Introduction to Quantum Field Theory

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Lecture 30 : Momentum in KG Theory

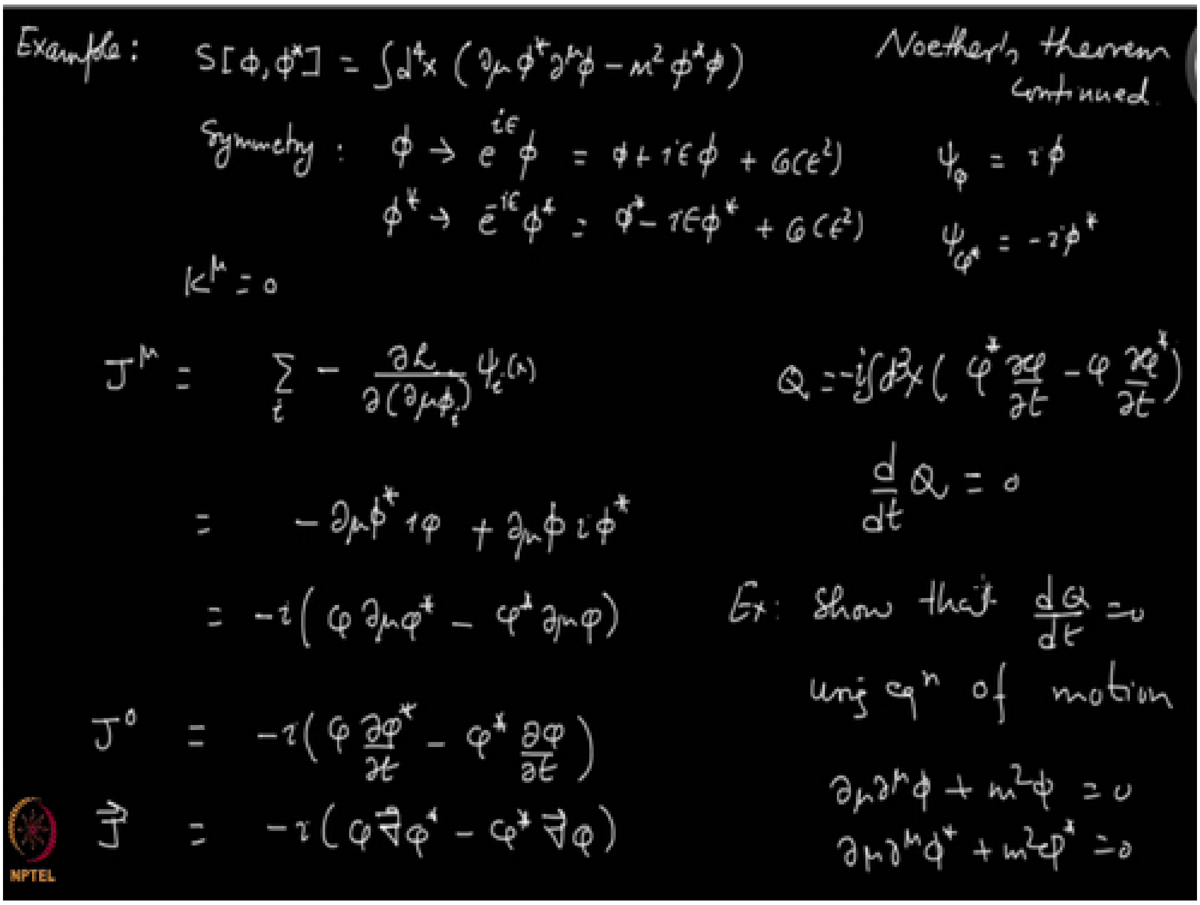


Figure 1: Refer Slide Time: 00:19

Noether's theorem and complex scalar field

Let us continue our discussion on Neother's theorem. Let me first remove this timeline. So, I had written down an expression for the conserve current which is here and of course even if I multiply j mu with some constant here that will still be conserved. So, because you will still get del mu j m equal to zero. But we will fix it to some level and you will see that I am going to change the signs later and we will see the reason why.

But for now we will just proceed like this. So let me give you one example of evaluating conserved current and corresponding conserved charge and for that I will look at complex Klein Gordon theory. So, let me write example here. So, you have two fields now phi and phi star you can treat phi and phi star as independent fields. So, d 4 x del mu phi star del mu phi - m square phi star phi. So, and the symmetry which I am interested in is this one. So, the symmetry is phi going to e to the i let me write earlier I think I wrote alpha but this time I will write epsilon to make the fact sorry just a second phi to make it explicit that I am looking at infinitesimal transformation. So, this says 1 + i epsilon phi + higher order pieces and similarly the phi star would be transforming as this sorry here should be phi simplify start that is correct and I think we have already discussed last time or yeah last time that the k mu would be 0 in this case.

Because the lagrangian density here that is not going to transform into something which will have a term which will be total derivative because this is internal symmetry it will not generate any del much e mu term . So, k mu is 0. So, all I have to do is plug in this formula for j mu there is a summation over i implied here over the repeated index i. So, if you have several fields phi i's under the symmetry transformation each of the field will transform in some way and these are the corresponding changes the coefficients of epsilon.

So, if you want I can make this more explicit by putting a summation. So, anyway there was a minus sign here. So, let it be like that I will put a sigma here. So, let us go here. So, for us phi i's are phi and phi star. So, we have 2 phi's and I should calculate j mu is equal to del L over maybe I should write here and I think I have been calling it psi. So, yeah.

So, before phi i psi i of x and then you have k mu which is 0 in this case. So, let us calculate what is psi of phi and psi of phi star I am using this notation. So, that instead of writing one and 2i imply psi corresponding to change in the phi and psi corresponding to change in the phi star. So, that is I phi from here you can read off and this is minus i phi star. So, here this will become and when I take the derivative of Lagrangian density with respect to phi del L over del mu phi del of del mu phi that will give you a del mu phi star.

So, you get a minus del mu phi star and then this one is i phi. So, i times phi and then you have minus and this one will give you del mu phi and the corresponding psi is minus I phi star. So minus makes its plus i phi star. So, which is i phi del mu phi star - phi star del mu phi. Again if you compare you will see with some book you will see that there is a there is a sign difference overall sign difference and we will see why there is difference but there is nothing wrong in here because clearly del mu j mu is zero whether you have a sign here or not.

Example

$$S[\phi, \phi^*] = \int d^4x \left(\partial_\mu \phi^*(x) \, \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x) \right) \tag{1}$$

$$\tag{2}$$

Symmetry

$$\phi \to e^{i\epsilon}\phi = \phi + i\epsilon\phi + \mathcal{O}(\epsilon^2) \tag{3}$$

$$\phi^* \to e^{-i\epsilon}\phi^* = \phi^* - i\epsilon\phi^* + \mathcal{O}(\epsilon^2) \tag{4}$$

$K^\mu = 0$

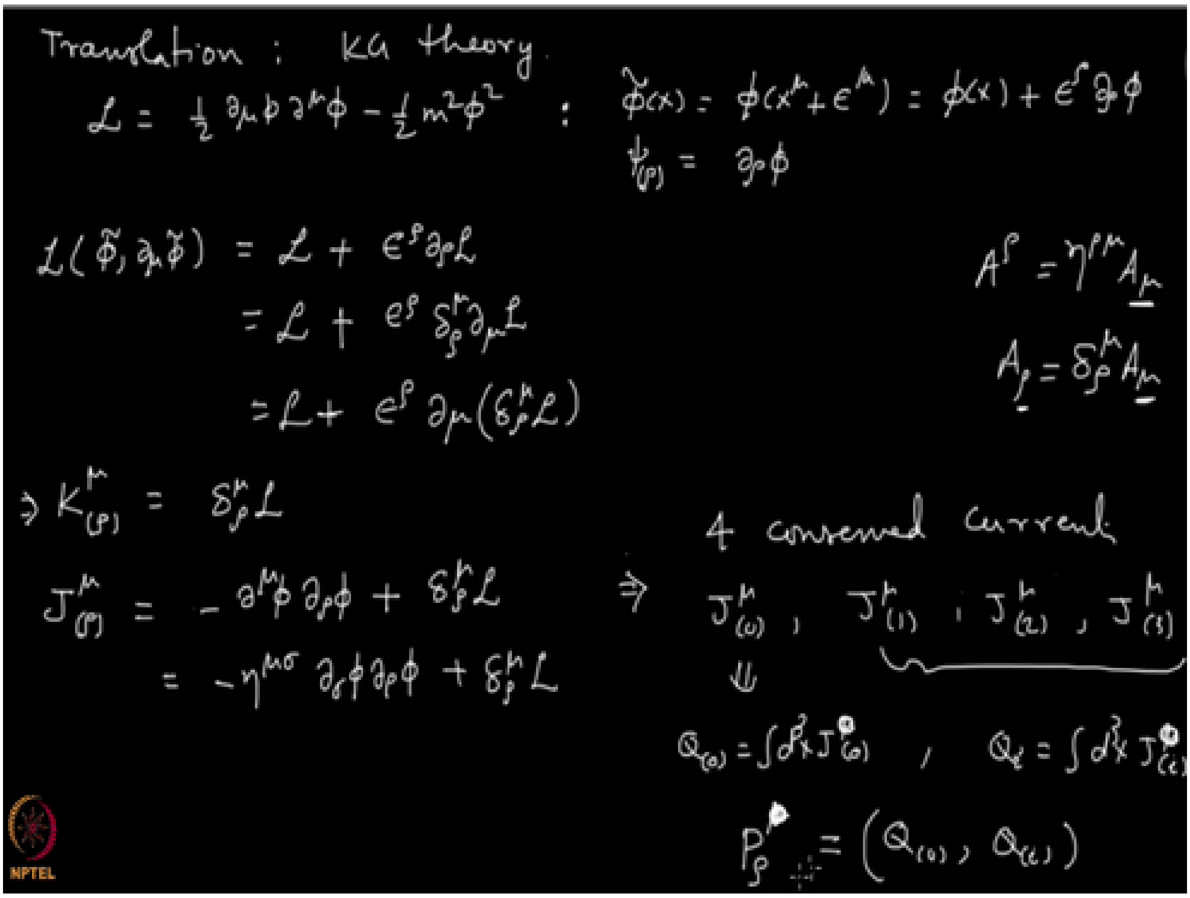
$$J^\mu = \sum_i \frac{-\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \psi_i(x) \tag{5}$$

$$\psi_\phi = i\phi \quad , \qquad \psi_{\phi^*} = -i\phi \tag{6}$$

$$J^\mu = -\partial_\mu \phi^* i\phi + \partial_\mu \phi i \phi^* \tag{7}$$

$$J^\mu = -i(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) \tag{8}$$

$$
\begin{aligned}
J^0 &= -i\left(\phi\frac{\partial\phi^*}{\partial t} - \phi^*\frac{\partial\phi}{\partial t}\right) &(9)\\
\vec{J} &= -i\left(\phi\vec{\nabla}\phi^* - \phi^*\vec{\nabla}\phi\right) &(10)\\
Q &= -i\int d^3x\left(\phi\frac{\partial\phi^*}{\partial t} - \phi^*\frac{\partial\phi}{\partial t}\right) &(11)
\end{aligned}
$$



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So, that is fine and then we can construct j0 and j0 would be minus i phi and this is del phi I have to put mu equal to zero. So, this becomes phi dot and this becomes this was phi star dot this is phi dot again your conserved current is this phi what will that be that will be gradient of phi star minus phi star gradient of phi because the special components of del mu that makes a gradient this is fine. So, you can write down now the conserved charge Q which is d cube x j 0 and j 0 is this.

So, I put a minus i phi del phi over del t - phi del phi star over del t something wrong here let us check phi. So, many things wrong what did I do wrong here phi this should have been starred this is fine. So, that is the charge and you know that dq over dt should be 0 but it will not be 0 for any field configuration it will be 0 for field configurations which satisfies the equations of motion.

So, show that indeed you get dq over dt to be 0 upon using equations of motion that you should do and equations of motion are simple in this case they are just del mu del mu del mu phi

+ m square phi equal to 0 and del mu del mu phi star + m square phi is equal to 0. So, use these equations put in the definition of Q and show that the total time derivative comes out to be zero and that would provide you a exercise on understanding how and what Q is?

So, that will not be difficult actually to show that. Now this was internal symmetry and things were simple and nice.

Exercise: Show that $\frac{dQ}{dt} = 0$, using equations of motion

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0 \quad (12)$$
$$\partial^\mu \partial_\mu \phi^* + m^2 \phi^* = 0 \quad (13)$$

Now I want to look at translation symmetry and Klein Gordon theory and we will talk more about translation in general but let us first have some experience with translation and in a simple setup and then we will talk more about it translation. So, here your Lagrangian density is half del mu phi del mu phi probably this is the thousandth time I am writing the same del mu phi del mu phi.

So, the symmetry we are talking about is phi tilde of x is equal to phi of x mu I am just writing mu explicitly here and I am looking at an infinitesimal transformation and I can do a Taylor expansion here plus epsilon mu del mu of phi or it is that is absolutely correct but I will make it rho it will be slightly comfortable and plus order epsilon square terms. So, that is the transformation we are looking at.

So, what is our psi of phi sorry what is our psi? We have 4 translations epsilon 0 epsilon 1 epsilon 2 and epsilon 3. So, I will put an index rho here and let me put a bracket. So, psi of the psi corresponding to each of those transformations is just the coefficient of the parameter epsilon. So, which is del rho of phi. Now let us take the um Lagrangian density with phi tilde and delta phi tilde and what I want to do is I want to find out the k mu.

So, for that I do this calculation and I will just give you the answer it is simple. So, I do not want to do the full algebra here. So, you get the density of course L + epsilon rho del rho of L of the Lagrangian density. Now we do not want this form what we want is epsilon rho. So, imagine let's imagine that we are taking only time translation not others. So, this would have been just epsilon 0 and del L over del t.

But what we want here is epsilon 0 times something which looks like del mu k mu. So, I want to convert this into del mu k mu kind of a thing. So, and that is not difficult let me tell you one thing. So, if you have to raise or lower an index on some object. So, if you have let us say a mu and you want to raise this index to let us say rho you use eta rho mu that raises it. But sometimes you may want to just change the index on this one.

So, you do not want a mu but let us say you want a rho then you do not use eta because eta will raise it suppose you want to keep the mu the new index again lower but change it. So, what you can do is you take a mu and write delta mu rho now this object is this one is a of rho because mu is contracted. So, this is how you can change the index by using a delta this one of course becomes a of rho. So, this is a of rho up and this is a of rho down.

So, you start from the mu index down in both the cases but here you end up with rho down and there you end up with rho up and that is what I am going to use here. So, I will write it as epsilon rho and this del rho I write as del mu this object is having an index down but then I should have a rho rather than mu. So, I have to have del mu of rho. So, this del rho is this these two things together and of course you had your L.

This is L plus and let me write it more neatly del mu of del mu rho L ok now this is in the form which we wanted this is a form of del mu k mu. So, k mu but you have vermont index rho because you have several or rather four transformations. So, each one has all the four terms are present here. So, what is k mu then our k mu corresponding to each of the transformation which we are labeling by rho is del mu rho times the Lagrangian density.

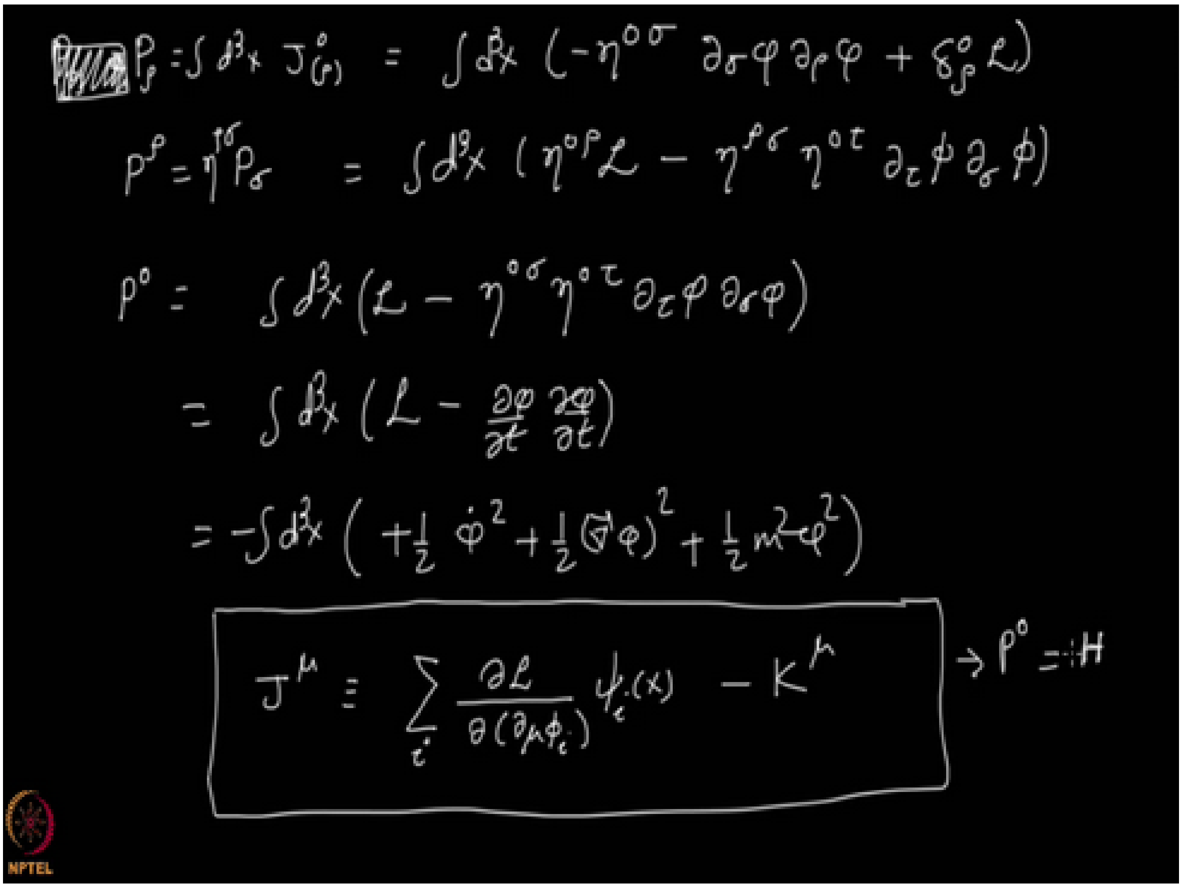


Figure 3: Refer Slide Time: 21:02

Translation symmetry in KG theory

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) \tag{14}$$

$$\tilde{\phi}(x) = \phi(x^\mu + \epsilon^\mu) = \phi(x) + \epsilon^\rho\partial_\rho\phi \tag{15}$$

$\psi_\rho = \partial_\rho\phi$

$$\mathcal{L}(\tilde{\phi}, \partial_\mu\tilde{\phi}) = \mathcal{L} + \epsilon^\rho\partial_\rho\mathcal{L} \tag{16}$$

$$= \mathcal{L} + \epsilon^\rho\,\delta^\mu_\rho\,\partial_\mu\,, \quad \mathcal{L} + \epsilon^\rho\,\partial_\mu(\delta^\mu_\rho\,\mathcal{L}) \tag{17}$$

Where we have used

And if you want to look only at time translation you can put only epsilon 0 to be non-zero and all the other epsilon 1 epsilon 2 epsilon 3 to be 0 and just look at the time translations. So, let us calculate now the current j mu for each of these four transformations if symmetry transformations or symmetries. So when you plug it in here we have other term also with the k mu. So, you get the following you get minus.

So, this is an exercise you should show that you get minus del mu phi del rho of phi + del mu rho times lagrangian density and just a second and this you can of course write as eta mu sigma del sigma phi del rho phi. So, we have got 4 conserved currents j mu of 0 j mu of 1 j mu of 2 this 2 means you are looking at translational symmetry along y axis and this 3 means you are looking translational symmetry along the z axis.

So, these are 4 and they will give you corresponding q's Q let me call it 0. So, that is d cube x j mu and similarly these 3 let me put them together some Q which will be j mu instead of putting a vector symbol here it is better to put an i and d cube x j mu of i. So, these are the charges 4 charges that we get and these charges are called together they form a 4 vector and they are called momentum. So, it together form what is called p mu. So, if you write Q 0 and Q i these are three Q 1 Q 2 Q 3 then they together form object which we call P mu and these are defined to be the to be the momentum of momentum carried by these fields.

This is what becomes the momentum operator, when you change the fields to quantum fields but this is the conserved quantity and momentum conserve charge is momentum for this theory. And in general you call it momentum whenever you are looking at space-time translations again you will also see why we call it momentum and you will see that when we are looking at the special Klein Gordon theory we are going to get the time component as really the energy and that is why this name makes sense. So, let us go ahead and. So, sorry I have made small mistake here. So, this is not correct mu I have anyway put to 0 when you are calculating the charge you put mu equal to 0. So, here the mistake is here it is it should be 0 and here it should be 0 and this is not the index the index is rho because it is this lower index which is relevant here mu there is no mu here. So, this is the momentum in the Klein Gordon theory.

$$A^{\rho} = \eta^{\rho\mu} A_{\mu} \quad ; \quad A_{\rho} = \delta^{\mu}_{\rho} A_{\mu} \tag{18}$$

that gives

$$K^{\mu}_{\rho} = \delta^{\mu}_{\rho} \ \mathcal{L} \tag{19}$$

$$\begin{aligned} J^{\mu}_{\rho} &= -\partial^{\mu}\phi \, \partial_{\rho}\phi + \delta^{\mu}_{\rho} \ \mathcal{L} && (20) \\ &= -\eta^{\mu\sigma}\partial_{\sigma}\phi \, \partial_{\rho}\phi + \delta^{\mu}_{\rho} \ \mathcal{L} && (21) \end{aligned}$$

4 conserved current

And let us define. So, what is p not defined let me just write. So, what is p of rho it is d cube x j 0 of rho and this turns out to be if you calculate minus eta 0 sigma del sigma phi del rho phi + del 0 rho times this let us check sigma is contracted only rho is free only rho is here looks correct and we can define this to be there is no need to put a bracket around this. So, let me call it P rho and I can define a P with an upper index rho and that will be P sigma rho sigma and this will be you can calculate this it is easy eta 0 rho times L - eta rho sigma eta 0 tau delta tau phi and delta sigma phi that will you can work out from here.

Let us check whether everything looks fine, rho is the free index see not all contracted rho is free. So, this looks fine to me and if you calculate now what is P 0 then you get integral d cube x. So, I put rho equal to 0 here then eta 0 0 is 1. So, you get l minus rho is 0 let me write rho eta sigma eta 0 sigma eta 0 tau del tau phi and del sigma phi now this is easy minus. So, this term contributes only when sigma is 0 and this term contributes only when tau is equal to 0 otherwise they are 0 which means these this tau and that sigma are forced to be to be 0.

So, I get del phi over del t del phi over del t and you remember this Lagrangian density has half phi dot square as the first term is from the del mu phi P's and this minus makes it minus half phi dot square. So, this is minus half sorry minus half phi dot square phi dot square half phi dot square and then this has already minus half gradient of phi square. And finally minus half m square phi square and which is just minus off this entire thing and you recall that the Hamiltonian is this piece.

So, we see that we have got p naught to be minus h and that is what I was referring to here where was it here. So, even if I change the entire sign it is still a conserved current and now you see that it makes sense to change the sign in my definition of conserve current because then I will get P 0 as the Hamiltonian. So, I will change the sign now. So, j mu I will write as del del mu phi i psi I of x and of course there is a summation over i and minus k mu and this is. So, there was a + k mu and I should have a minus, minus sign here. Now when I do this P naught will come out to be positive. So, with this definition of j mu the signs will come out P 0 will be H with the sign.

$$J_0^\mu, J_1^\mu, J_2^\mu, J_3^\mu \tag{22}$$

$$Q_0 = \int d^3x\, J_0^\mu \quad , \quad Q_i = \int d^3x\, J_i^\mu \tag{23}$$

$$(Q_0, Q_i) = P_\rho \tag{24}$$

Conserved charges

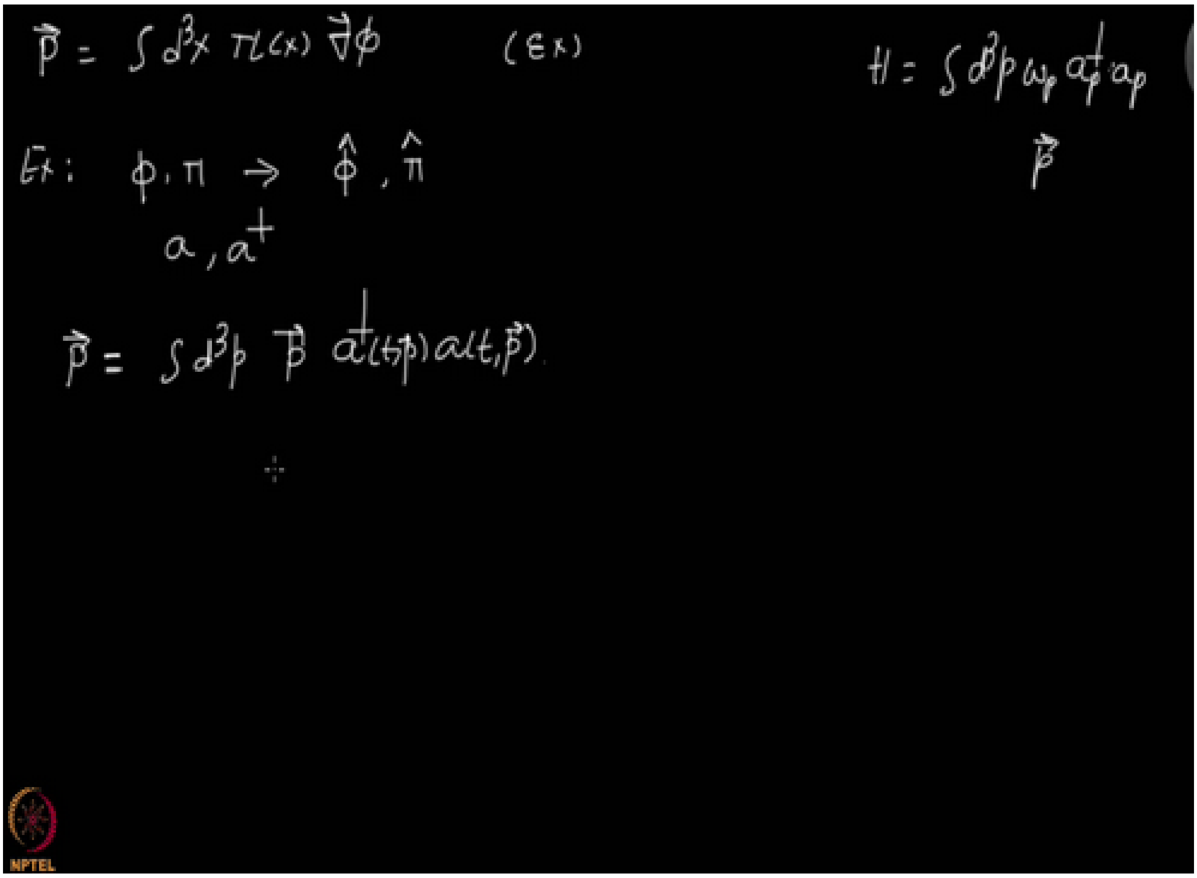
$$\begin{aligned}
P_\rho &= \int d^3x J_\rho^0 = \int d^3x \Big(-\eta^{00}\partial_0\phi\partial_\rho\phi + \delta_\rho^0\mathcal{L}\Big) && (25)\\
P^\rho &= \eta^{\rho\sigma}P_\sigma = \int d^3x(\eta^{0\rho}\mathcal{L} - \eta^{\rho\sigma}\eta^{0\tau}\partial_\tau\phi\partial_\sigma\phi) && (26)\\
P^0 &= \int d^3x\Big(\mathcal{L} - \eta^{00}\eta^{0\tau}\partial_\tau\phi\,\partial_\sigma\phi\Big) && (27)\\
P^0 &= \int d^3x\Big(\mathcal{L} - \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t}\Big) && (28)\\
P^0 &= -\int d^3x\Big(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\Big) && (29)\\
P^0 &= -H && (30)
\end{aligned}$$

Changing the sign of J^μ

$$J^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\psi_i(x) - K^\mu \tag{31}$$

Exercise:

$$\begin{aligned}
\vec{P} &= \int d^3x\, \Pi(x)\vec{\nabla}\phi && (32)\\
H &= d^3p\, \omega_p a_p^\dagger a_p && (33)\\
& && (34)
\end{aligned}$$



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Exercise:

$$\phi, \Pi \rightarrow \tilde{\phi}, \tilde{\Pi} \tag{35}$$

express them in terms of $a, a^\dagger$

$$\vec{P} = \int d^3x \, \vec{p} \, a^\dagger(t,p) a(t,p) \tag{36}$$

And of course you can also ask what is P and I leave it as an exercise for you to check that this is pi of x times gradient of i and this this gives you the momentum. Now you recall long back we have used the expression of momentum when we were dealing with Klein Gordon theory the quantum theory of Klein Gordon fields and at that time I had just guessed what P would be based on Lorentz symmetry.

And I had said that P just looking at the Hamiltonian. So, you recall if you look at the Hamiltonian what we had written it was d 3 P and there was omega here omega P an aP dagger aP and of course some constant here and then I have said that because this is, it was kind of easy to guess that this you can replace by P to get the momentum operator. So, this is the time you can check this. So, exercise is this you take this expression of P which is a classical expression in the classical field theory.

And promote phi and pi to quantum operators. So, replace them by phi that phi hat and pi hat and express these in terms of a's and a daggers and then show that this operator P which you get in the quantum theory is indeed this. So, I leave it as an exercise to do we will continue our discussion more on Neother's theorem in our next video in the next video.