Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 28 : Noether's Theorem The Proof

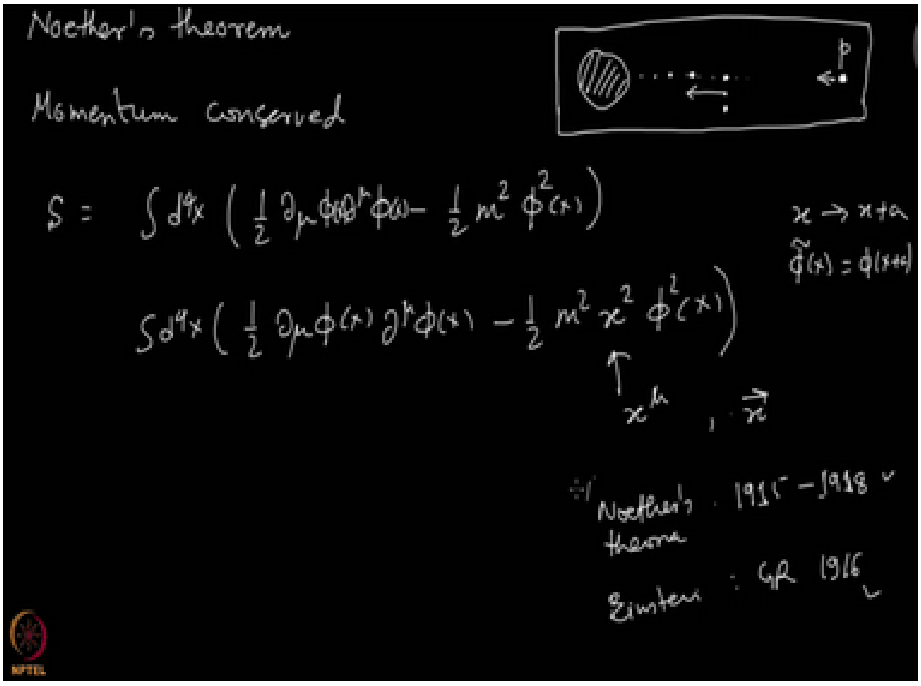

Figure 1: Refer Slide Time: 00:25

Today in this video we are going to start the discussion on Noether's theorem and we are going to derive this theorem which is a very profound theorem. But before I do so, I want to talk about symmetries not just in quantum field theory context but in the context of classical mechanics that you already are familiar with. So, you know that momentum is conserved. So, this is something which we have learned in our school that momentum; an energy is conserved.

And in your classical mechanics course you would have learned why these conservation laws hold and you must have learned that it is the invariance under space translations that leads to momentum conservation and invariance under time translations meaning if you have symmetry under translations and under time translation and under space translations then you have momentum and energy conservation laws.

So, for example you know that if you have a free particle I am just taking one example and if the space is homogeneous all these points are equivalent. Then if this guy has some momentum I mean you can assign it a momentum p which will not change with time because it is a consequence of space being homogeneous or having translation invariance. But suppose I want to spoil this symmetry I do not want to have space translation invariance.

Then of course momentum would not be conserved and that is also familiar to you. So, what you can do is you can put a massive object here say earth or sun or whatever and now different points in space are different. Now this point is not same as this point anymore. Because here when the particle is at this point it will experience a different strength of force compared to this one. So, different points are now different and you know that momentum of a particle which is falling on earth for example its momentum will keep changing with time.

So, if you were to release something at rest then momentum is zero but then it is going down its momentum keeps increasing. So, momentum is not conserved because you have destroyed the homogeneity of space by introducing a massive particle here or massive object here which is earth or whatever again you can of course spoil this translational invariance in many other ways. But if you were to take both of these together as part of the system this the particle and the earth then this entire thing together taken as a system is living in a space which is homogeneous.

So, if you were to take the entire thing and translate to here nothing changes and that is why you have conservation of momentum not individually for this particle and the earth but both taken together. So, momentum of the system would be conserved and we have been talking about different theories. So, one theory which we have been looking at is that of Klein-Gordon theory and we have seen that this is invariant under space-time translations.

Let me write this. So, this is the situation that you had when you had a free particle here it is equivalent to that like this system has translation invariance. So, what would be the analog of this situation where the translation is broken. So, let me spoil the translation invariance in this action. So, I will do something. So, that it gets spoiled. So, you can think of an action which is like this you can construct many more which will destroy the symmetry.

Momentum conservation $\rightarrow$ translation symmetry

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi(x) \, \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \right) \tag{1}$$

If we destroy the symmetry

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi(x) \, \partial^\mu \phi(x) - \frac{1}{2} m^2 x^2 \phi^2(x) \right) \tag{2}$$

Let us keep this term unchanged let us not spoil this one. If I have such a term then the translational invariance is gone. So, you can try to check whether action remains invariant when you take x going to x + a and phi of x factorial of x is phi of x + a. So, you will see because of this term it is going to be different clearly this x square here it is distinguishing different space time points because at x equal to 0 it is some you have something else at x is equal to some other value it has something you have the fields are multiplied with some other values.

So, different space time points are now behaving differently and you will not expect a momentum conservation here because of this broken symmetry. So, here is an example of how you can destroy certain symmetries in fact you can in fact destroy Lorentz transformations Lorentz symmetry also. So, now you have x square which is Lorentz invariant now imagine instead of this if you had some something else which does not respect Lorentz transformations at all. So, x mu or so, this will not be scalar anymore or whatever you can make it worse by putting just this.

So, that will spoil your Lorentz symmetry as well. So, whatever symmetry you want to destroy you can easily destroy by putting terms which do not respect the symmetry. So, now what I want to do is talk about conservation laws in quantum field theory. And that work is due to Noether and she found this profound theorem which goes by her name Noether's theorem and I was looking at some dates and I found that this was a work which she did in 1915 and I think one paper in 1918 I might be wrong but I am not trying to be very accurate in the details.

But this is the period in which this theorem was proved by her and so, let me write Noether's theorem and it was around the same time when Einstein, Albert Einstein was developing GR. So, he wrote his papers on GR starting 1916. So, this is the time when he gave GR. So, you see this is at the same time when this GR came about and Noether's theorem came about and both Einstein and Noether's have had profound impact on the way we do modern physics.

That entire thing thought process of doing modern physics is heavily influenced by the works of Einstein and Noether. So, all that have we have been doing. So, far that way of doing is heavily influenced by instant because it is all based on symmetry and Noether gave us the conservation laws that come out of symmetry and that is what we are going to talk about today

Transformation

$x \to x + a \quad ; \quad \tilde{\phi}(x) = \phi(x + a)$

the symmetry gets destroyed, similarly we can destroy any symmetry

1 Noether's theorem

So, here is the theorem. So, we are going to be looking only at I mean the theorem concerns invariance of action which we have been writing as S under a global continuous symmetry. So, global meaning the parameters which parameterize the transformations of the symmetry they are constants they do not depend on space time points and the symmetry is continuous it is not a discrete symmetry.

So, this is what theorem concerns about and because it is a continuous symmetry I can go close to identity meaning no transformation at all and look at infinitesimal transformations. So, I will be looking at infinitesimal symmetry transformations. So, let us assume that you have several fields in the problem phi 1, phi 2 and so forth let us say n of them which I will collectively denote as phi a and these could be scalar fields or vector fields or spinor fields or other fields whatever there.

So, let us take a infinitesimal symmetry transformation such that under this transformation a field configuration phi a of x. So, you have let me write in one dimension this is t, x. So, but I can only draw in 2. So, I have chosen this entire line to denote t, x and you are given some field configuration, something. So, I do a transformation and it goes to phi tilde a of x which is related to the original configuration by this.

So, the field gets modified at each space time point and the modification is equal to epsilon psi a of x. So, if you have and of course there will be other epsilon squared terms if you are looking at or you can in fact just say that I have transform field by this much and this is the symmetry transformation now because it is a symmetry transformation the action should not change. So, whatever the value of action was S is a functional it depends on the entire field configuration it gives you a number when you specify the entire field configuration.

So, S of phi a tilde is equal to S of and because it is a symmetry transformation I have assumed then this has to be just this it cannot change the value of the function cannot change under a symmetry transformation or rather that is when you say it is a similar transformation when action does not change and it is we should keep in mind that the field that appears here I have not assumed anything about it I have not assumed that it satisfies equations of motion this is a general field configuration there is no further assumption on it and because it is a symmetric transformation this will remain the same.

So, no assumptions have been made about five satisfying equations of motion that is good. So, let us see here now that is point number one we will this we are going to use I wish I could write maybe I should write it on the next page again.

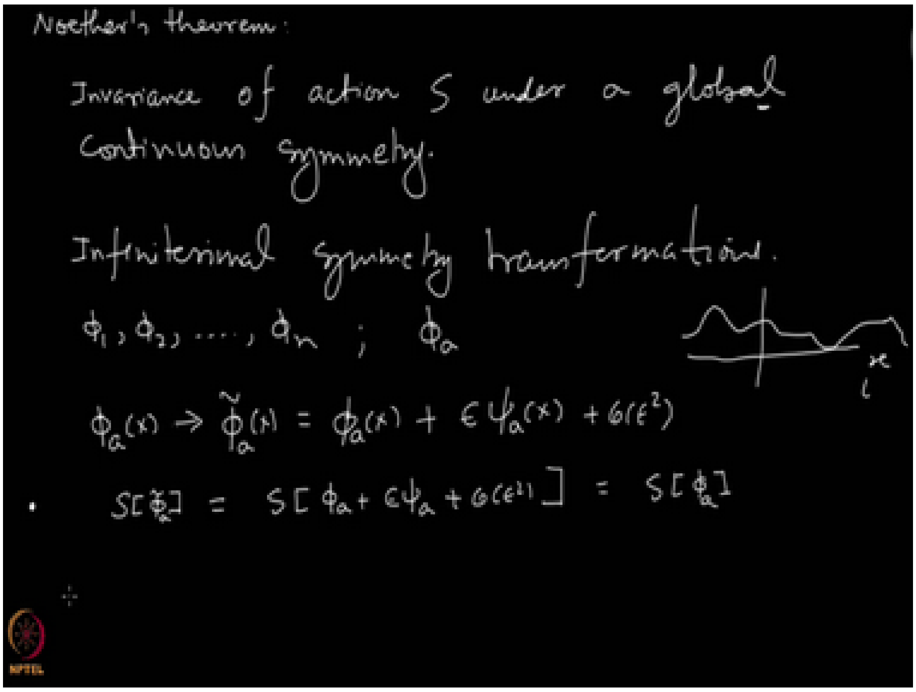

Figure 2: Refer Slide Time: 08:59

Invariance of action S under a global continuous symmtery.
Infinitesimal symmetry transformation

$$\begin{aligned} &\phi_1, \phi_2, \cdots \phi_n \quad ; \phi_a &(3)\\ &\phi_a(x) \to \tilde{\phi}_a(x) = \phi_a(x) + \epsilon\psi_a(x) + \mathcal{O}(\epsilon^2) &(4)\\ &S[\tilde{\phi}_a] = S[\phi_a(x) + \epsilon\psi_a(x) + \mathcal{O}(\epsilon^2)] &(5)\\ &S[\tilde{\phi}_a] = S[\phi_a(x)] &(6)\end{aligned}$$

under symmtery transformation

$$\begin{aligned} S[\phi_a + \epsilon\psi_a(x)] &= S[\phi_a] &(7)\\ \phi_a(x) \to \phi_a(x') &= \phi_a(x) + \epsilon\psi_a(x) &(8)\end{aligned}$$

not a symmtery transformation

$$\begin{aligned} S[\phi_a + \epsilon\psi_a(x)] &= S[\phi] + \mathcal{O}(\epsilon) &(9)\\ \mathcal{O}(\epsilon) &\sim \int d^4x \Big(\text{fields and derivatives and } \epsilon(x)\Big) &(10)\end{aligned}$$

So, under the transformation I have S of phi a + epsilon psi x this psi a is made out of the fields phi and their derivatives that is all that can happen. So, if you are changing a field the new pieces which will appear will be either made up of phi's or their derivatives is nothing else and epsilon is the small parameter of transformation and this we said that this is same as S of phi because it is a symmetry.

Now let us do something else now let me consider transformation which is not a symmetry maybe let me write it this way. So, consider a transformation where phi a goes to phi a prime maybe I should show the x and what is phi prime phi prime is related to phi in the following way it is I am sorry I want to do again sorry. So, here this is symmetry and what was the transformation I looked at it was this phi a of x goes to phi a prime of x sorry phi a tilde of x which is phi of x + epsilon psi of x.

So, this was the symmetry this was the symmetry transformation and that is the the consequence of interaction does not change. Now I want to look at not this transformation but a slightly different transformation where phi a goes to phi prime of x which is phi of x + epsilon of x psi of x this psi and this psi are same here epsilon was a constant it was a continuous parameter but a constant continuous parameter but here I have made epsilon a function of x.

So, this is still small this is still an infinite infinitesimal parameter but that parameter depends on space time points. Now we are saying that we are looking at global symmetries. So, this is a symmetry transformation but this is not. So, this is not a symmetry transformation. Now let us ask what happens to the action when I do such a transformation. So, if I start from phi x then there is some value of the action and what happens when you do a small change like this.

So, what we want to know is what is this object? How it is related to S of phi you can put up our epsilon square pieces also. So, because you have changed the field configuration which is proportional to an infinitesimal parameter epsilon the action should also change by an infinitesimal amount. So, it will change by some term which will be order epsilon and of course there will be higher order pieces also or epsilon square or absolute q and so forth.

Now I am interested in this object how much it has changed. Now let us write down the most general expression of this term. So, because S is a functional of the fields this object also has to be functional that that is that is of course obvious thing that it has to be a functional. So, and construct it out of the fields and its derivatives. So, this piece will be something of this form and something constructed out of fields and derivatives.

And of course this has to be an order epsilon term. So, epsilon also should appear inside and only one power of epsilon you cannot have more powers of epsilon here because the first term is order epsilon if you are looking at some other term which was or epsilon square that one will have epsilon square terms but this one will have only linear epsilon dependence. So, that is what we are able to say but then we have constructed this transformation in a particular manner such that if I take epsilon to be constant.

Then this transformation collapses to that transformation because this epsilon becomes the constant epsilon and psi are same I should not have missed a meaning in the limit epsilon becoming a constant this transformation should become a symmetry transformation. And this term should vanish because the moment you have symmetry transformation you should get only S of phi the value should not change.

So, it means that this further we can say that should be of the following form. It should be made up out of the fields and derivatives and epsilon dependence should be such that it should reflect this fact. And I can ensure that by saying that epsilon can depend only on derivatives because if you have a derivative let me go slow. Because if you have a derivative of epsilon then the moment you say epsilon is constant all the derivatives will go to zero derivatives of a constant object is zero.

So, from this we conclude that as far as the epsilon dependence is concerned it can be in this form. So, derivatives of epsilon what are those derivatives you could have tell me of epsilon or you could have two derivatives or a number of derivatives on epsilon. So, you have d4 x epsilon del mu epsilon of x times other objects which are made out of the fields and the derivatives plus you could have del mu del nu epsilon of x and of course you can have other things multiplying made out of fields and derivatives and so forth.

So, this much we are able to say but then you realize that this is the mu index is free here and I should contract to make it a Lorentz scalar. So, whatever quantity you put in here should have a vector index mu. So, let me call it k 1 mu and k when mu is the object which is made out of the fields and derivatives here you have two indices mu and nu. So, this object is a second rank tensor to make turn it into a scalar because your action is a scalar law and scalar I should have object with two indices mu and nu which I can contract with this mu and u to construct a scalar and so forth.

So, what I have said is that this is equal to maybe here same thing S of phi a + integral d 4 x del mu epsilon x k 1 + del mu del nu epsilon sorry x k 2 mu nu and so forth. But all the terms are linear in epsilon let me see if I wanted to say anything more here now and then of course there will be order epsilon square terms. So, this fact that the action should take this form depends on the fact that we are looking at a transformation which collapses to a symmetry transformation when we make the parameter a constant.

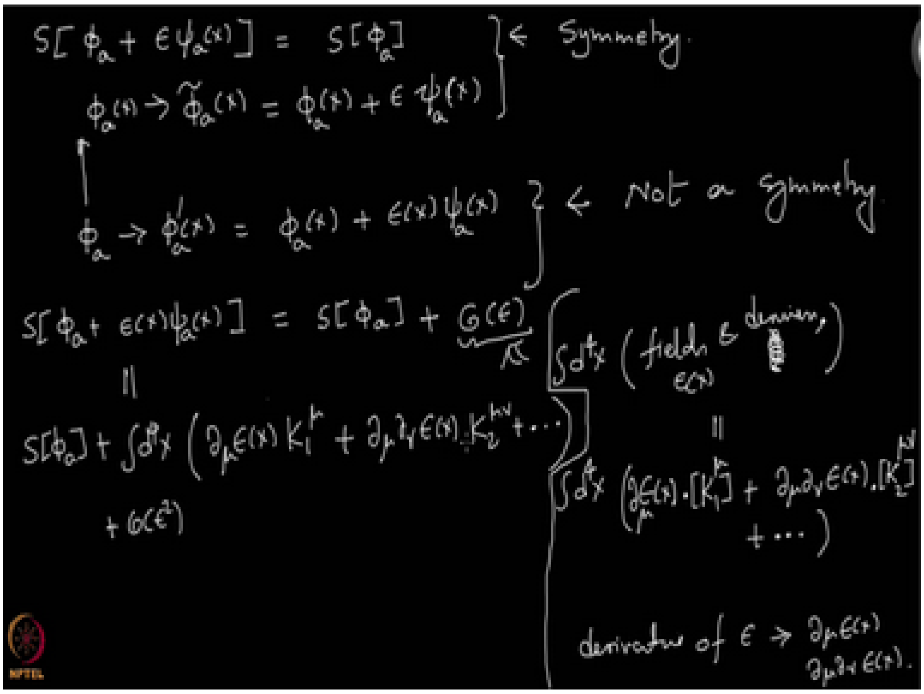

Figure 3: Refer Slide Time: 14:18

In the limit $\epsilon(x)$ becomes constant, it becomes a symmetry transformation

$$\mathcal{O}(\epsilon) = \int d^4x \bigg(\partial_\mu \epsilon(x)[K_1^\mu] + \partial_\mu \partial_\nu \epsilon(x) + [K_2^{\mu\nu}] \bigg) + \cdots \tag{11}$$

$$S[\phi_a] + \int d^4x \bigg(\partial_\mu \epsilon(x)[K_1^\mu] + \partial_\mu \partial_\nu \epsilon(x) + [K_2^{\mu\nu}] + \cdots + \mathcal{O}(\epsilon^2) \bigg) \tag{12}$$

$$S[\phi_a'] = S[\phi_a + \epsilon(x)\psi_a(x)] \tag{13}$$

$$= \quad S[\phi_a] + \int d^4x \, \epsilon(x) \, \partial_\mu J^\mu + \mathcal{O}(\epsilon^2) \tag{14}$$

Use equation of motion

$$\int d^4x \, \epsilon(x) \, \partial_\mu J^\mu = 0 \tag{15}$$

Now what I can do is I can take these derivatives and remove it from epsilon and put on the k. So, that the epsilon is freed and that I can do by integration by parts. So, I have to do integration by part once and that will go there and of course it picks up a minus sign and similarly I can do integration by parts and shift the derivatives on k and so forth. So, let me write it down. So, my action becomes now which is just. So, phi + epsilon x psi of x and this I have shown that this is equal to S of phi a + integral d 4 x and if I pull out the derivatives from epsilon and transfer to the other case I will have this form epsilon of x and then you will have del mu j mu and the j mu is constructed out of the k's and k 2's. So, if I pull the derivative here it becomes minus del mu k 1 mu. If I do it twice it becomes the two derivative get transformed on here and that entire thing you can write it as like this and this is because you see all the both the indices are contracted.

So, if you are not sure that it is obvious please do it in do the one step in between and transfer the derivatives here and here the two derivatives will transfer here and then pull out a single derivative like the tell me here and whatever remains is your j mu. So, what we have concluded is that under this transformation the order epsilon term changes like this or the order epsilon piece is this it has this specific form that epsilon is multiplied with del mu j.

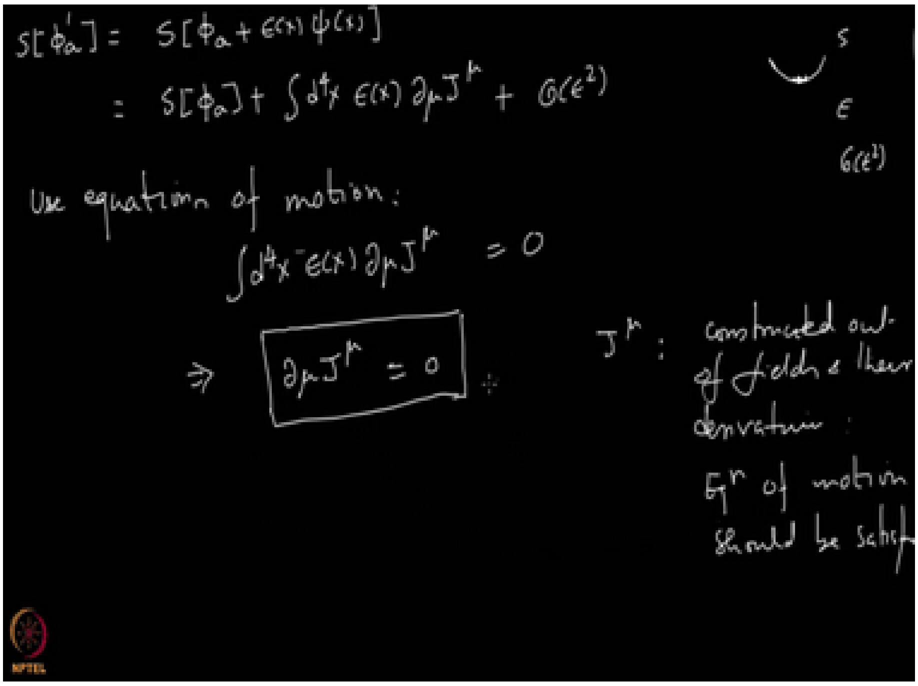

Figure 4: Refer Slide Time: 25:45

So, this is all we can say based on symmetry. So, all that has gone into this argument is just this symmetry information and then we have looked at an arbitrary transformation around the field configuration phase phi a now I cannot say anything more based on just symmetry but now what I will do is use equation of use equations of motion now I say that I am interested in those field configurations that satisfy the equations of motion and ask what happens then I can say more. So, if your field configuration phi is does satisfy equation of motion then you know that S is stationary at that point S takes S is stationary there.

So, if the field configuration satisfies equations of motion you are here. So, I am assuming this represents the value of S different points correspond to different field configuration and for field configuration that satisfy question motion you are here and the moment you make slight changes to field configuration from this you either reach here or reach there. And what is special here the special here is that if you change the field configuration by order epsilon terms the action does not change by order epsilon by changes by order epsilon square that is what it means to be stationary the first derivative is zero.

So, if I use equations of motion meaning if I satisfies equations of motion then this transformation which is written here under that transformation the action will become S phi plus this term cannot be there because of what I have said here and only order epsilon square terms will be there. So, your change will be order epsilon square and not order epsilon. So, if you use equations of motion then you have and where you remember that j mu is constructed out of fields that satisfy the fields and the derivatives.

And of course now they have to satisfy the equations of motion those fields which are entering into the j then only this statement is true. But here we never said what epsilon really is except for the fact that it should behave properly. So, that all the boundary terms that are generating when I am doing integration by parts those terms can be discarded. When you do integration by parts you generate boundary terms and I have said that the assume that the epsilon behaves in such a way that those terms can be discarded.

So, if you take epsilon to be 0 at the boundaries then you can discard the boundary terms but other than that I have not made any assumption about epsilon. So, what does that imply is it says that because this result this integral has to give you zero no matter what epsilon is this can work only if del mu j mu is identically zero not otherwise. So, this implies that del mu j mu is equal to 0 and this is a conservation law this is saying that the current is conserved and j mu let me write specifically constructed out of fields.

And the derivatives and equation of motion should be satisfied because if equations of motion are not satisfied you are stuck at this point beyond that you cannot say anything this state this statement relies on the fact that your action is not changing. So, that is the important thing which we have found that we have a conserved current and let me now manipulate this a little bit to arrive at a conserved quantity.

$$\partial_\mu J^\mu = 0 \tag{16}$$

Where J^μ is constructed out of fields and their derivatives

Equation of motion should be satisfied

$$\partial_0 J^0 + \partial_i J^i = 0 \quad , \quad \frac{\partial J^0}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{17}$$

Upon integrating

$$\frac{\partial}{\partial t} \int d^3x J^0 = - \int d^3x \vec{\nabla} \cdot \vec{J} \tag{18}$$

$$\frac{\partial}{\partial t} \int d^3x J^0 = 0 \tag{19}$$

$$Q = \int d^3x J^0 \tag{20}$$

and

$$\frac{dQ}{dt} = 0 \tag{21}$$

This is a conservation law, $Q \rightarrow$ conserve charge

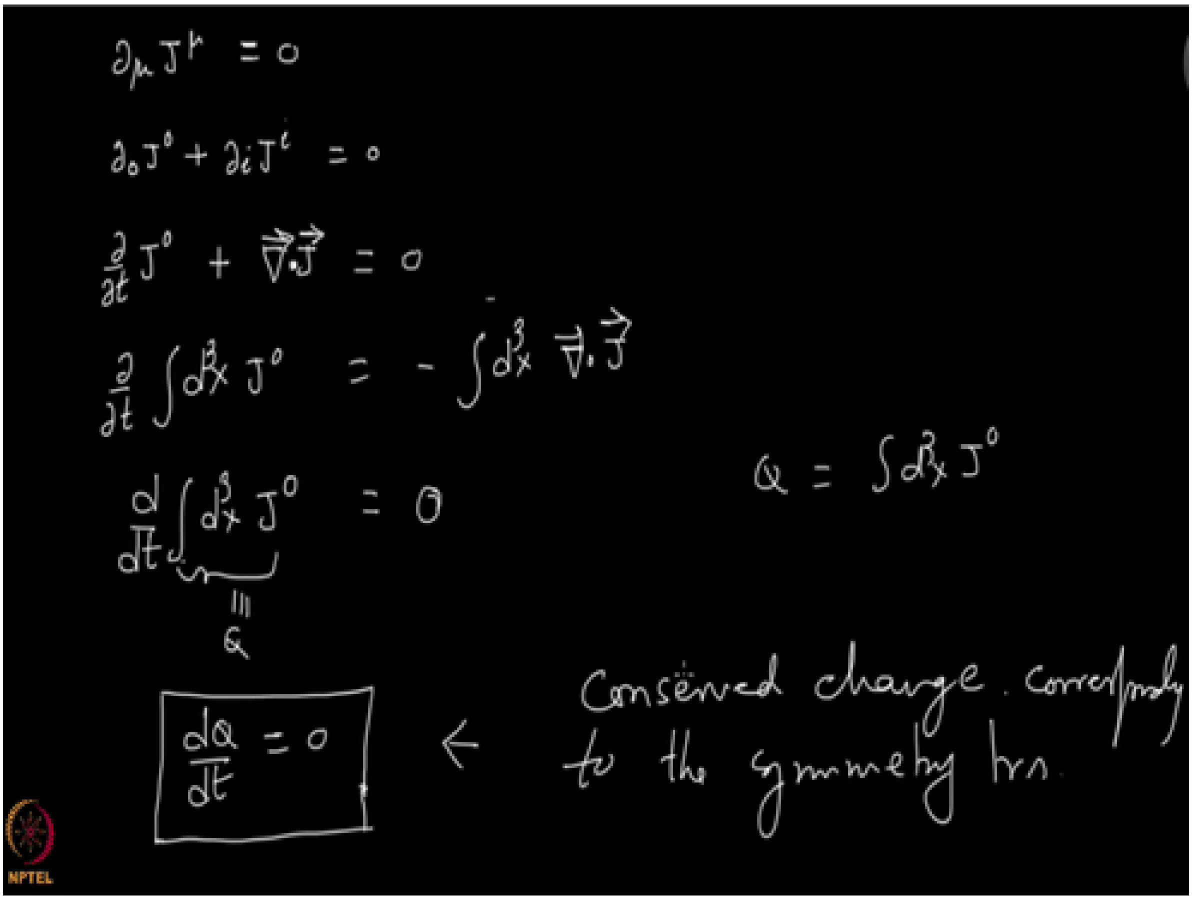

Figure 5: Refer Slide Time: 32:59

Conserved charge corresponding to the symmetry transformation

So, you have del mu j mu equal to zero and of course different field theories will have different j mu's and different within a field theory different symmetries will have correspondingly different j mu's. So let us write this explicitly this is just del 0 j 0 + del i j i and that is 0 this is just your partial derivative of time I can write it this plus this is gradient del i this thing is gradient. So, your divergence of j which is just saying that the time rate of change of this j naught at that is just the divergence of j.

So, now this is a local equation because at each space time point this is true now what I will do is I will do the integral over all space and time. So, I have del over del t integral d cube x j0. So, this I can take inside the derivative because this is integral over space and that is a derivative of time. So, these two can be written in like there is no problem. So, they commute. So, you have j naught and that this is equal to minus divergence of j.

Now here once you have done the integral over all the space there is no reference to any space points and this is no more a function of x there is no x in here and the partial derivative I can replace by total derivative there is no distinction now between d over dt and del over del t. So, I can write this as d cube x j 0 and on the hand side I assume that the current j there is no current flowing at infinity. So, what I am saying is I am looking I am turning this which is a volume integral into a surface integral.

So, instead of d cube x divergence of j I write it as a surface integral of j and if I assume that there is let us say the fields are vanishing at the boundary which is at infinity then this will be zero. So, if I take the fields to behave properly I can write this hand side to be zero let me call this object as q. So, q is integral d cube x j 0. So, this says that dq over dt is equal to 0 meaning q is a conserved quantity.

As fields evolve in time no matter what they do S as long as they satisfy the equations of motion dq over dt is going to be zero and this is the conservation law and q is called a conserved charge and we have proof Noether's theorem that if you have a continuous symmetry there will be a corresponding conservation law saying that dq over dt equal to 0. We will stop this video here and we will continue our discussion on Noether's theorem in the next video.