Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 3 : Quantizing Schrodinger Field continued

---

1 Recap

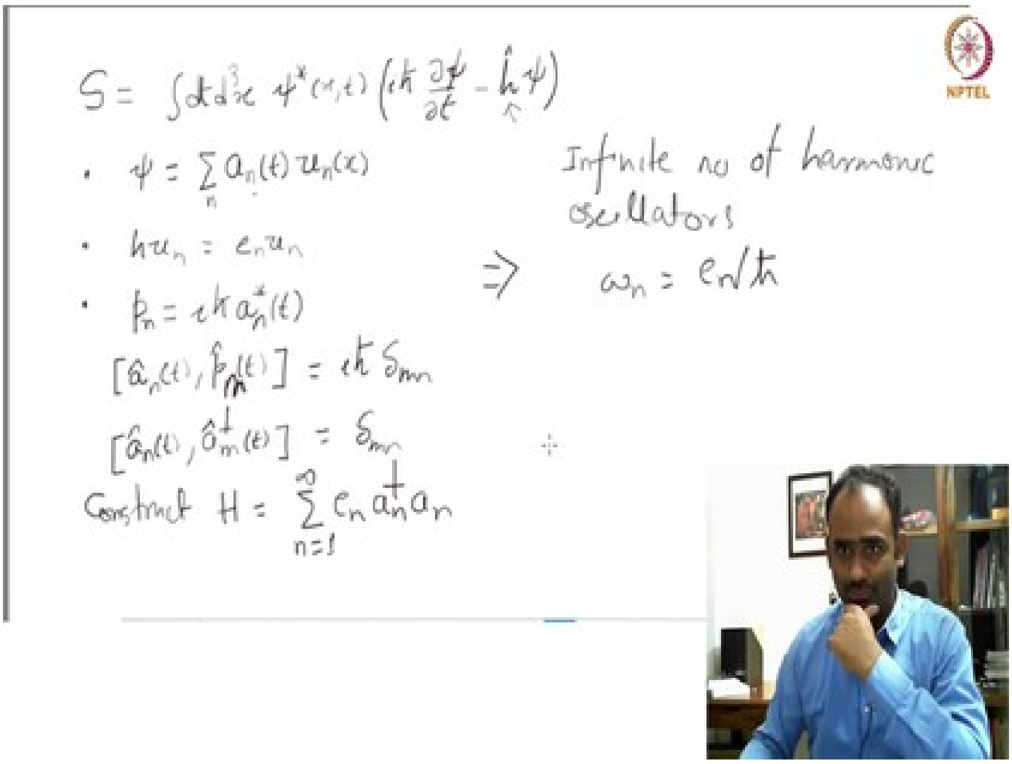

Figure 1: Refer Slide Time: 00:24

Let us start with a quick recap of what we did last time. So, we were looking at a system which is described by the following classical action. And then you have a psi star - h hat psi again if you take the action and write down the Lagrange equations you will get the Schrodinger equation. And then for this in finite dimensional system it is infinite dimensional because you have a field rather than fixed number of coordinates here we could do the following.

We could expand psi as x that is what we did where the u ns are the eigen functions of this Hamiltonian of the single particle wave single particle quantum theory this one. So, let me write that down also. So, h hat do not need let us not write a hat here it is h psi is sorry what I want to write is this h operator acting on u n gives you e n u n. So, these u ns are eigen functions of this operator h.

So, we did this expansion and then we took the a's as the generalized coordinates and found out that the corresponding conjugate momenta are ih bar a star t. So, that is what we found and then we said we want to make a quantum theory out of this classical theory. So, I will replace the the coordinates in this theory the a's right here and the a's and the p's can replace them from being functions to operators and impose commutation relation.

So, if I do that we saw that we get let me put a hat now because I am saying these are operators and if I impose the commutation relation then this says ih bar and sorry it is better to put it like m here this is m. So, you have delta m n and we saw that this leads to the following a n hat t a m dagger t. So, a n hat and a m dagger they give you delta m n. So, that is the commutation relation between a m and a m dagger.

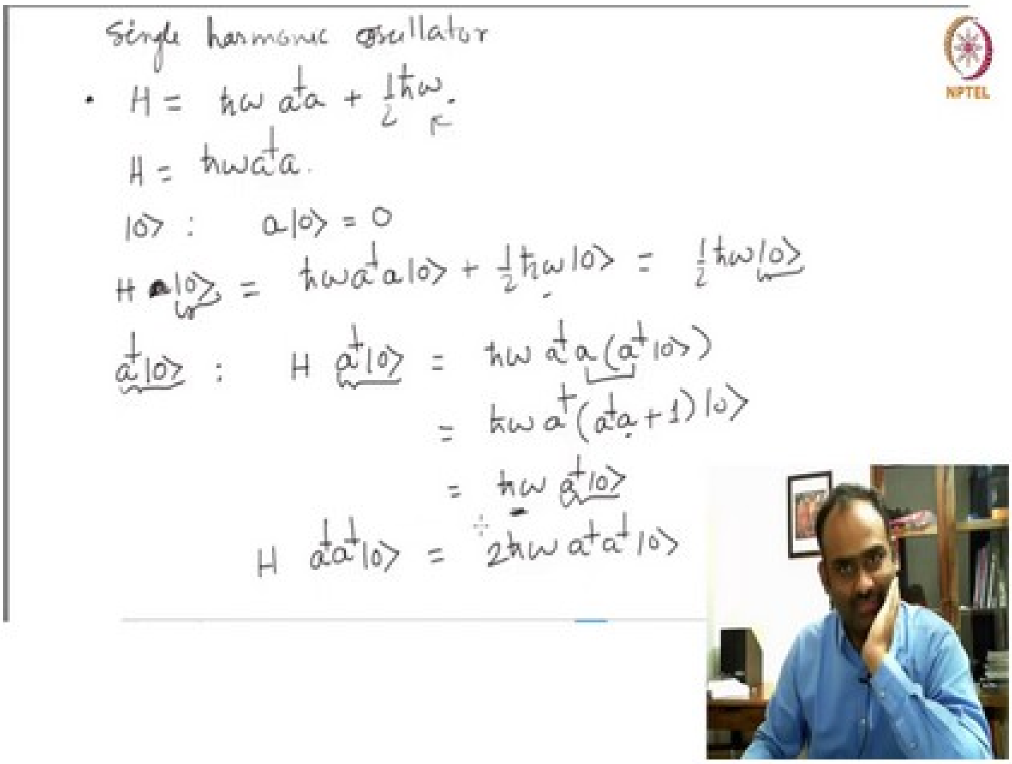

Figure 2: Refer Slide Time: 06:17

$$S[\psi(\vec{x},t)] = \int dt d^3x \left[\psi^*(\vec{x},t)\left(i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t)\right)\right] \tag{1}$$

$$\psi(\vec{x},t) = \sum_n a_n(t)u_n(\vec{x}) \tag{2}$$

$$hu_n = e_n u_n \tag{3}$$

$$p_n = i\hbar a^\dagger(t) \tag{4}$$

$$[a_n(t), p_m(t)] = i\hbar\delta_{mn} \tag{5}$$

$$[a_n(t), a_m^\dagger(t)] = \delta_{mn} \quad (6)$$

Construct the hamiltonian

$$H = \sum_{n=1}^{\infty} e_n a_n^\dagger a_n \quad (7)$$

Infinite number of harmonic oscillators with frequency $\omega_n = \frac{e_n}{\hbar}$

And then we constructed the Hamiltonian for this theory let me remove the timeline anyway. So, if you construct the Hamiltonian of this theory this quantum theory it turns out to be the following which is really a sum of infinite number of harmonic oscillators. So, the description of this system is this quantum system is that of infinite number of harmonic oscillators harmonic oscillators and these oscillators have different frequencies.

So, the frequencies are omega n is e n over h bar whereas y is the Planck's constant and also from the Hamiltonian it is clear that these different oscillators they are not interacting with each other even though the h hat here or the h here has a potential it has a V in it as you saw last time. But anyway after quantizing this system becomes that of harmonic oscillators which do not interact and there are infinite of them because the summation runs from one to infinity.

So, to proceed further it will be useful to recollect what you already know about harmonic oscillators. So, that is what I am going to do.

1.1 Single harmonic oscillator

$$\begin{aligned} H &= \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega && (8)\\ H &= \hbar\omega a^\dagger a && (9)\\ |0\rangle &: a|0\rangle = 0 && (10)\\ H|0\rangle &= \hbar\omega a^\dagger a|0\rangle + \frac{1}{2}\hbar\omega|0\rangle && (11)\\ &= \frac{1}{2}\hbar\omega|0\rangle && (12) \end{aligned}$$

$$\begin{aligned} Ha^\dagger|0\rangle &= \hbar\omega a^\dagger a(a^\dagger|0\rangle && (13)\\ &= \hbar\omega a^\dagger(a^\dagger a + 1)|0\rangle && (14)\\ &= \hbar\omega a^\dagger|0\rangle && (15) \end{aligned}$$

So, brief recollection of that single harmonic oscillator let us recall what we know already. So, I believe that you have already learned this. So, you know that for a simple harmonic oscillator the Hamiltonian is h bar omega a dagger a plus half h bar omega that is the Hamiltonian for the harmonic oscillator. So, right now I am looking at one harmonic oscillator. So, if you want to drop this term what we can say is let us say I am not interested in this term.

So, I subtract out from the Hamiltonian this ground state energy because this is the one which gives you the ground state energy. So, I will remove that part and I will just take the Hamiltonian to be h bar omega a dagger a I can do this here there is no problem. Now you define a state get 0 by saying that this state is such that if you take the operator a and act on this then it annihilates it that is the definition of the ket zero.

This ground state this gets zero and then you may ask what is the is it an eigenstate of the Hamiltonian. So, let us see let us ask what is this for a moment I will keep this entire full Hamiltonian here with the zero point term also. So, I have h bar omega sorry I am sorry this is not I am asking about Hamiltonian acting on this. So, you have h bar omega a dagger a acting on this plus half h bar omega acting on this because it has two pieces.

So, what you get here is this a acting on the ket zero will kill this state this gets zero that is how it has been defined. So, this piece goes away and what you are left with is half h bar omega at zero. So, clearly this state ket zero is an eigenstate of the Hamiltonian with energy half h bar omega if it is an eigenstate the eigen values energy. So, that is the energy of this ground state this state and you already know it is a ground state.

And then you also recall how to build up states in this theory you operate with a dagger acting on ket zero and let me give you an example of this. So, you ask what this is and you can start trying to figure that out by operating with the Hamiltonian on this state because maybe it is an eigenstate of the Hamiltonian and it is. So, you have now I am going to drop the this term half h bar omega I am not going to carry any more.

So, you have half h bar omega a dagger a and then you have this piece a dagger ket zero. So, what is this? This is h bar omega a dagger that is fine now if you could bring this guy this a next to the ket zero next to the vacuum then it will annihilate it right. So, that is what I am trying to do but for doing that I have to use commutation relations. So, the commutation relation between a and a dagger and a a dagger is 1. So, I will have let me look at this piece. So, this piece I will write in the bracket.

Now let us write round bracket it will give you a dagger a +1 acting on ket zero. Now when this operator acts on this ket zero it kills it. So, that term is gone you are left with only this. So, you are left with h bar omega a dagger ket zero. Now this a dagger ket zero is what you started with. So, clearly the a dagger keat zero this state is an eigenstate of the Hamiltonian with energy this much.

You already know that if I act with the string of a daggers on the ket zero I go up the ladder and they all are eigenstates. So, if you take a dagger a dagger twice and ask what is the energy of that state you will find 2h bar omega right. So, that is something I believe is familiar to you. Now we can use the same thing in our present context because our system is of after all a sum of lots of harmonic oscillators and they are not talking to each other. So, this formulation this method will work.

$$Ha^\dagger a^\dagger |0\rangle = 2\hbar\omega a^\dagger a^\dagger |0\rangle \tag{16}$$

Define: $a_n |0\rangle = 0 \qquad \forall\, n$

$$\begin{aligned} H|0\rangle &= 0 & (17)\\ Ha_m^\dagger |0\rangle &= \sum_n \hbar\omega_n a_n^\dagger a_n a_m^\dagger |0\rangle & (18)\\ &= \sum_n \hbar\omega_n a_n^\dagger (a_m^\dagger a_n + \delta_{mn}) |0\rangle & (19)\\ &= \hbar\omega_m a_m^\dagger |0\rangle & (20) \end{aligned}$$

$$\sum_n \hbar\omega_n a_n^\dagger a_n a_m^\dagger a_s^\dagger |0\rangle = (\hbar\omega_m + \hbar\omega_s) a_m^\dagger a_s^\dagger |0\rangle \tag{21}$$

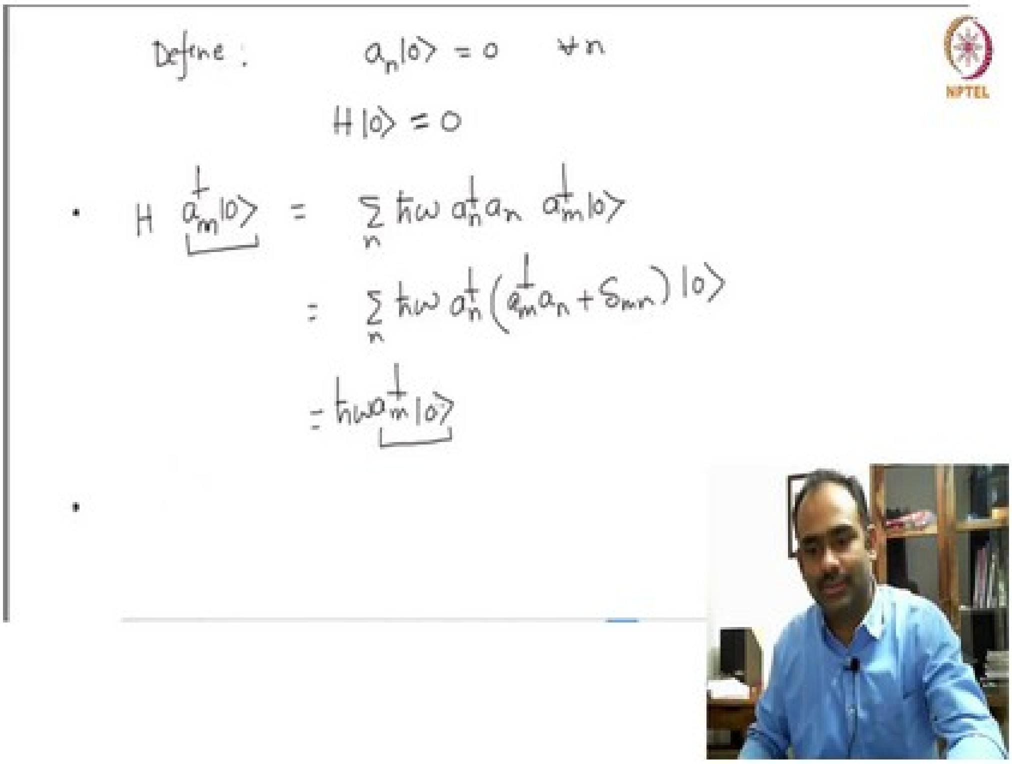

Figure 3: Refer Slide Time: 12:38

So, what we do is we define a ket zero to be that state which is annihilated by every n. So, this is 0 for all n. So, it is annihilated by a 1, a 2, a 3 all of them let us see and of course you can see that your h acting on ket zero. Now h is the Hamiltonian corresponding to infinite number of oscillators this will also be zero. Because your h contains an a on the right hand side and that a there is a summation over of course n but a n is going to annihilate so that is why you get this.

Now let us do the same thing as we did for single harmonic oscillator. So, we take a state a n let me write it m just for ease and act on. So, this is m and am dagger ket zero and ask is it an eigenstate of the Hamiltonian and that is easy. So, you have summation over n h bar omega a n dagger a n and then we have a m dagger ket zero right. So, you have summation over n h bar omega a n dagger now if I could take this a n 2 next to the vacuum here that will kill it as you saw before and that is what we want.

So, let us do it but this time it is not just 1 but you have a delta function. So, you have to put a delta m n and ket zero if you recall the combination relation I think we can go back a few here that is good. So, now this annihilates the vacuum. So, this term I can forget the only term which is relevant now is this one now this delta m n will force the n here to become m and the summation will go away. So, only that term is picked up you get a m dagger ket zero and of course you have n h bar omega. So, you see that this state is an eigenstate of the Hamiltonian with energy h bar omega. So, if you excite any of the harmonic oscillators to the first excited state that is that state where one oscillator is excited is an eigenstate of the Hamiltonian of your theory that is what it means. How about this maybe I should go to the next page how about this. So, I start with a n 1 a n 2 instead of m and n I am writing n 1 and n 2 that is maybe there is no need to do that for now. So, let us write a m a s and we have two daggers ket zero you can ask whether this is an eigenstate of the Hamiltonian and if so, what is the energy of this state. So, I will put h bar omega a n dagger a n here and there is a summation over n. So, what does that become I have been missing the n's here.

Here there should be an m there should have been an n let us find omega n and you have a n dagger. Now I want to take it through this string. So, I can just give you the answer this is a simple exercise you can do you have to just use the commutation relation and you will get the following. You will get this that this state is an eigenstate. So, you will get a m dagger a s dagger acting on ket zero again that is something which is going to happen and then you have h bar omega m + h bar omega s.

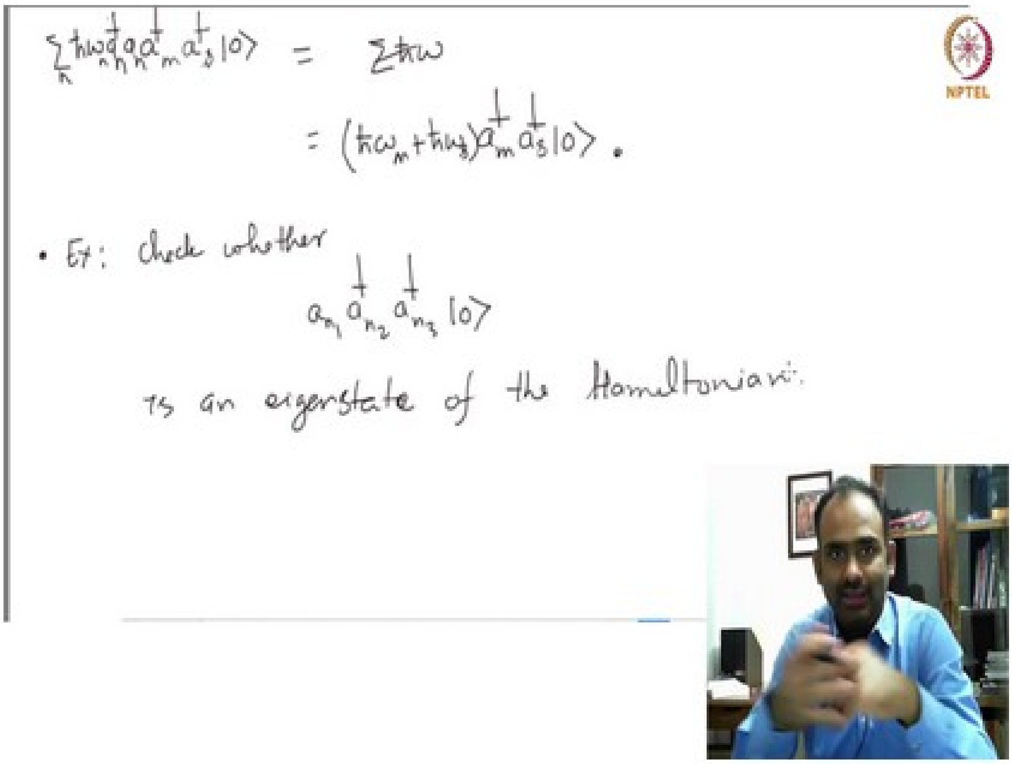

Figure 4: Refer Slide Time: 16:26

Exercise: Check wheather

$$a_{n_1} a_{n_2}^\dagger a_{n_3}^\dagger |0\rangle \tag{22}$$

is an eigenstate of Hamiltonian?

So, it will be just the sum of these two energies. So, that is another thing I will give you one small exercise to do check whether let us say a n 1 a n 2 a n 3 this one whether this one is an eigenstate of the Hamiltonian. So, I have not put a dagger here I put the daggers on these two but not on this one whether this is an eigenstate of the Hamiltonian. So, at least we have an understanding of the system what it looks like, how to construct states of this.

What I will next do is try to show you that this system which we have is equivalent to another system which will be the first which will be your first quantized theory. So, this is the second quantize theory because you have you know done two quantization's one you have started already with Schrodinger equation and then on the top of it you have quantized a's. So, I will take this system and it show that it is equivalent to another first quantized system.

And then I will make some remarks about why this description is useful and after one after I have done that I will leave this description and we will start doing quantum field theory for Klein Gordon fields. So, I have started this description using Schrodinger equation because this is a familiar equation and one can immediately jump into the field theory by quantizing something which we all already know about. So, let us meet in the next video.