Introduction to Quantum Field Theory

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Lecture 27 : Theory of Scalar Fields

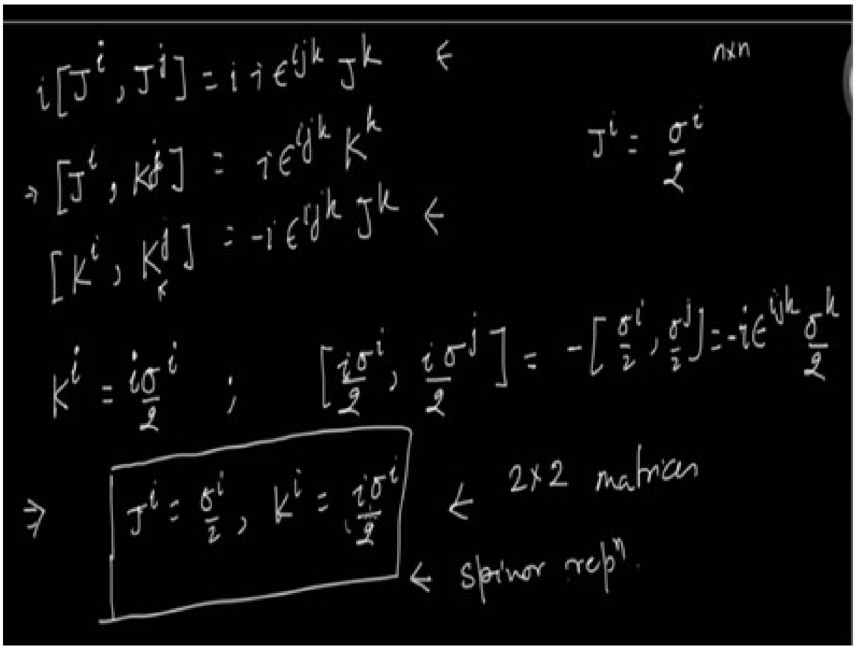


Figure 1: Refer Slide Time: 00:35

Let us continue our discussion on symmetries. But before that let me revisit what we were doing last time and I think I forgot to say some things and let me remediate that thing now. So, if I remember correctly, I only talked about these two commutation relations; the first one and the last one and I showed that if you take the generators to be this half times Pauli matrices for the J's and i times half the Pauli matrices for the boost generators and the first and the third commutation relations are satisfied.

But that is not sufficient we have to ensure that all the commutation relations are satisfied and I forgot to say anything about the second set of commutation relations this one. But we can see that this is also satisfied if you plug in the J's and the K's which are here you will see that this is satisfied. And what you can do is you can take the first equation multiply on both the sides with i.

So, here if I multiply i here you can put it next to this and that is what our K is and extra i that I multiply here I can put it with this one and that makes it a K and this is what you have here. This is a K which means these forms the correct representation of the generators. Let me undo this. So, this argument is fine now.

$$[J^i, J^j] = i\epsilon^{ijk} J^k \tag{1}$$
$$[J^i, K^j] = i\epsilon^{ijk} k^k \tag{2}$$
$$[K^i, K^j] = -i\epsilon^{ijk} J^k \tag{3}$$

Let us resume today's discussion on symmetries. So, we have till now talked about several kinds of symmetries and it will be a good idea to just classify them properly. So, some of the symmetries that you have encountered were of the following types where if you take a field phi of a, so I am assuming there are several fields in the problem and under your symmetry transformation this field could get transformed to this where the transform field is related to the original field configuration by this relation.

This is one of the transformations that you have seen, this is space translation where you take the field and translate the entire configuration by vector a which means you transform the fields, transform the entire field configuration by a 1 in x direction, a 2 in y direction and a 3 in z direction and that gives you your new field. So, this kind of transformations or symmetry transformations, these are continuous transformations.

They are continuous because they are parameterized by a continuous parameter a or in this case three continuous parameters. So a is or ar whatever is a constant parameter and you can change its values continuously, so that is why it is a continuous transformation. If your transformation is not parameterized by one or more continuous parameters, then it is a discrete transformation. And an example would be for example you take Klein-Gordon field and you look at a transformation which gives you the new field phi tilde of x which is related to the original field or old field by this relation. Here you have discrete transformation or even you could have thought of this one. Again, there is no continuous parameter, so it is a discrete transformation.

Now there is a distinction between this kind and these two. These two transformations, the first and the third, they are related by changing the argument in the space time arguments, not this one, this one. So, here there is a change in the argument. So, here x you have here x + a, here you have x, here you have –x. So, these are space time symmetries. These we could say space time symmetries.

So, I say this one and this one are space time symmetries, spacetime transformations and if your action is invariant under it, it will be space time symmetry of your problem. And this one is clearly not a space time symmetry, does not involve any change in the space time points. So, your new field configuration is not related to the original field configuration by some space time transformation.

These ones are called internal transformations or internal symmetries. If your action is invariant under such a transformation you say your system has an internal symmetry. So, let me give you one more example of internal symmetry. So, think of phi a x, now instead of t, x I am writing this x. If this goes to phi tilde a x, where U is the unitary matrix, then this is an internal symmetry.

So, this one and this one, these are internal symmetries; internal symmetry transformations provided that keeps the Lagrangian unchanged. Let me give you some more examples of space time transformations. Of course, you have seen already that you can look at transformations like this where lambda is Lorentz matrix here or in fact we saw in the last video that there are more general class of fields which will transform under some representation of the Lorentz group. And your fields could be transforming under those representations or in fact I was writing them by inverse. So, this is another space time transformation. One more I can tell. So, you could have the transform field to be related to the untransformed field by this where alpha is a constant and we have seen that this transformation is a symmetry of Klein-Gordon theory.

So, this is also space time symmetry because here the argument of the space, the space arguments they are scaled up. So, there is the scaling transformation. So, there are several such transformations that we have and they are space time transformations or internal transformations, internal symmetries.

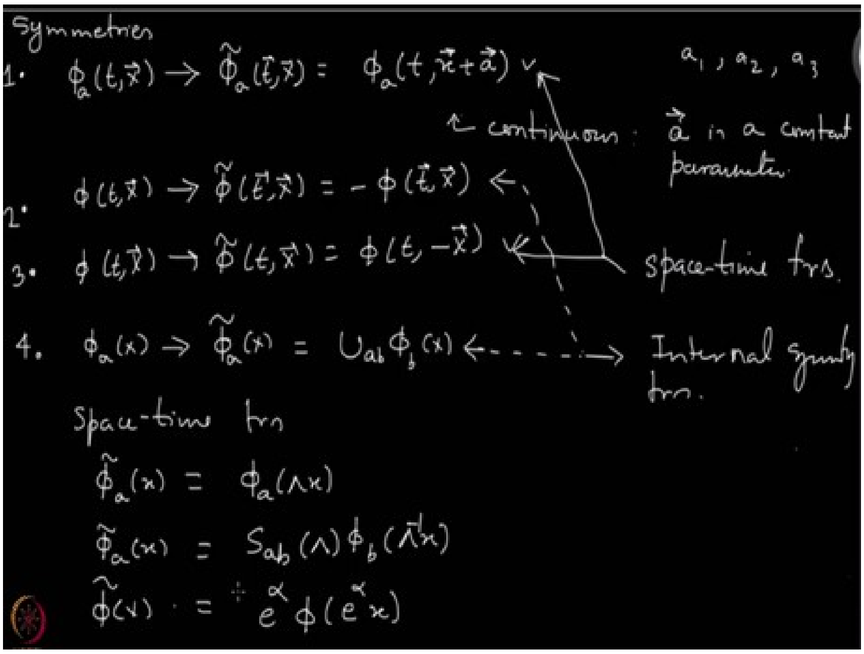


Figure 2: Refer Slide Time: 02:41

Where,

$$J^i = \frac{\sigma^i}{2} \tag{4}$$

$$K^i = \frac{i\sigma^i}{2} \tag{5}$$

Symmetries

1. $\phi_a(t,\vec{x}) \rightarrow \tilde{\phi}_a(t,\vec{x}) = \phi_a(t,\vec{x}+\vec{a})$ This is a contiuous symmetry because it is characterised by continuous parameter - a
2. $\phi(t,\vec{x}) \rightarrow \tilde{\phi}(t,\vec{x}) = -\phi(t,\vec{x})$
3. $\phi(t,\vec{x}) \rightarrow \tilde{\phi}(t,\vec{x}) = \phi(t,-\vec{x})$
4. $= U_{ab}\phi_b(x)$
5. $\tilde{\phi}_a(x) = \phi_a(\Lambda x)$
6. $\tilde{\phi}_a(x) = S_{ab}(\Lambda)\phi_b(\Lambda^{-1}x)$
7. $\tilde{\phi}(x) = e^{\alpha}\phi_{(}e^{\alpha}x)$

Continuous symmetry

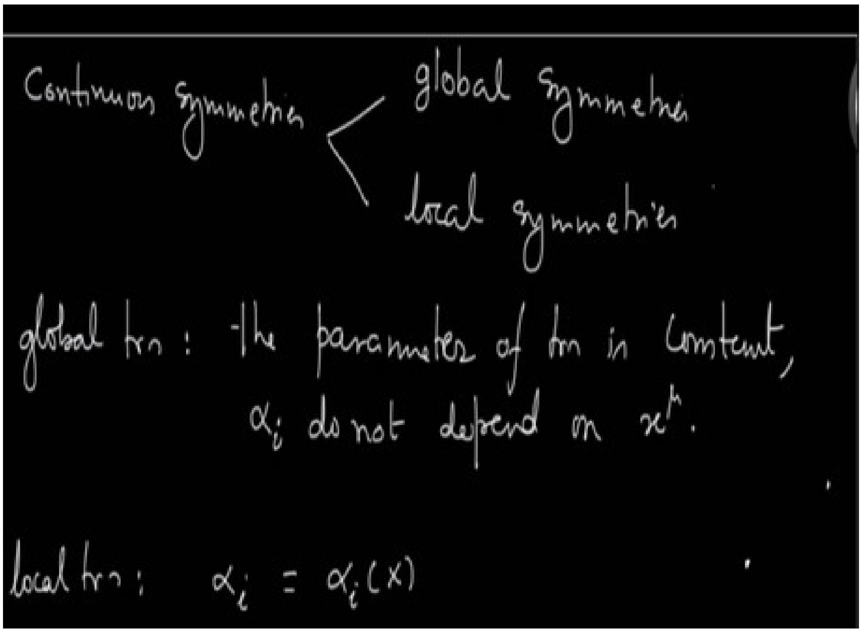
Now, I want to make a further classification. The continuous symmetries these could be further divided into two, one set is called global symmetries, another is local or gauge, local symmetries or gauge symmetries. Now, the distinction between these two is the following. Of course, in both of them you have a one or more than one continuous parameter which parameterizes the transformation.

But in the case of global transformation the parameter of transformation is constant. By constant I mean it does not depend on space time points, is constant that is if I denote the parameter by alpha or alpha i if there are more than one, then the alpha i do not depend on this. Alpha i do not depend on x or x mu. So, global, global means what everywhere right, so when you do a transformation, the transformation is done in the same manner at all space time points.

So, it is a global transformation. The transformation does not change from this space time point to that space time point, so it is global in the sense that it is the same transformation everywhere that is why it is called global transformation as opposed to local transformation here the parameters alpha i which parameterizes the transformation, they do depend on the space time points.

So, they depend on the space time points and that is why they are called local, local means relating to that particular point. So, the transformation would be different here and different at that point and different at that point. So, the transformations depend on those points. So, they are locally changing the fields and they may be changing fields in different manner at different points. So, that is called a local transformation.

And as you may notice that all the continuous symmetry transformations that I have written down till now in these lectures, they all have been global transformations. So, it will be good idea to give you one example of local transformation so that it is clear what we are talking about.



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1. Global symmetry

2. Local symmetry

Global transformation- The parameter of transformation is constant, α_i do not depend on x^μ.

So, an example of local transformation or gauge transformation. So, I think I wrote at some point of time the section you do not need to know really what it is but just for completeness you know this is the action for free Maxwell fields where F mu nu is del mu A nu – del nu A mu. So do not have to worry if you are not familiar with this already, we do not need to know much about this.

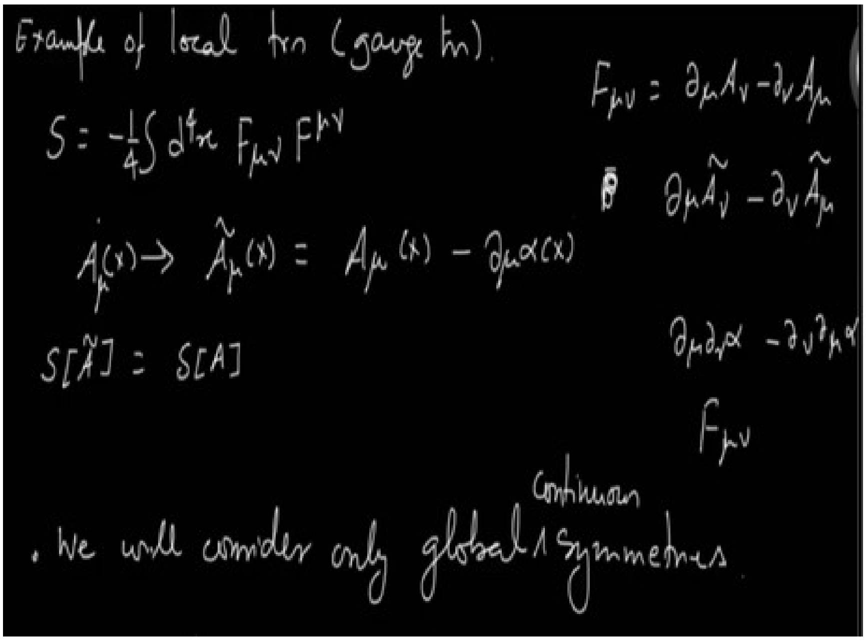
All I am going to show you is that this theory has a local symmetry. So, think of this following transformation. So, suppose A mu of x goes to this new field configuration which is related to the original field configuration by this. So there is a parameter alpha which I can change, but that parameter is a function of x and that is why this is a local transformation because it depends on x.

Now, you see that the action S as a functional of A tilde is going to be same S as the functional of A because here you have del mu alpha and F mu nu this is you del mu A nu – del nu A mu, so this piece del mu so what I am saying is construct del mu A nu tilde –del nu A mu tilde this is what you have to do. You have to replace the fields by the transformed fields in the action.

When you do so, this piece will give you del mu del nu turn from here del mu del nu alpha and that one will give you del nu del mu alpha, but these two are same, so they cancel because of the minus sign and that is why your F mu nu is unchanged under this transformation and because your action is built out of two F's, your action will not also change under this transformation.

So, this is an example of a local symmetry transformation. So, we will talk about only global continuous symmetries in this course, we will not be concerned with local symmetries. So, we will consider only global symmetries, global and of course continuous symmetries. So, of course the symmetries will leave no imprints on the theory.

So when you are looking at the objects that you construct in this theory, they will know that there is an underlying symmetry in the theory and there are many ways in which these symmetries will manifest themselves and let me show you first set of consequences of the symmetries which is the following.



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Local transformation-The parameter of transformation is function of x^μ, the transformation varies at different points, $\alpha_i = \alpha_i(x)$

Lets take a local transformation

$$S = -\frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\mu\nu} \tag{6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

This theory has local symmetry

So if field configuration, let us say you have several fields and let us label them by phi a. If phi a satisfies equations of motion, then if under the symmetry transformation phi a goes to phi tilde a of x, then phi tilde x also satisfies the same equation of motion. Again, it is very straightforward to understand why that should happen, but let me give you a few examples before I give the argument that why this has to hold true.

So, let us start with our Klein-Gordon theory. So, for Klein-Gordon theory you know that the equations of motion are this or I can write it more, I mean I can again I did with the indices like eta mu nu del mu del nu this operator del square the same square. Now, let us take field configuration which you know that satisfies this and later I will anyway give you an argument in general, but let us choose one particular solution.

So, I know that if I take phi of x to be e to the i k dot x, this does satisfy equation of motion that we can very quickly check. So, substitute phi of x in here or in this one, so you have eta mu nu when you take the derivative with del nu it will pull out i k nu and again phi of x because it is an exponential and then when you again take the derivative it will pull out now i k mu and again the phi get back and then m square and then you have of course phi of x. And here e to the ik dot x here the solution, I mean the K's are constraints, so constraint is that K 0 square – K 3 by 2 square is m square, this constraint you have an act on K and here you see that i squared is –1 and this makes it – k square + m square phi and k square is m square, so –k square is –m square and then this is 0. So, indeed that phi satisfies this equation of motion. So, this is true a solution.

$$S = A_\mu(x) \to \tilde{A}_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x) \tag{8}$$

Now construct $\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$

$$\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu = \partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha \tag{9}$$

Hence

$$S[\tilde{A}] = S[A] \tag{10}$$

We will consider only global continuous symmetry.

If $\phi_a(x)$ satisfies any equation of motion then $\tilde{\phi}_a(x)$ also satisfies the equation of motion for KG theory.

Now if you take new field phi tilde of x, which is e to the ik and change the x to x + a where this a is basically a mu because we have x mu here, but anyway I am writing like a dot product, so it is fine and you can immediately verify that this also satisfies this equation of motion and because when the derivative acts it does not do anything to this one because it only pulls out, it acts only on this part.

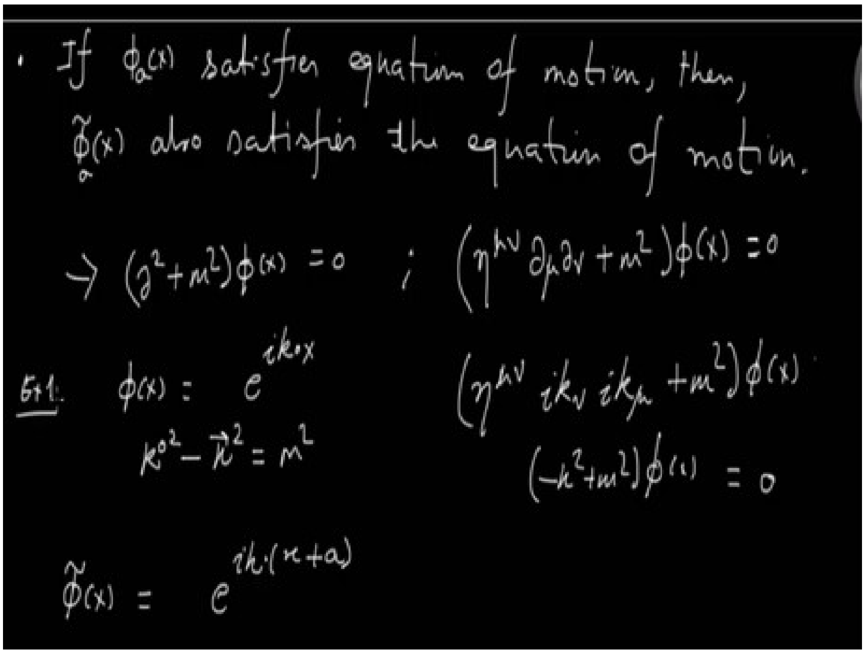

Figure 5: Refer Slide Time: 17:05

So, whatever I did here goes through and you realize that phi tilde of x also satisfies the same equation of motion. So, I mean this is interesting to note, but you can guess that this is a consequence of the fact that we have an underlying translational symmetry. Let me give you another example which is a little more fun to work out. So, let us say example 1. And you can create more examples for yourself to practice. Anyway, so let me take another field configuration which is obtained by taking my original field configuration and doing this transformation. So instead of x i take lambda x which is same as e to the i k mu lambda mu nu x nu. Now, let us ask whether this phi tilde of x also satisfies the same equation of motion, meaning is this true? Let us check.

So, I have here for this operator I have eta rho sigma del rho del sigma + m square phi tilde of x which is e to the ik mu lambda mu nu x nu, remember you cannot use mu and nu here all again because then you will start making mistakes, all those things which are contracted you should keep them different. Now, let us look at this when this derivative acts on here, you get eta rho sigma and then you get, let me just write and then I can explain.

So, when these two derivatives are acting on this it should be acting on with the first derivative and then the second, but does not matter in this case. So you get ik mu lambda mu nu, but then this x nu when gets acted upon by del rho it will give you delta nu rho, right del x nu del rho this is delta nu rho. And the second piece will give you ik, now I should use a different; instead of mu I will use mu prime and for nu nu prime, otherwise it will be a mistake.

So, I get again ik mu prime lambda mu prime nu prime delta nu prime sigma again for the same reason this one, I should have said first this one and then that one but it does not matter in this case, plus of course m square and you have your this thing, again let me write it phi tilde instead of writing the full expression. So, what is this now? So, you have i square is -1, then you have lambda mu nu this I have taken, then you have eta sorry that is not what I want to do.

So, first I will make this index row, so this becomes lambda mu rho, these two together, then I have eta rho sigma that I have taken. Then I have this one and this makes nu prime to be sigma. So, I get lambda mu prime sigma and then you have k mu k mu prime + m square phi tilde of x. I hope everything is in here, yeah and this condition you remember right. This is the condition for Lorentz transformation

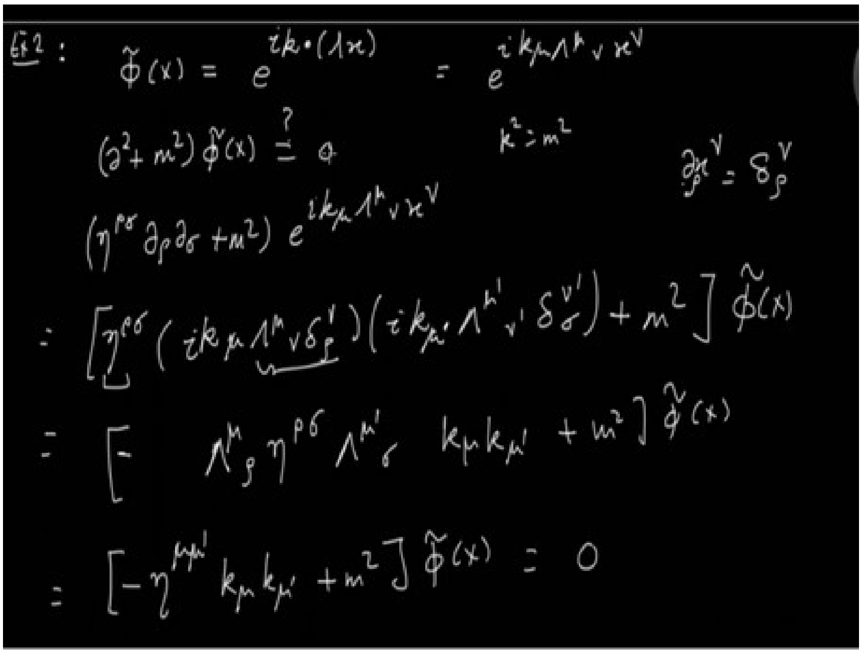

Figure 6: Refer Slide Time: 21:47

Example(1)

$$
\begin{aligned}
(\partial^2 + m^2)\phi(x) &= 0 \quad ; \quad (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(x) = 0 && (11)\\
\phi(x) &= e^{ik\cdot x} && (12)
\end{aligned}
$$

$$
\begin{aligned}
(\eta^{\mu\nu} ik_\mu k_\nu + m^2)\phi(x) &= 0 && (13)\\
(-k^2 + m^2)\phi(x) &= 0 && (14)
\end{aligned}
$$

$$
k^{02} - \vec{k}^2 = m^2 \qquad (15)
$$

I mean not the condition, this should give you again eta and it is minus eta mu mu prime and you have k mu k mu prime + m square which is this is just –k square. And again I should have said that here the k is constraint the components of k are constrained to be this and then you see that this is –k square which is –m square and this is m square which gives you 0 together. So, it is clear that phi tilde of x also satisfies the same equation emotion.

And again, no surprise that this should be a consequence of the underlying Lorentz symmetry that we have in the Klein-Gordon theory. So, let me give you the general result now which will allow us to find solutions or at least say that if you have one solution to whatever theory you are looking at you can generate other solutions by looking at what is underlying symmetries that theory has.

$$
\phi(x) = e^{ik\cdot(x+a)}
$$

also satisfies the equation of motion

Example(2):

$$\tilde{\phi}(x) \;=\; e^{ik\cdot(\Lambda x)} = e^{ik_\mu \Lambda^\mu{}_\nu x^\nu} \tag{16}$$

So, the general result is this. If we have a symmetry where you know your phi of a goes to phi tilde of a x and this leaves the action unchanged, then if phi a satisfies equation of motion, then so does phi tilde of a. And proof is easy. So, under the symmetry transformation that the theory has your phi a goes to phi tilde of a and whatever phi tilde of a is in terms of phi's and right now I am not writing, I am just saying whatever that transformation is.

Now take that phi a an arbitrary configuration, right now I am not assuming that phi a satisfies equation of motion. So, it could be any field configuration, it does not have to satisfy the equation of motion. Now take that field configuration and do a little variation of the field configuration. Let me write this kind of phi. So start from here and do a small change to it, so at every space time point there is a change, an epsilon is a small parameter you have pulled out.

So if you do this, then under the same transformation that you have, you will, we start from phi and then the transformation goes to phi tilde. If I start from a configuration which is slightly different from this one, then what I expect is a field configuration which is slightly different from this one. So, I expect that I will get, so there will be this kind of relation.

So. I am of course assuming that the transformation is smooth so that a small change in this field configuration from phi to phi plus epsilon phi does not create a finite jump or otherwise it will be a trouble. So, you want again it to be or the change to be for your epsilon. So, we are assuming smoothness here and since we have assumed that this is a symmetry transformation and it means that if I take S phi tilde of a then that should S of phi a, now I am going to use this.

So that means if I take S phi tilde a, I am suppressing x right now, plus epsilon psi a. So meaning I am looking at this field configuration, then the difference between this and that is this order epsilon square terms that are generated, then the action will also have this property that phi a, this phi and that phi are different. In this, I am suppressing x and this is because of this field. So, the action will differ by our epsilon square terms.

So, now let us see. And if I assume that phi a satisfies the equations of motion, then the value of the action in the vicinity of this field configuration will not change right, the action is stationary when the equations of motion are satisfied or when the field satisfies the equations of motion. So, the change will be of second order in epsilon. So, this if I assume that the S of phi a satisfies the equations of motion, then this object here is just S of phi a plus order epsilon square.

And of course, we have got epsilon squared terms already here, you can combine them here and this is just a consequence of using the equations of motion now. So, let me write phi a satisfies equation of motion. So, if phi a did not satisfy equation of motion, then I could not write this, then this is not true because then there will be an order epsilon term also be there.

$$\begin{aligned}
& \partial^2 + m^2)\tilde{\phi}(x) && (17)\\
=\;& (\eta^{\rho\sigma}\partial_\rho\partial_\sigma + m^2)\, e^{ik_\mu \Lambda^\mu{}_\nu x^\nu} && (18)\\
=\;& \left[\eta^{\rho\sigma}(ik_\mu \Lambda^\mu{}_\nu\, \delta^\nu_\rho)(ik_{\mu'}\Lambda^{\mu'}{}_{\nu'}\, \delta^{\nu'}_\sigma) + m^2\right]\tilde{\phi}(x) && (19)\\
=\;& \left[-\Lambda^\mu{}_\rho \eta^{\rho\sigma}\Lambda^{\mu'}{}_\sigma k_\mu k_{\mu'} + m^2\right]\tilde{\phi}(x) && (20)\\
=\;& \left[\eta^{\mu\mu'}k_\mu k_{\mu'} + m^2\right]\tilde{\phi}(x) && (21)\\
=\;& 0 && (22)
\end{aligned}$$

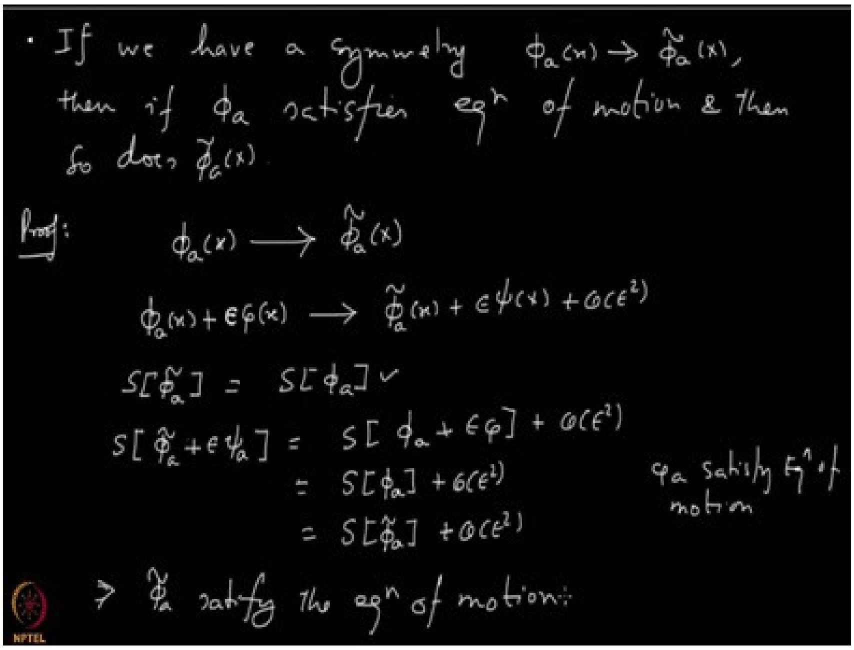

Figure 7: Refer Slide Time: 28:01

General result If we have a symmetry $\phi_a(x) \rightarrow \tilde{\phi}_a(x)$, thus if $\phi_a(x)$ satisfies the equation of motion then so does $\tilde{\phi}_a(x)$

proof

$$
\begin{array}{rcll}
\phi_a(x) & \rightarrow & \tilde{\phi}_a(x) & (23) \\
\phi_a(x) + \epsilon\varphi(x) & \rightarrow & \tilde{\phi}_a(x) + \epsilon\psi_a(x) + \mathcal{O}(\epsilon^2) & (24)
\end{array}
$$

$$
\begin{array}{rcll}
S[\tilde{\phi}_a(x)] & = & S[\phi_a(x)] & (25) \\
S[\tilde{\phi}_a(x) + \epsilon\psi_a(x)] & = & S[\phi_a(x) + \epsilon\varphi(x) + \mathcal{O}(\epsilon^2] & (26) \\
 & = & S[\phi_a] + \mathcal{O}(\epsilon^2) & (27)
\end{array}
$$

Because ϕ_a satisfies equation of motion, action is stationary, $\tilde{\phi}_a(x)$ also satisfies the equation of motion.

But if the equation of motion are satisfied, then order epsilon term are absent but then S of phi a is same as S of phi tilde a that is due to symmetry, this one. So, this says that if you look at S values if you go away from configuration phi tilde a to configuration which differs by order epsilon pieces, then action changes by order epsilon square, not by order epsilon term, meaning it is stationary.

If it is stationary, then it means that phi tilde a satisfies the equations of motion and that is the proof. So phi tilde a satisfy the equation of motion and that is what you have seen in all the examples that I gave before these ones. So, this is the proof of it. And in the next lecture. we will continue further discussing the consequences of continuous symmetries or more precisely global continuous symmetries and we will start looking at Noether's theorem; that is the plan for the next video.