Introduction to Quantum Field Theory

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Lecture 29 : Noether's theorem continued

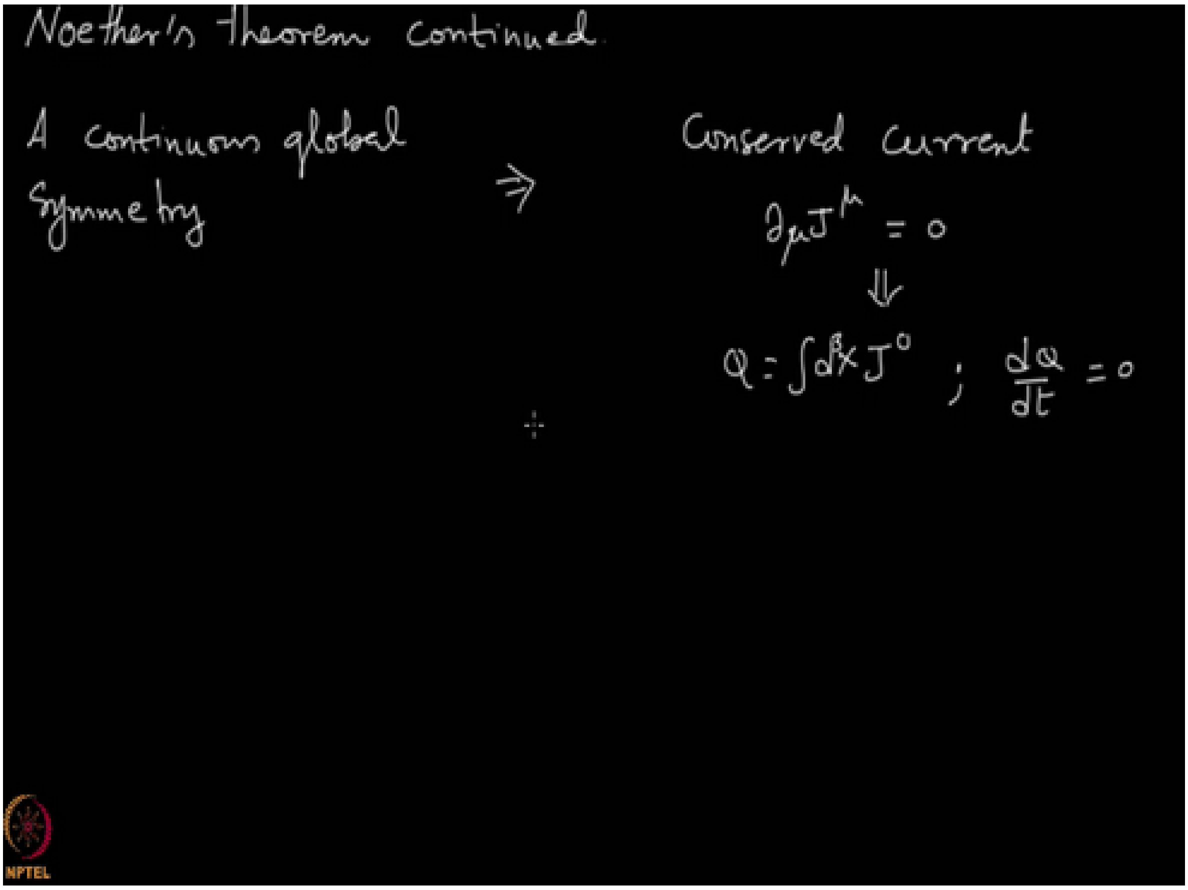


Figure 1: Refer Slide Time: 00:17

Last time I proved Noether's theorem which is a statement that if you have a continuous global symmetry then it implies a conserved current. So, continuous global symmetry that implies a conserved current that is you will have a quantity j mu whose when you take the 4 divergence that will give you zero. And this you saw that further implies that you can construct a charge q which is obtained by integrating over all space the j mu j 0 and which for which dq over dt is 0. So, the q is a conserved charge.

So, each every global symmetry that your theory would have that would lead to a conserved current and a corresponding charge and I also gave some examples of how you can destroy certain symmetries by putting some terms maybe I think I will do this explicitly. So, that we have at least one instance one example where we know how to spoil things also again I think I was referring to this one. So, if you have let me go to the next stage. So, let us take this action which is a functional of field phi. So, I am looking at Klein Gordon theory. Now remember that it is only the field over which you have control. So, if S is the function of phi and it takes different values depending on what field configuration you give to it. So, if you change the field configuration the value of S would change you do not have any control over x you cannot choose to do something to x.

It is an integration variable the only thing you have access to is the phi and I said that I want to choose a different theory not the Klein Gordon theory with a spoiled symmetry and I said I will keep all this the same but the second term I would change. So, I put an explicit x square here. Now let us see how the symmetry is spoiled. So, you know that this theory has translation symmetry meaning if I take the field phi of x and transform it to a phi tilde of x which is related to phi of x by this translation meaning the field at point x in the transform field is related to the field original field with the arguments translated.

So, that is the field transformation that we have and if I now just to remind you in this case we had S of phi tilde is equal to S of phi because the time and space translations what is the symmetry and let us look at this one now. So, if I calculate S of phi tilde you will get d 4 x tell me maybe I should just give the answer because I think we have done already these kind of manipulations before. So, I will just give you the answer. So, del mu phi of x so I have put in this expression replaced phi by phi tilde.

So, you will have in this expression phi of x plus a and then I change the dummy variables the variable x. So, that I get phi of x and then you get del mu phi of x this you will be able to easily do, I hope I have not made any mistake in this m square a x phi of x you should not do anything to this. Remember all you can do is change the field by phi tilde and then make this substitution but you cannot do anything to x you should not start putting x you should not change this to x + c or something like this you should not do only fields you can transform.

And once you do that and change the variables back to x you would get this. And this is clearly not equal to S of phi. If the change was a total derivative then we could have concluded that S of phi tilde is S phi because the total derivative you can ignore because on the boundary you can say the fields vanish but this is not the case here. So, you see that I have spoiled the symmetry of translation you should check whether you we have also spoiled or not spoiled the Lorentz symmetry.

So, you should check whether this is still a symmetry or not for this action please do this part. So, good. So, let us now apply Noether's theorem to Klein Gordon theory. That is what we want to do. So, we want to find the conserved current. But before we find the conserved current we should ask which symmetry we are talking about. So, let us list down the symmetries which we know already about this theory and see whether that symmetry would lead to a conserved charge or not. So, you know time translations; that is a symmetry then we know space translations they are symmetries.

We have already done this before. We already know that boost is a symmetry we know that rotations are symmetry. And actually yeah let me go further. So, and then you know time reversal is a symmetry and then you know space reversal is symmetry of Klein Gordon theory. We also saw that phi of x going to phi tilde of x which is equal to minus phi of x that is also symmetry and maybe there are more symmetries which we have not talked about maybe there are more to be discovered which we have not discussed here.

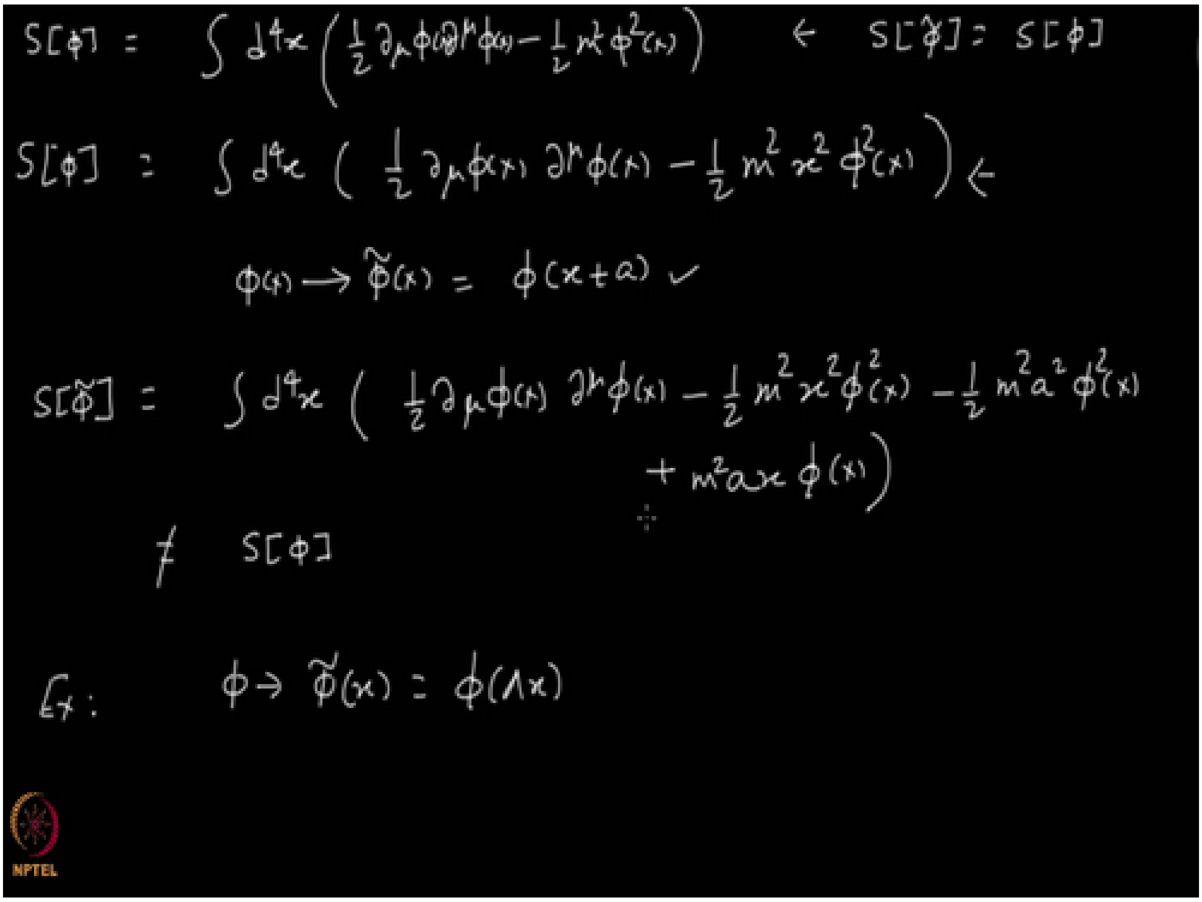
So, by more symmetries high I mean more symmetries of Klein Gordon theory because that is the theory I am looking at different theories will have different kinds of symmetries in them. It is not necessary that the symmetries which you have in this theory will be present in another theory that you can write. So, let us look at these symmetries. So, time translation let us see, lets ask whether it is a global continuous symmetry whether all the symmetries that I have listed are the global continuous and whether they will lead to a conserved charge.

A continuous global symmetry $\rightarrow$ conserved current

$$\partial_\mu J^\mu = 0 \tag{1}$$

$$Q = \int d^3x J^0 \tag{2}$$

$$\frac{dQ}{dt} = 0 \tag{3}$$



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So, time translation yes it is a global continuous symmetry. So, it will lead to a conserved current of its own space translation. So, these are three global continuous symmetries they would lead to three conserved charges and usually we call them momentum and the charge corresponding to time translation is called energy or the Hamiltonian. Now boost is also a continuous global symmetry. So, that would also lead to three conserved currents and correspondingly three charges.

Rotation is of course continuous global symmetry that would lead to ofcourse conserved charges, time reversal is it a global continuous symmetry? no it is not because this is phi of x phi tilde of x is equal to phi tilde of t x is equal to phi of minus t x. So, clearly there is no continuous parameter that parameterizes this transformation. So, it is not a global continuous symmetry it is a discrete symmetry.

So, you do not expect a conserved charge for this. Similarly, for space reversal; sorry again there is no parameter which parameterizes this one no continuous parameters. So, it is a discrete symmetry and again you do not expect any conserved charge this transformation of course it is there is no again no continuous parameter. So this is also discrete symmetry. So, this also does not lead to and if you could find other continuous symmetries which are present in the Klein gordon theory you would have corresponding currents and charges but we will only talk about these ones in our discussions. So, till now our discussion was in the context of when we were talking about Neother's theorem it was general but now we want to write an explicit expression for conserved current. See earlier we just said that we will have this kind of expression and we can read of as the coefficient of epsilon what is del mu j mu maybe I should go back and show you. So, you see here if you did a transformation phi going to phi prime where phi prime remember was not a symmetry transformation because it was phi prime was related to phi by this expression where epsilon is x dependent and only when epsilon depends on sorry only when epsilon is a constant this was a symmetry.

So, for this transformation we saw that we could based on general arguments we could say that the change in the value of the action would be this much at or epsilon. And then we can read off del mu j mu if you could express it like this. But now what I am going to do is give you an explicit expression in a special class of theories and that special class is the following where action is this. So, your Lagrangian density here and you have some fields phi i and Lagrangian density depends on first derivatives and may be second derivatives and maybe on higher derivatives.

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) \right) \tag{4}$$

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2x^2\phi^2(x) \right) \tag{5}$$

$$\phi(x) \to \tilde{\phi}(x) = \phi(x+a) \tag{6}$$

So, that is the general action that you could construct where there are several fields labeled by I but the restriction I am going to put is I will not include all these higher terms. So, my action would be just containing the fields and the first derivatives that is the class to which I want to restrict to and all the known interactions obey this fall into this category. So, all known interactions in nature have action of this form whether you are looking at electro weak symmetry strong interactions whatever you are looking at they all will always have first derivatives they don't have higher derivatives.

Now when we are looking at a translation symmetry the action S phi of tilde is going to be the same as S of phi and remember in writing this we do not use equations of motion this has to be true for any generic field configuration. So, whatever field configuration phi you take and if you do a symmetry transformation to phi tilde this should hold without using equations of motion that is important but for this to hold true you do not need that the Lagrangian density should remain unchanged.

So, even when Lagrangian density goes to Lagrange it transforms to Lagrangian density plus epsilon times some total derivative term that would still leave the action unchanged here epsilon is constant and I am here thinking of infinitesimal transformation that is why you see an epsilon here this should be clear because if you put such a term in the action and do the integral d 4 x this one you can ignore. Because this integral d 4 x del mu k mu that is a volume integral over four volume you can write it as integral d sigma mu k mu.

So, what we have done is we have turned the volume integral into a surface integral this is something familiar to you. and here it is boundary in space in space time and because we are going to assume that the fields vanish at the boundary this term can be ignored so this can be dropped. So, a total derivative change in the Lagrangian density is fine. Maybe here I should write. So, sigma is the boundary and d sigma mu is the boundary surface term boundary element.

And because I am assuming all the fields and the derivatives to vanish at the boundary it will give you a vanishing contribution. So, that is allowed of course but it may happen that in certain cases the Lagrangian density does not change at all. So, the k mu is absent in some in some cases. So, it may happen that Lagrangian density as a function of phi. Remember Lagrangian density is a function not a functional is as a function of phi tilde is same as this and the k mu is zero there is no q mu and typically it will happen.

When you have internal symmetry because internal symmetry is not going to touch space time points and if it doesn't involve space time then it cannot generate a space time derivative. Let me make this more concrete by giving example.

$$S[\tilde{\phi}] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{1}{2}m^2x^2\phi^2(x) - \frac{1}{2}m^2a^2\phi^2(x) + m^2a\,x\phi(x) \right) \tag{7}$$

$$S[\tilde{\phi}] \neq S[\phi] \tag{8}$$

translation symmtery spoiled

Exercise: Check for

$$\phi(x) \to \tilde{\phi}(x) = \phi(\Lambda x) \tag{9}$$

Noether's theorem for KG field

So, let us take a Lagrangian density which is a function of the fields and their first derivatives. So, the range in density is a function because if you specify the value of field and the first derivatives at a point you get a number. A function is that is a quantity which gives you a number if you specify some variables for example f of x is x square. So, if you choose a value of x you get a number here.

So, similarly here if you choose a value of given a field if you choose a value of x then that is the variable you have and these are remaining variables and that gives you a a number for the Lagrangian density. So, it is a function now I look at the symmetry transformation this. So, this is specified some theory specified to you and let us allow for multiple fields and then also I specify a symmetry that this theory has and I say.

So, there is an infinitesimal symmetry if you tell me what psi of x is then the symmetry is completely specified by this information. But let us keep it general we do not specialized to any particular symmetry we say this is the symmetry and then clearly if I am looking at the derivative terms which are going to appear in the Lagrangian density I can already write down the expression for it.

So, it will be just del mu of phi i of x + epsilon, epsilon is constant because we are looking at global continuous symmetries. So, the derivative acts only on this piece. Now I will do a Taylor series expansion. Taylor expansion, so, I should be calculating the Lagrangian density at phi tilde del mu phi tilde this is the transformed Lagrangian density as a function of transform fields and this would be to order epsilon to the 0 it should remain unchanged.

Because of this if I put epsilon equal to 0 you the phi tilde and phi tilde are exactly the same. So, I expect it to be the same original Lagrangian density and then corrections of order epsilon and what are those? So, because it is a function of these variables I will take a derivative with respect to these variables. So del L over del phi times the change in phi and that is epsilon times psi plus del L over this thing del of del mu phi I and then you get this change.

So, epsilon del mu psi I and of course r epsilon square terms which I which I omit and as I said that in general this would be the Lagrangian density can change by a total derivative if you have a symmetry. So, in general I would be able to write it as epsilon times del mu k mu for some k mu. Now remember here we have talked about the symmetry but the way we have formulated Noether's theorem we want to promote epsilon to continuous local symmetry meaning we want to take epsilon and make it x dependent.

And then read off the total derivative term that gets generated so that is what I will do. So, make epsilon a function of x. So, if I do. So, then I am not looking at symmetry transformation but I am looking at a different transformation which is let me write here phi prime of x I distinguish by the symmetry transformation with constant epsilon with this transformation by putting a prime here plus epsilon x psi of x that is psi I keep the same.

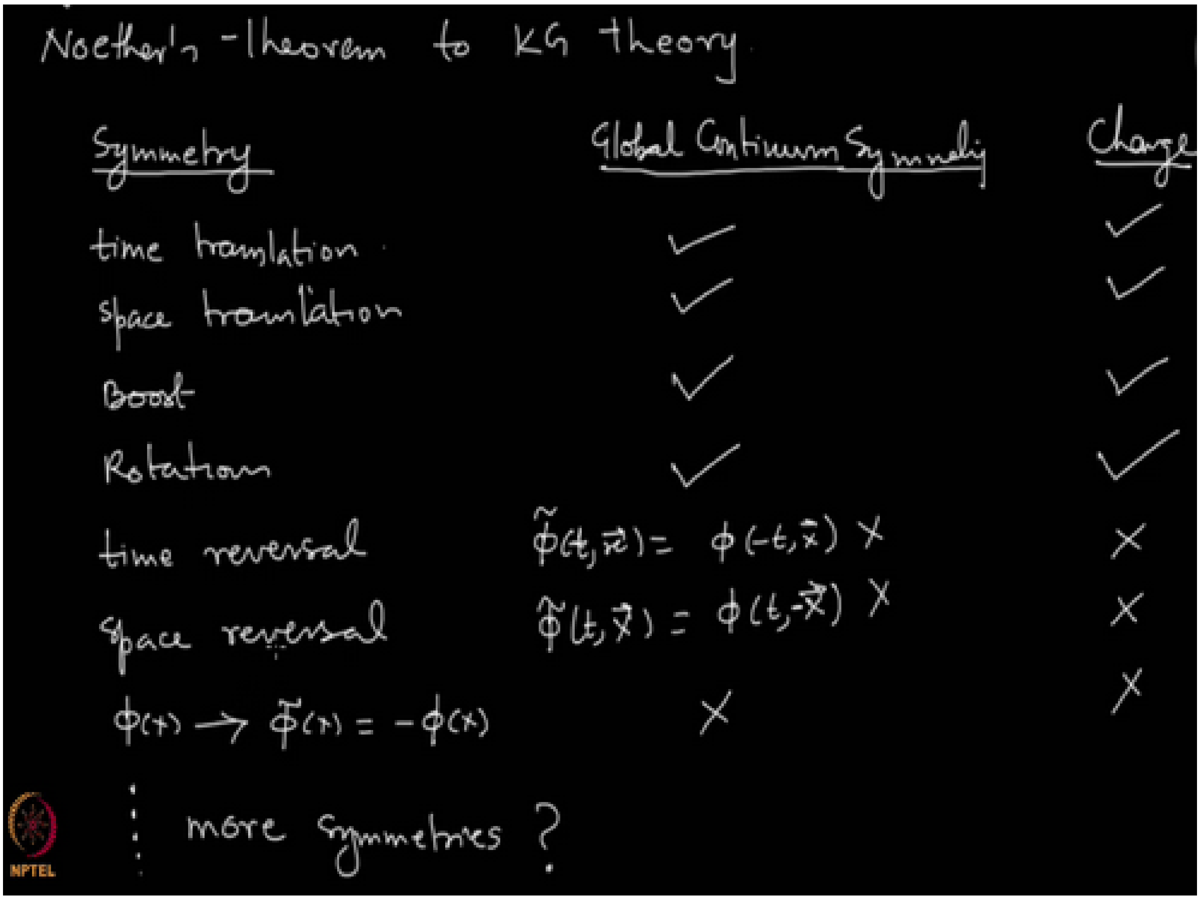
So, that when I take epsilon to become a constant I reach onto a symmetry transformation and similarly del mu phi I prime ok. So, let us now I think I need to go to the next page which is not the best thing to do I think we can remember this. So, we just need to remember this and I am going to the next page. So, let us see what is L of phi prime uh. So, L of let me write here. So, we have L of y prime is just a second there is someone at the door. So, sorry for the disturbance what was it. So, L of phi prime is what happened something changed yeah is going to be L of phi plus those order epsilon terms. So, I am going to write now l of 5 prime minus l what is the difference. So, L of and phi prime of x this difference is equal to that is what you have del L over del phi i times the change and the change is epsilon times psi i of x.

And then the change to the derivative and then the the change in the del m phi that is del mu of epsilon times psi that is just the Taylor series expansion plus of course higher order terms are there. Now I can do a simple thing this term is fine let us let me write it anyway no it is boring to write. So, many times. So, let us look at this piece this one I will write as I will just put the derivative here.

Symmteries	Global transformation	Charge
Time translation	✓	✓
Space translation	✓	✓
Boost	✓	✓
Rotation	✓	✓
Time reversal	$\tilde{\phi}(t, \vec{x}) = \phi(-t, \vec{x})$	×
Space reversal	$\tilde{\phi}(t, \vec{x}) = \phi(t, -\vec{x})$	×

So, you generate a term epsilon of x and the derivative acts on this one. So, you get epsilon of x del L over del del mu phi i times del mu psi and then there is another term in which the derivative x on the epsilon I think I should have there is no need to let me just write the answer that is fairly simple. So, you can you can do it. So, all you have to do is let me write the answer and then we can understand.

So, you have epsilon of unnecessarily deleted it is, let me let me write it anyway I should not change my plans in between. So, epsilon del L over del del u phi i times epsilon of x and del mu psi of x + phi i and psi i of x and del mu epsilon of x that is good. Now let us go back and check here when we were looking at the symmetry transformation. Then we had this piece epsilon del mu k mu and what was k mu? k mu was just the coefficient of epsilon from these two terms which is del L over del phi times psi and del L over del del m phi times w psi and that is the piece which you see here.



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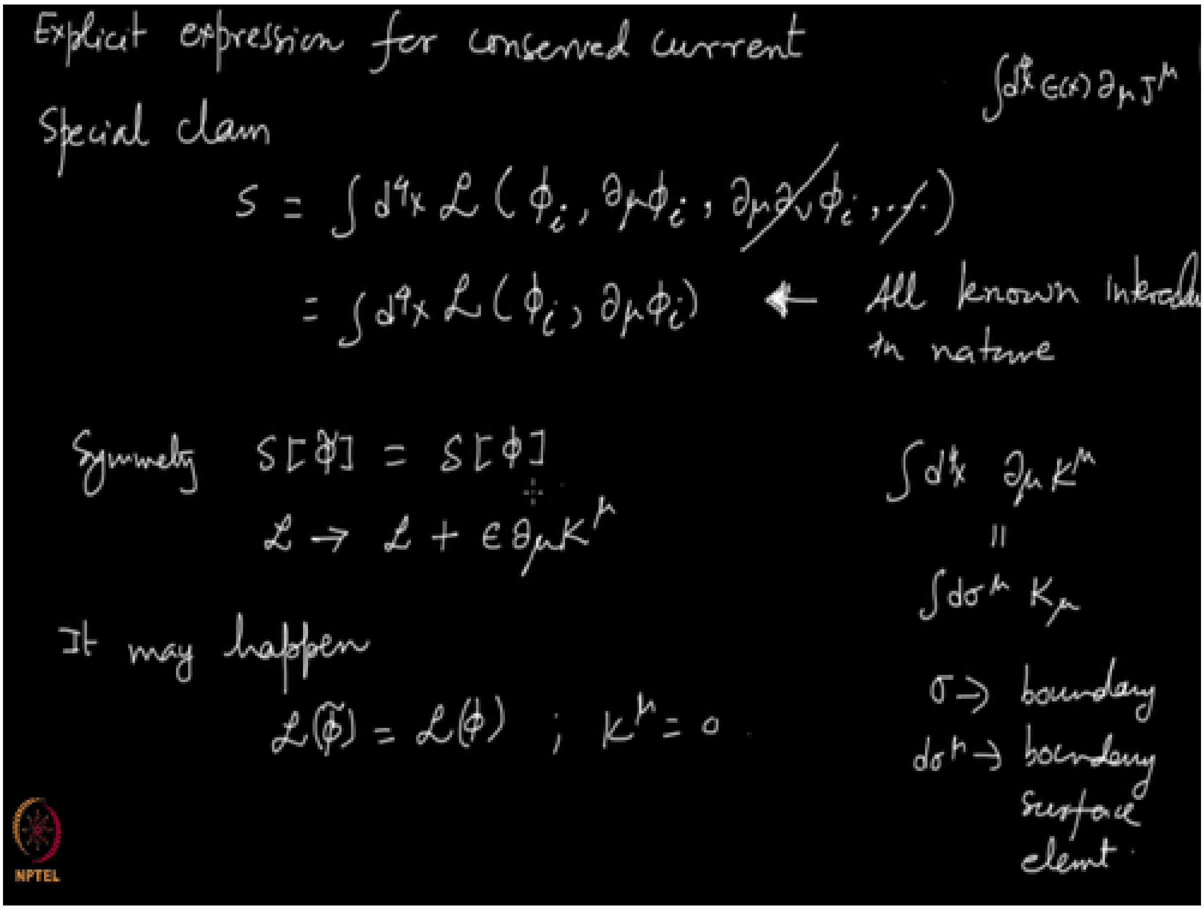


Figure 4: Refer Slide Time: 12:32

If you pull out this epsilon outside then this is exactly what you saw for k mu in that previous speed not k del mu k mu sorry we should not I made a mistake I said k mu but you have del mu k mu. So, these two terms I have taken care of and now we have this last piece. Now what I will do is I will transfer the derivative from here to this one which will give you a minus sign. So, minus del L over del del mu phi i and you get a del mu plus the derivative with all these terms together del mu epsilon of x del del mu phi i let us check whether everything is fine.

So, let us do this one. So, when del mu acts on epsilon. So, you get del mu epsilon times this piece which is this term and then when del mu acts on the remaining two you get epsilon outside. So, which means I have missed epsilon here. So, you get epsilon of x times del m of this entire thing and these two these two cancel and these one these first two terms anyway have used up in writing del mu k mu . So, these two together correctly give me the last term in this equation and the first two are already came. So, everything is here and the reason I have arranged it this way because you see this is a del mu total derivative term here and then when I look at the action under this transformation which is not a symmetry transformation this total derivative term I will be able to put to 0 I can ignore this term.

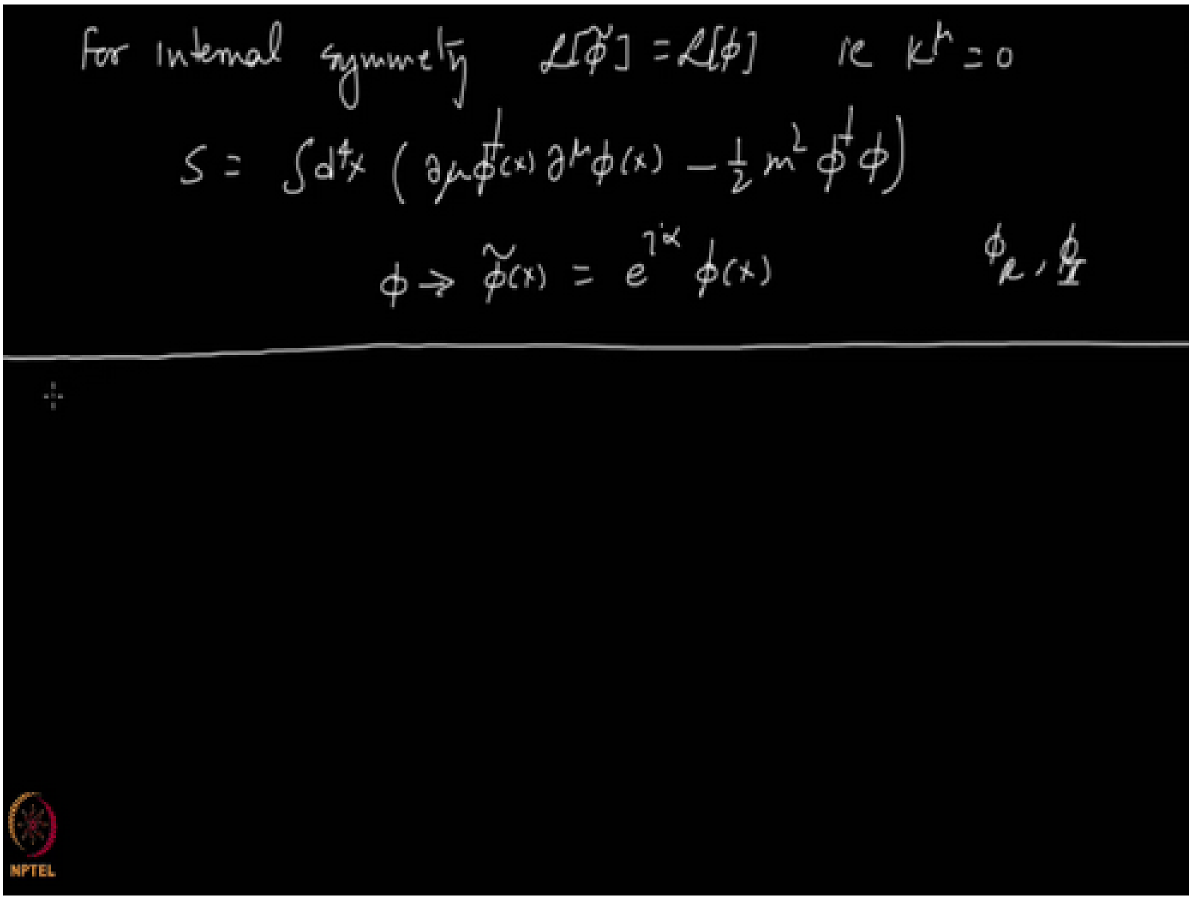


Figure 5: Refer Slide Time: 19:30

Explicit expression for conserved current
Special class

$$S = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu\phi_i, \partial_\mu\partial_\nu\phi_i \cdots) \tag{10}$$

$$S = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu\phi_i) \tag{11}$$

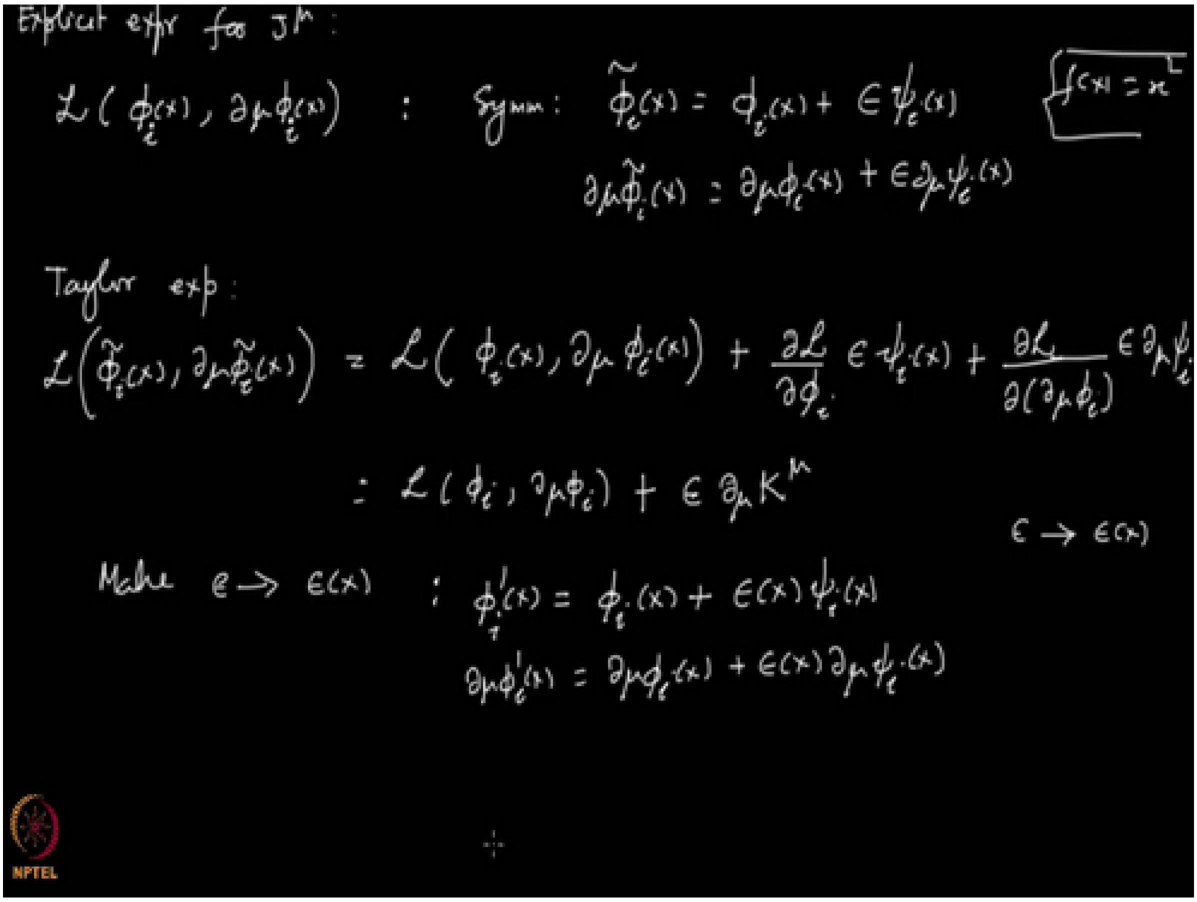


Figure 6: Refer Slide Time: 22:00

All known interaction will fall in this class

Symmetry:

$$S[\tilde{\phi}] = S[\phi] \tag{12}$$
$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon\,\partial_\mu K^\mu \tag{13}$$
$$\int d^4x\,\partial_\mu K^\mu = \int d\sigma^\mu K_\mu \tag{14}$$

Where

$\sigma \rightarrow$ boundary, $d\sigma^\mu \rightarrow$ boundary surface element
It may happen

$$\mathcal{L}(\tilde{\phi}) = \mathcal{L}(\phi) \quad ; \quad K^\mu = 0 \tag{15}$$

for internal symmetry, $\mathcal{L}[\tilde{\phi}] = \mathcal{L}[\phi]$ i.e. $K^\mu = 0$

$$S = \int d^4x \left(\partial_\mu \phi^\dagger(x)\,\partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^\dagger(x)\phi(x) \right) \tag{16}$$
$$\phi(x) \rightarrow \tilde{\phi}(x) = e^{i\alpha}\phi(x) \tag{17}$$

Explicit expression for J^μ

$$\mathcal{L}(\phi_i, \partial_\mu \phi_i) : \tag{18}$$

$$\text{Symmetry :} \quad \tilde{\phi}_i(x) = \phi_i(x) + \epsilon\,\psi_i(x) \tag{19}$$
$$\partial_\mu \tilde{\phi}_i(x) = \partial_\mu \phi_i(x) + \epsilon\,\partial_\mu \psi_i(x) \tag{20}$$

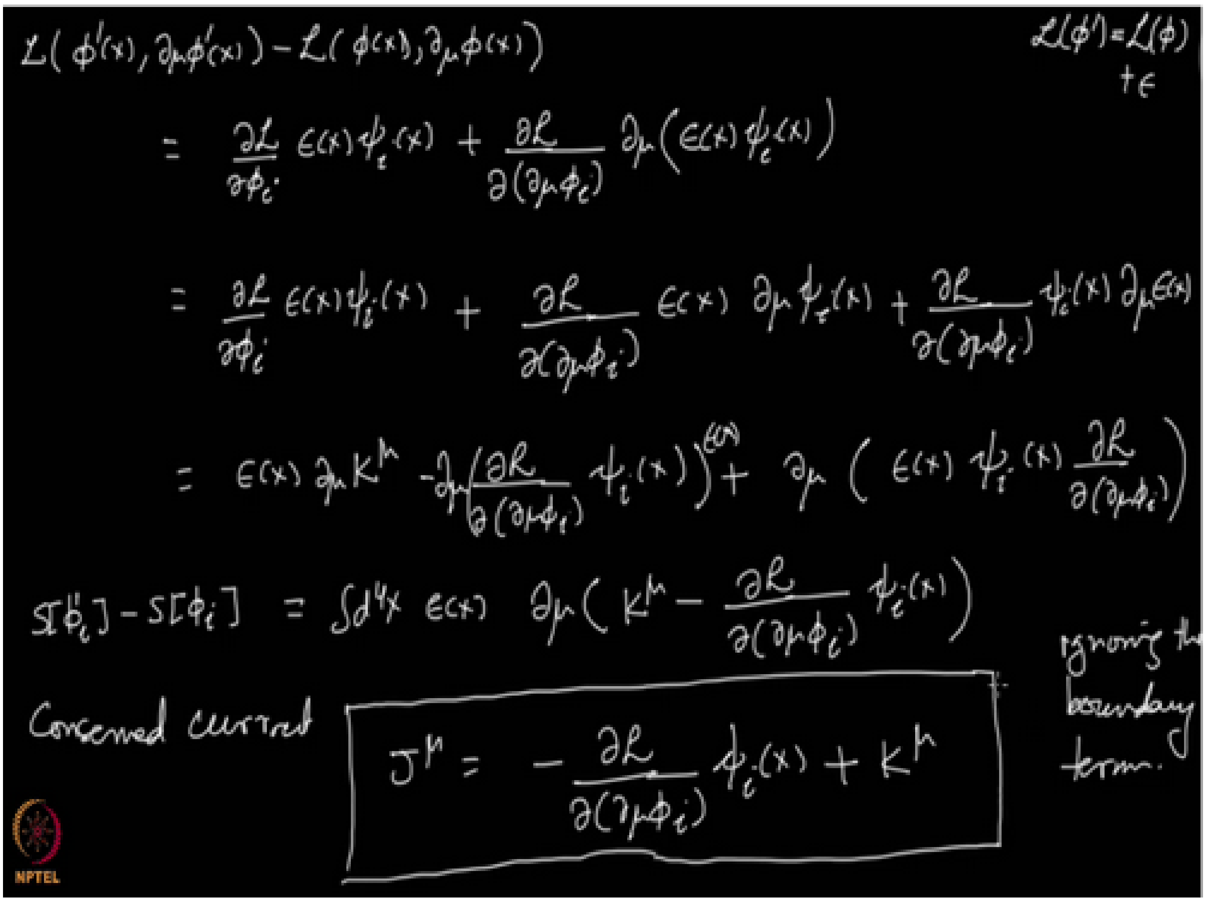
Taylor expanding

$$\mathcal{L}(\tilde{\phi}_i, \partial_\mu \tilde{\phi}_i) = \mathcal{L}(\phi_i, \partial_\mu \phi_i) + \frac{\partial \mathcal{L}}{\partial \phi_i}\epsilon \psi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\epsilon \partial_\mu \psi_i(x) \tag{21}$$
$$= \mathcal{L}(\phi_i, \partial_\mu \phi_i) + \epsilon \partial_\mu K^\mu \tag{22}$$

make $\epsilon \rightarrow \epsilon(x)$

$$\phi_i'(x) = \phi_i(x) + \epsilon(x)\,\psi_i(x) \tag{23}$$
$$\partial_\mu \phi_i'(x) = \partial_\mu \phi_i(x) + \epsilon(x)\,\partial_\mu \psi_i(x) \tag{24}$$

So, what we have now is S of phi i prime minus S of phi i is equal to integral d 4 x epsilon of x I am just substituting the Lagrangian density del mu. So, this also has a del mu and this also has a del mu del m acting on the whole thing and you have epsilon x here. So, I can write it as del mu k mu minus times psi i of x where psi x is the change due to the symmetry transformation. And this I am ignoring the boundary term ignoring the boundary term.



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So, now we just use our proof of Noether's theorem that were we read off j mu by looking at this thing the coefficient of epsilon x was del mu j mu. So, we get a conserved current j mu is equal to k mu or maybe I can write this one first psi i of x + k mu. So, that is our expression for the conserve current in the case of this special class of actions where only the first derivatives are present and I have made no assumptions about this.

So, they are the fields are general they are not necessarily scalar fields and this psi i mean I have not made any assumptions about what psi i can be I am not using scalar. So, there are n number of fields and they can be whatever they want and this is a generic transformation the k the capital K is the one which is the total derivative term which is generated when you look at the symmetry transformation. So, you should contrast this one this is different that is not due to symmetry transformation because epsilon is x dependent.

$$\mathcal{L}(\phi_i', \partial_\mu \phi_i') - \mathcal{L}(\phi_i, \partial_\mu \phi_i) = \frac{\partial \mathcal{L}}{\partial \phi_i}\epsilon(x)\psi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\mu\Big(\epsilon(x)\psi_i(x)\Big) \tag{25}$$

$$\mathcal{L}(\phi_i', \partial_\mu \phi_i') - \mathcal{L}(\phi_i, \partial_\mu \phi_i) = \frac{\partial \mathcal{L}}{\partial \phi_i}\epsilon(x)\psi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\epsilon(x)\partial_\mu\psi_i(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\psi_i(x)\partial_\mu\epsilon(x) \tag{26}$$

$$\mathcal{L}(\phi_i', \partial_\mu \phi_i') - \mathcal{L}(\phi_i, \partial_\mu \phi_i) = \epsilon(x)\partial_\mu K^\mu - \partial_\mu\Big(\frac{\partial R}{\partial(\partial_\mu \phi_i)}\psi_i(x)\Big)\epsilon(x) + \partial_\mu\Big(\epsilon(x)\psi_i(x)\frac{\partial R}{\partial(\partial_\mu \phi_i)}\Big) \tag{27}$$

Now

$$S[\phi_i'] - S[\phi_i] = \int d^4x\, \epsilon(x)\partial_\mu\Big(K^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\psi_i(x)\Big) \tag{28}$$

Ignoring the boundary term,

Conserved current

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\psi_i(x) + K^\mu \tag{29}$$

Conserved charge

$$Q = \int d^3x\, J^0 \quad ; \quad \frac{dQ}{dt} = 0 \tag{30}$$

Anyhow we have found the conserved current j mu is this much and that is what we wanted to know and now with this you can construct the conserved charge . So, let me write here itself. So, all you have to do is q is integral d cube x j zero that is the conserved charge and d dq over dt will be 0 good. So, we will continue our discussion further on Noether's theorem and we will talk about some examples.

We will talk about space time translations we'll talk about Lorentz transformations and all these things in the later videos.