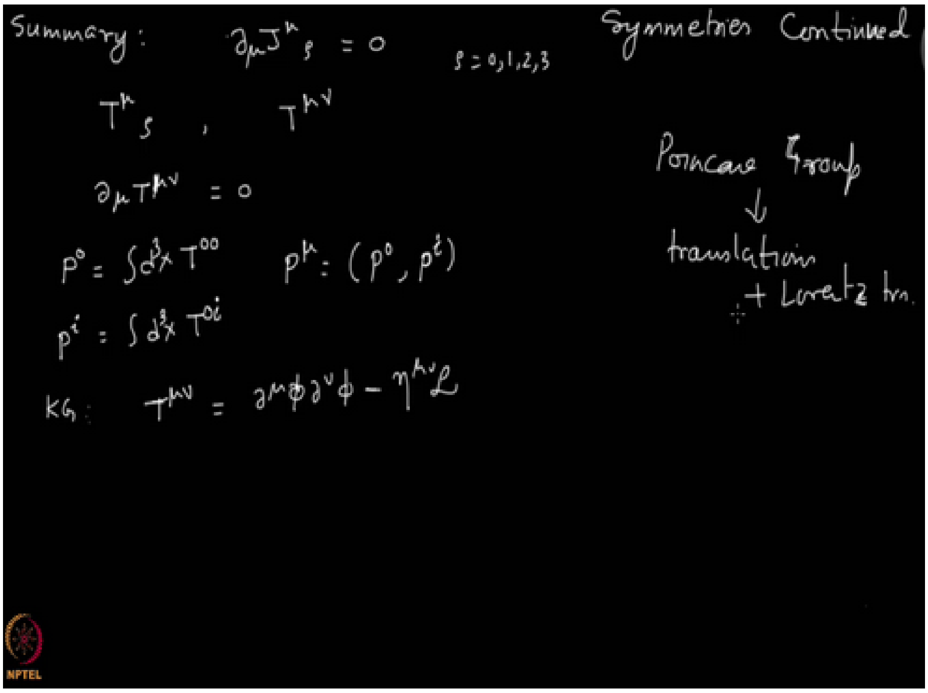


Figure 1: Refer Slide Time: 00:25

Summary:

Let us continue our discussion on symmetries last time we were discussing about the translation symmetry let us go back and see yeah and we had found the corresponding current j mu for space-time translations and also the corresponding charges which give you the momentum. So, let me summarize here. So, we found last time for each translation that there was a conserved current del mu j mu which is 0 and these conserved currents were labeled by rho's. So, rho runs from 0 to 4, 0 to 3. So, rho is when rho is 0 you are talking about space time translations when rho is 1 you are talking about the current j mu which corresponds to translation along x axis and so on. Generally it is instead of writing j mu rho we write T mu rho that is the standard notation and we also introduce T mu nu.

So, you just raise the lower index up with the help of metric tensor eta mu nu. So, that is what gives you T mu nu. So, it is really del mu T mu nu is equal to 0 that is the relation you have and mu is the index which corresponds to the this index on the derivative which contracts and nu is the index which labels which transformation you are talking about. So, if it is 0 you are talking about time translations. So, that is what we found and then we can constructed conserved charges by looking at T 0 nu.

So, you have, we found that we have P 0 which is basically the Hamiltonian or gives you the energy which is integral d cube x T 0 and nu is 0 for that one because it is about time translations and then we also had P i where i could be 1, 2 or 3 and that is again the first index has to be 0 why because it is the j 0 which gives you the conserved charge. So, the first index is the one which contracts with del mu with the del operator.

$$\partial_\mu J^\mu_\rho = 0, \quad \rho = 0, 1, 2, 3 \tag{1}$$

So, that is the index which you have to put 0 here and of course the i index is the second index which labels translation you are doing and we also saw that we club them like this. So, which means that these 4 charges P 0 P 1 P 2 and P 3 they form a 4 vector that is the purpose of this index. So, if you look at how the transformational Lorentz transformation they will transform like a 4 vector otherwise there is no meaning of putting a mu index here.

So, that is what P mu is it is a object which is a 4 vector you can also see this in another way if you recall when we were doing the quantization of Klein Gordon theory we had the Hamiltonian and we had guessed the momentum operator this one. And now we have explicitly found this in the last or previous or last to last class video no here that was an exercise this one. And if you take this operator and act on a state which is some single some state in the Klein Garden theory you see that this gives you the momentum of that state for momentum of that state.

So, really the P mu the mu here is really a 4 vector index. So, you have to treat it as a 4 vector this is a 4 vector this was another way of convincing you that what I am saying is correct and we had also found explicitly an expression in the Klein Gordon theory of this current which is called energy momentum tensor. And this turned out to be symmetric under mu and nu interchange even though rho was at a different footing because it was just labeling the direction.

And mu was on a different footing it was the one which was contracting with this one which was really the index of the current j mu but they turned out to be symmetric the object T mu nu turned out to be symmetric under the interchange which is a bit surprising at this stage and we are going to see we are going to see later why that happened. Now what I want to do in next video is to look at Lorentz symmetries.

So, I want to look at the full Poincare group. Recall Poincare group has both translations and Lorentz transformations. So, that is what we want to look at but before giving a proof of I mean before doing explicit calculations for finding the conserved current in the case of Poincare group or the Lorentz group we are going to do a bit of hand waving and build up our expectations of what we should get when we do explicit computations.

So, that is what I am going to do in this video I will just do some hand waving and actually we are going to get very close to the actual results. So, that is the goal let us see now.

Instead of J^μ_ρ, we will write $T^\mu_\rho, T^{\mu\nu}$

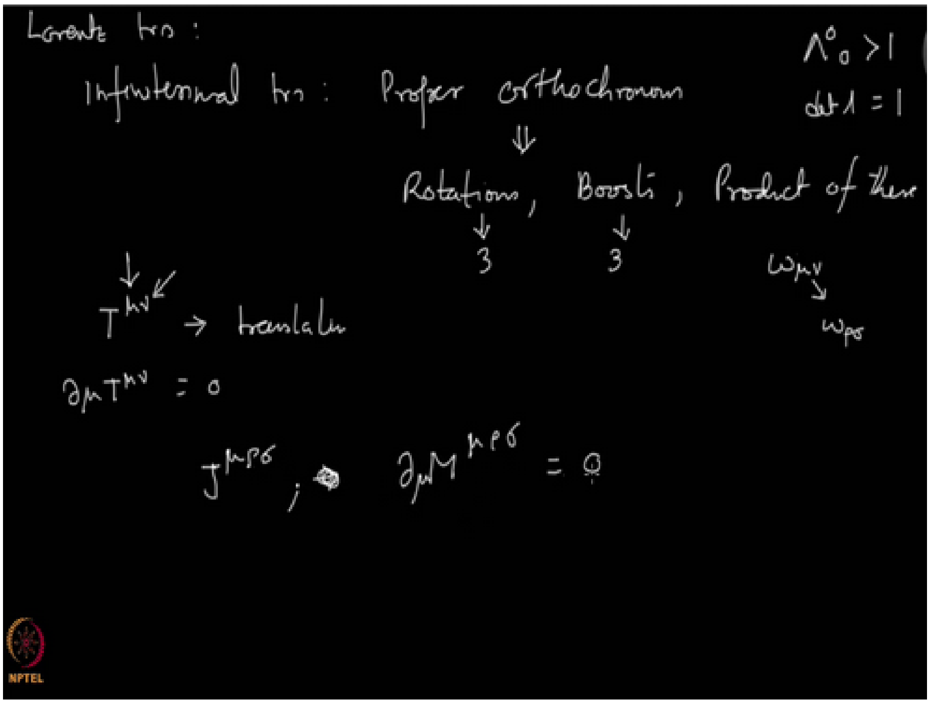
$$\partial_\mu T^{\mu\nu} = 0 \tag{2}$$

$$P^0 = \int d^3x\, T^{00}, \quad P^\rho = \int d^3x\, T \tag{3}$$

$$P^\mu = (P^0, P^i) \tag{4}$$

In KG theory

$$T^{\mu\nu} = \partial^\mu\phi\, \partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{5}$$



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Now we want to look at poincare group

So, first when I am looking at Lorentz transformations I am going to be looking at infinitesimal transformations which means I will be close to the identity which further means that you will be restricted to proper orthochronous Lorentz transformations because you are close to identity you remember your lambda 0 0 has to be greater than 1 and what was the other thing determinant of lambda is 1. So, this is because; this is the one which is connected to identity.

So, when you are doing infinitesimal transformations you are going to be moving within this group parameterized by this group or rather restriction imposed by these conditions. So when we are doing infinitesimal transformations we will be within the proper orthochronus part of the group and if you recall all the transformations in proper orthochronous Lorentz transformations they were either rotations or boosts or product of these two.

So, all we need to really study is rotations and boosts that is the thing here in this part of the group and we have been we have already seen that there are six parameters which parameterize this transformation. So, 3 for rotations and 3 for boosts and these are these we have been writing as omega mu. Omega mu is anti-symmetric and that is why this forms six independent entries that is good. Now what should I expect for the conserved current when I am looking at Lorentz transformations.

So you see we got T mu nu whenever when we were looking at translations mu is the one which contracts with del mu. So, that del mu T mu nu is 0. So, I should have some object which will be instead of writing j let me write j for a while. So, I should get some current j mu for each of the Lorentz transformations and there are six of them and they are parametrized by omega mu nu. So, you can choose different values of mu and nu and the omega will give you the transformation parameters.

So, you expect just like here you had new which was labeling the 4 translations I will have let us instead of mu let me write omega rho sigma I should have two indices rho sigma which will be the equivalent of nu here parameterizing which transformation we are talking about. So, we expect an object which will carry 3 indices and usually it is denoted by M mu rho sigma. More common notation is M mu sigma and this mu index will contract with the derivative and this should be the conserved current.

So, I have not said anything about what M is and what is it is explicit expression but I do know that because I am looking at Lorentz transformations I expect a conserved current which will carry these 3 indices at least that much I can say and actually I am going to we will be able to say more just based on our experience with translations. So, that is good. So, we have the current mu rho sigma and if I go on to let me go to the next slide. So, I have been able to argue that I am expecting to get something of that sort it does not say much except for the fact that it will have 3 indices that is what I have figured out. So, when I go from current to the charge I have to look at j 0 integral d cube x remember that. So, what i should do is take this M and put the first index to 0 and do an integral over space. So, the charges will be M the conserved charges will be M rho sigma 0 rho sigma d cube x and you have the integral over this.

Now these charges will correspond to all the different transformations which we can do the boost and the rotations. So, let me just bifurcate this. So, you expect that the charges corresponding to boosts will be this M 0 that is anyway there and boosts are parameterized when you are looking at omega rho sigma they were omega 0i. So, I put omega rho 0i these were the boosts and omega ij they were the rotations.

So, for rotations I will get these ones M 0 ij. So, these are corresponding to the boosts and these are corresponding to the rotations. So, this is how far we can go with this now recall that when we were doing when we studied classical mechanics and you were learning about rotations and translations you learn that momentum is conserved and corresponding to rotations angular momentum is conserved.

Poincare group → Translation + Lorentz transformation

Lorentz transformation → Infinitesimal transformation, proper orthochronous

$$\Lambda^0{}_0 > 1, \text{and} \det \Lambda = 1$$

, either rotation, boost, product of rotation and boost. 6 parameter of $\omega_{\mu\nu}$

$$T^{\mu\nu} \rightarrow \text{translation} \quad (6)$$

$$\partial_\mu T^{\mu\nu} = 0 \quad (7)$$

For Lorentz transformation the expectation for Noether current,

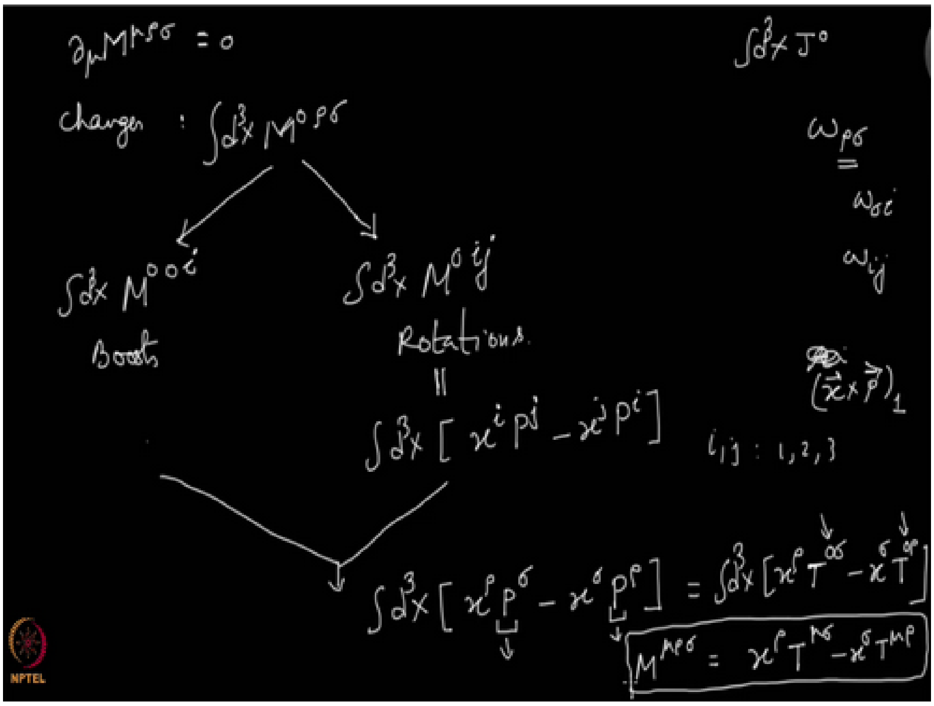
So, here because you have rotational symmetry you should also expect some angular momentum some definition of angular momentum here which will be conserved and that thing should come out of this actually this should be the angular momentum just like you had the momentum before T mu nu this should be the object which will give you angular momentum. But angular momentum is r cross P or x cross P that you have known for long.

Which means if I am looking at angular momentum coming out of this is the let us say this is the charge which gives you angular momentum then you expect that its structure should be of this kind this will be something like this. So, x i P j - x j P i x cross p, so, if you look at i'th element of this or first element of this then this is y cross P z minus z cross P y and that is what I am writing here. So, you see that the rotations will give you angular momentum the charge here which will be of this form.

So, that is fine which means that these P's are the operators which we have found here already these ones P mu and P mu nu is written in P mu is encoded in the T mu nu doing putting the mu index 0 and doing the integral you get P mu. So, we see that the charge corresponding to rotations will be constructed out of x and momentum operator that is good but we can say actually more here the indices are i and j which run from one to 3.

So, i and j run from one to 3 but we remember that we are working in a in a theory which has Lorentz symmetry then we can expect that the things which we write should transform nicely under Lorentz transformations and this object is not having a nice Lorentz transformation it is only nice under rotations. So, we expect to see something nicer and that to get that nice objects i should take this and combines with the boosts because these are the ones which are related to omega 0i and here this is only ij.

So, if I combine these two I expect something nice and I can already guess what it should be. So, of course because I am asking for a charge. So, it will be integral d cube x and it has to have these two indices rho and sigma but now I can almost guess what M 0 rho sigma would be? It would be this. So, I know this part and I want to make it nice with the nice Lorentz transformations.



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$$J^{\mu\rho\sigma} \quad ; \quad \partial_\mu M^{\mu\rho\sigma} = 0 \tag{8}$$

Now the charges will be

$$\int d^3x \, M^{\mu\rho\sigma} \tag{9}$$

For boost

$$\int d^3x \, M^{00i} \tag{10}$$

For rotation

$$\int d^3x \, M^{0ij} \tag{11}$$

Rotation will give us angular momentum as charges

$$\int d^3x \, [x^i p^j - x^j p^i] \quad i,j \to 1,2,3 \tag{12}$$

Combining with boost

$$\int d^3x \, [x^\rho p^\sigma - x^\sigma p^\rho] \tag{13}$$

$$= \int d^3x \, [x^\rho T^{0\sigma} - x^\sigma T^{0\rho}] \tag{14}$$

Thus

$$M^{\mu\rho\sigma} = x^\rho T^{0\sigma} - x^\sigma T^{0\rho} \tag{15}$$

So, I write rho P sigma minus x sigma P rho. So, this is something what I would expect based on Lorentz transformations and I can say further now this d cube x and then you have x rho and remember what yeah I should have actually this is what I am hand waving. So, I am fine but you see that the P is the charge and charge is obtained by doing integral over space of the current. So, what I should have written is not really this but the corresponding current.

So, this is not what I have done very nicely. So, this one it is not. So, good but let me correct it. So, what I should have written was this. So, I should have written this T 0 rho now this is fine. So, this is actually our expectation based on all the arguments and hand waving that I have been doing and then I have the expression of d cube x times the charge sorry d 3 x times the current.

And I have been able to guess what the 0th component of the charge would be you see this is still 0 and that is 0 and it is easy to now guess what the general expression for M mu rho sigma would be you just put in place of 0's here a mu. So, you guess that M mu rho sigma which is the current corresponding to Lorentz transformation should have the form something like this. So, it will be a product of x times the energy momentum tensor that you obtain from the translations.

And this is what we should expect it will come out to be correct almost except for a small difference and we will talk about that in detail when we do the derivation. But we have built up our expectations and we just have to do it carefully and find it out. Let me see if I wanted to say anything more, no. So, this is all I wanted to say and we will do it properly in the next video.