# Introduction to Quantum Field Theory

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## Lecture 26 : Theory of Scalar Fields

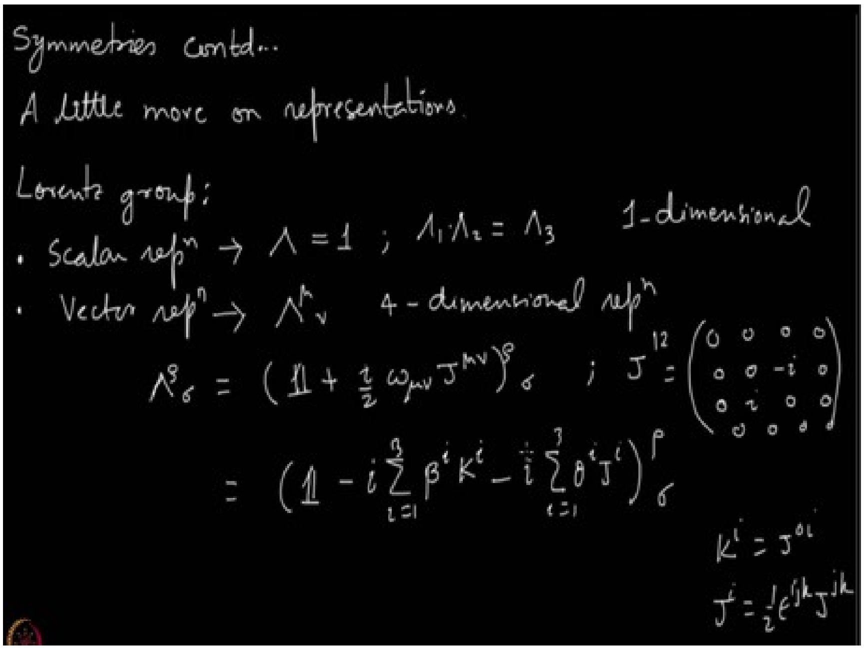



Figure 1: Refer Slide Time: 00:13

Let us continue our discussion on symmetries. And I will in this video talk a little bit more about representations. So, a little more on representations, it will be mostly what I have said before but maybe a few points here and there and hopefully this repetition will also help in understanding this. So, we have looked at Lorentz group before and we have already seen some representations of the Lorentz group.

So Lorentz group, and the two representations that we have encountered so far are scalar representation and vector representation. These we have explicitly seen. So, in the case of scalar representations things are easy, all the matrices lambda, the transformation matrix lambda. They were all mapped to identity, they were all equal to 1. And this forms the representation because this choice satisfies the multiplication law.

So, for example if you have lambda 1 as one element of the group, lambda 2 as another element of the group and multiplication of these three gives you some element which let us call lambda 3 and you have such multiplications defined for all the elements of the group.

This multiplication will be trivially satisfied if all the lambdas; lambda 1, lambda 2, lambda, 3 in this case the group is continuous, so it does not make so much sense to count them like this.

But nevertheless, if you were to identify all the matrices with 1, say all of them are equal to 1, then the multiplication rule will be trivially satisfied because here if lambda 3 is 1, lambda

2 is 1, lambda 1 is 1; then the product is also 1 times 1 is 1, so it is going to be satisfied. So, clearly this forms the representation and that we have seen before and then we have seen another representation that is vector representation.

And here the matrix lambda was given by lambda mu nu. And this is a four-dimensional representation, meaning the matrices are 4 cross 4. Here the representation is one dimensional. So, also if you recall that if I try to write down what an element lambda mu nu of this Lorentz's group is near identity, meaning if I am close to the identity then these elements can be written as the following where identity here is 4 cross 4 matrix, identity matrix plus i over 2 two omega mu nu.

Sorry, I have made a slight mistake. Let me rewrite it. Instead of writing mu nu I will write rho sigma. So, an element lambda near identity is the following identity plus i over 2 omega mu nu, these omegas are the parameters which parameterize each element and as you may recall these are 6 independent parameters because we found that omega is antisymmetric under mu nu interchange and then you have the generators which are J mu nu.

So, this is a sum 6 terms and then you take the rho sigma element. The first one gives you just delta rho sigma and then these are the remaining infinitesimal terms because we are looking at a group element infinitesimally away from identity and you will recall that for example explicitly if you look at J 12, these are the 4 cross 4 matrix because we are looking at a four-dimensional representation.

So matrices have to be 4 cross 4 and this we will recall or you can go back and see there it was of this form. See the diagonal entries have to be 0 because if you put omega as 20, I mean identity is here right, so these have to be zero because identity is the one which has all the diagonal entries equal to 1. So, here they are 0 and anyway J was antisymmetric under mu nu interchange, which is not reflected here anyway, fine.

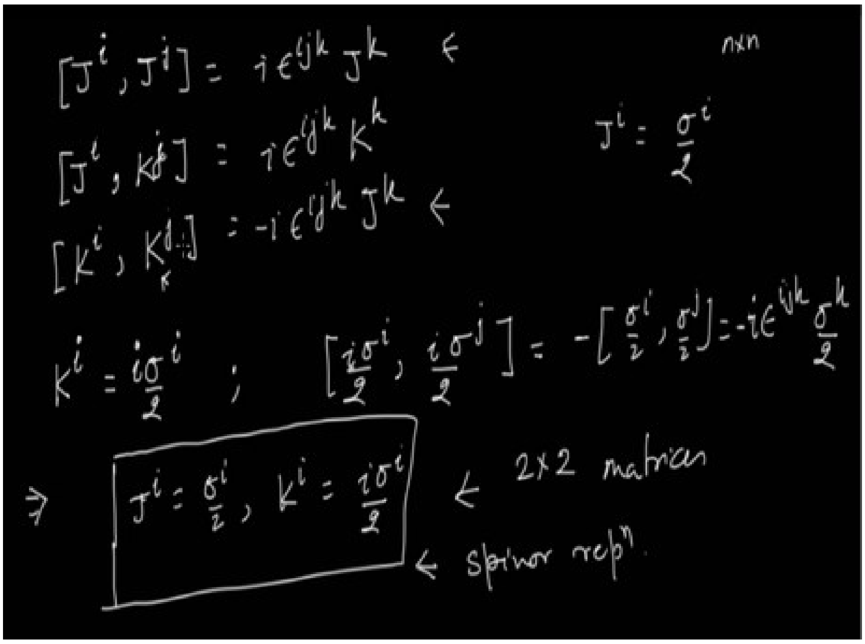



Figure 2: Refer Slide Time: 07:57

A little more on representation.

Lorentz group

- Scalar representation $\rightarrow \Lambda = 1 \quad ; \Lambda_1 \Lambda_2 = \Lambda_3$

- Vector representation $\rightarrow \Lambda^{\mu}{}_{\nu}$ 4- dimensional representation

So, that is one thing. Now, you recall that we also wrote these generators slightly differently and wrote them as this, so the parameters we instead of writing omega mu nu's, we can also choose 6 parameters beta 1, beta 2, beta 3 which correspond to boosts and the generators are K i and i ranges from 1 to 3 and again 3 and then you had these 3 rotations rho sigma.

See this rho sigma gives you the elements of these matrices J and K and J and K are built out of J mu nu. So, if you recall the K i was J 0 i and J i was half epsilon ijk J jk, the summation over j and k. And you also know that you need to know the commutators between these generators, they define the algebra of this group. And we had found that if you look at the generator commutations, so you have J's and K's and I will write down the commutation relations for all of them. So, if you take the commutation between the two J's you get. Now, let us write down what we get when we look at the commutators of J and K, you get and here K. Here is K, not J and then also if you look at the commutation of two boost generators that is what we had found.

And of course, we were looking at four-dimensional representations at the time of the defining representation here and we had an explicit expression for K and J in terms of this, so they all turned out to be four dimensional because we were looking at four-dimensional representation which is a vector representation. But then you recall that we said we can look at any dimensional representation.

So, what it means is if I could find matrices which are not 4 cross 4, but have some other dimensions, which also satisfy exactly the same relations. Meaning if I could find J's and K's of matrices which I can label as J and K, which are n cross n and which exactly satisfy this relation, then I can use those and form my elements of the group because after all the lambdas are just linear combinations of these K's and J's in addition to that unity.

So, I would have an n dimensional representation of lambda in that case if I succeed in finding such matrices. So, now what I will do is I will give you one such representation. So, I am not going to talk about how to systematically find the representations of Lorentz group, but I am going to give you one representation which is somewhat easy to see from the relations here. Let us focus on the first one, first commutation relation.

So this is the whole algebra, but let us look at the first. This one, I already know that if I look at poly matrices or sigma over 2 where sigma the poly matrix, the sigma over 2 satisfy this commutation relation. Remember the SU 2 algebra is identical to this one. It is the same, exactly the same. So, it is clear that if I take J i's to be sigma i over 2 at least the first relation is going to be satisfied.

And if I succeed in finding K's which are also 2 cross 2 matrices, sigma are 2 cross 2 matrices. If I succeed in finding 2 cross 2 matrices corresponding to the K's which satisfies this and this relation also, then my job is done and that is what I am going to do now. And it is easy to guess. You see this third one is almost like the first one except for an overall minus sign and minus sign is easy to arrange. So let us see.

If I take, let us do a little bit of it in trial and see whether things work out. Let us say I take K i to b sigma i over 2, clearly that will not work minus sign, but let me put an i here that might work because here you have i and here you have i and these you know you have to multiply them together and of course then take the difference with the multiplication in different order.

Close to identity, the elements will be,

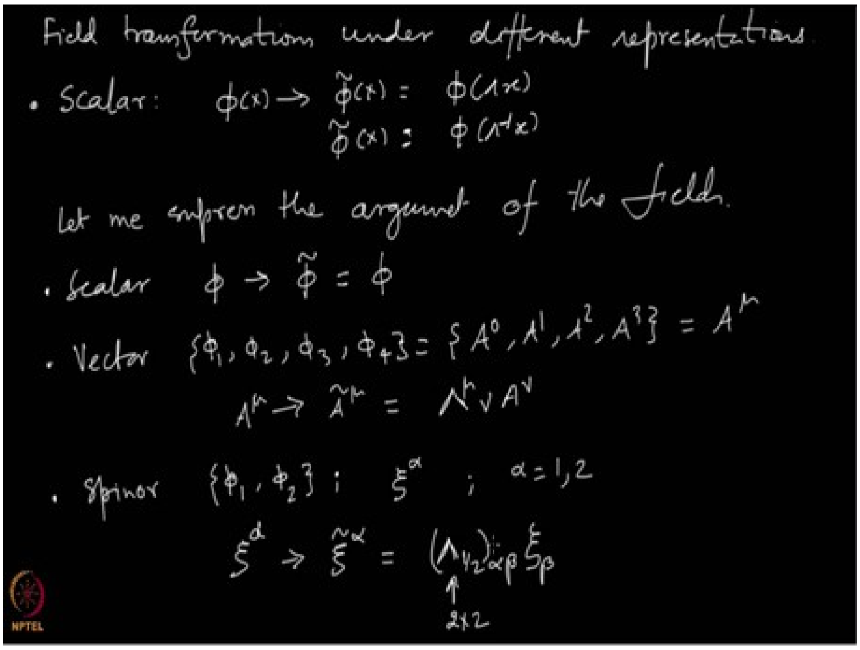



Figure 3: Refer Slide Time: 16:38

$$\Lambda^{\rho}{}_{\sigma} = \left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}\, J^{\mu\nu}\right)^{\rho}_{\sigma} \quad ; J^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

$$\Lambda^{\rho}{}_{\sigma} = \left(\mathbb{1} - i\sum_{i=1}^{3}\beta^i K^i - i\sum_{i=1}^{3}\theta^i J^i\right)^{\rho}_{\sigma} \tag{2}$$

$$K^i = J^{0i} \tag{3}$$

$$K^i = \frac{1}{2}\epsilon^{ijk} J^{jk} \tag{4}$$

The commutator

$$[J^i, J^j] = i\epsilon^{ijk} J^k \tag{5}$$

$$[J^i, K^j] = i\epsilon^{ijk} k^k \tag{6}$$

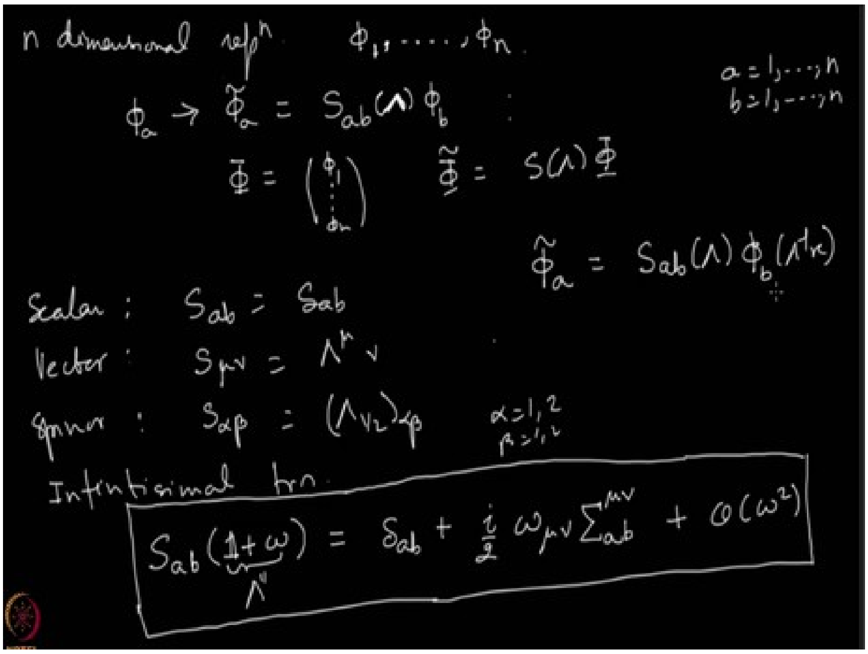
$$[K^i, K^j] = -i\epsilon^{ijk} J^k \tag{7}$$

If we take

$$J^i = \frac{\sigma^i}{2} \tag{8}$$

$$K^i = \frac{i\sigma^i}{2} \tag{9}$$

But as far as factor of this i is concerned, it gets multiplied twice, so it generates a minus sign, so this might work, i or minus i might work, let us see. So if I take i sigma i over 2, so I substitute

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this thing here in K i And take its commutation relation with i sigma j over 2 so that is what I am putting in here and let us see whether I get this. So J's are still sigma i over 2 right? So, I am saying J will be sigma i over 2 and I am trying to see whether this I can use for the remaining generator.

Let us see whether it works. So, this is let us just compute the left hand side and see whether we get the right hand side. So, this i will give you i squared which is −1 and then you get sigma i over 2 sigma j over 2 and this is minus epsilon, the commutation relation of these two is exactly this one, first line, so I get i epsilon ijk k sigma k over 2 and you see this I get sigma k, not i sigma, so this is good because this is what I wanted here.

So, I have got exactly the right hand side, start from the left and I arrive at the right hand side. So, it means that I can choose the following representation; meaning if I choose J i to be sigma i over 2 and K i to be i sigma i over 2 these 2 cross 2 matrices will form representation. So, these 2 cross 2 matrices form a representation of the algebra, but then that algebra, the generators you can put in here.

And you can get an element lambda rho sigma, which is at an infinitesimal distance or which is infinitesimally away from the identity. So, you get the group elements. So here I have given you an example of 2 cross 2 representation and this is called a spinor representation. So, now we have one more in our list which is spinor representation, so we have three representations.

Let me see what else I wanted to say? And of course, these are not the only representations; you can look into textbooks and you will find that there are many more representations, but the three which I have told you they all are irreducible representations and even with these three you can create an infinite number of representations which are trivially found by constructing reducible representations. I will talk a little more about that in a while but let me proceed.

let's take the third commutator

$$\left[\frac{i\sigma^i}{2}, \frac{i\sigma^j}{2}\right] = -\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = -i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{10}$$

So if we chose

$$J^i = \frac{\sigma^i}{2} \quad , \quad K^i = \frac{i\sigma^i}{2} \tag{11}$$

2x2 representation or spinor reprsentation, three representation for Lorentz group for now, irreducible representation.

Field transformation under different representation

• Scalar:

$$\phi(x) \;\; \rightarrow \;\; \tilde{\phi}(x) = \phi(\Lambda x) \tag{12}$$

$$\tilde{\phi}(x) = \phi(\Lambda^{-1} x) \tag{13}$$

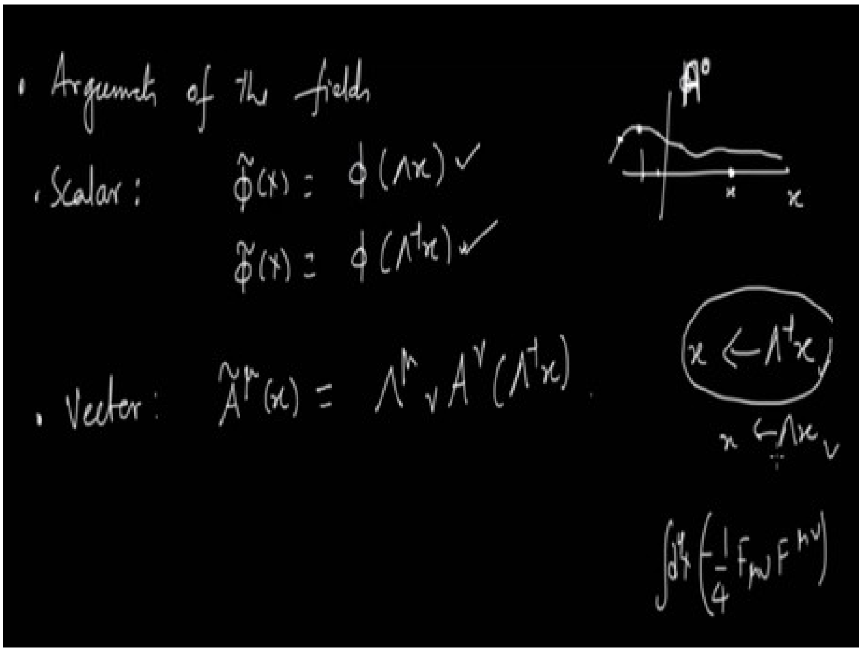



Figure 5: Refer Slide Time: 30:53

let me suppress the argument of the fields

So, as far as the group is concerned, we have found the representations but we are interested in fields, we are doing field theory. So, what we are interested in now is looking at how fields transform under different representations. So, let us look at first scalar. We have already talked about this and here you say that phi x goes to phi tilde of x where the transformed field is related to the original field by this or you could have also written it like this that does not matter.

It will not matter if you put this in the action, this transformation your action will still remain unchanged so that is a symmetry, whether you put a lambda or lambda inverse. Let me do one thing let me just suppress these arguments, the x and lambda elements, these things I will suppress for a while and I will come back to them after few minutes.

So, let me suppress, I suppress not because it is not important and that everything will be fine, I just want to retain your attention towards the things which I am trying to clarify here and I do not wish to have this additional complication arising from the arguments at the moment and that is why I am suppressing. I will make everything correct in a moment. Suppress the argument of the fields.

So, for scalar then I will write as a little with some inaccuracy that phi goes to phi tilde which is same as phi. How about vector fields? So, here you have basically four fields; phi 1, phi 2, phi 3, and phi 4 which you know you will usually write them as A 0, A 1, A 2, A 3; think of the 4 vector field corresponding to the electromagnetic fields, so this one or even in short a mu where mu runs from 0 to 3.

So, these are the four fields A 0, A 1, A 2, A 3 or phi 1, phi 2, phi 3, phi 4 whichever way you like it. And how does A mu transform? A mu transforms in the following way. So see, remember I have to talk about representation. So, A mu goes to A tilde mu which is lambda mu nu A nu so that is the transformation. These components A 0, A 1, A 2 and A 3 they are mixing with this matrix.

So, this is a linear transformation because multiplying with a matrix is a linear transformation and this is how they mix. So, A tilde 0 is lambda 0 nu A nu where this is summation over nu. So, all the four components mix here. And we have now today seen a two-dimensional representation which has a spinor representation and here because it is two dimensional it will act on fields, I mean it will act on a field which has two components let me call it phi 1 and phi 2.

So, like here you had four components because your matrix lambda was 4 cross 4. So 4 cross 4 thing multiplying on a four column. So, here you will have a matrix corresponding to lambda as a 2 cross 2 matrix. So, here let me write as phi alpha, does not matter up or down, where all phi's takes values 1 and 2. So, what will be the transformation rule for this? It will be phi alpha, let me not write phi.

So, generally I write phi here but for spinors let me make it more use a different symbol, so it is psi, there are two psi's, so because alpha takes values 1 and 2 the same thing, phi 1, phi 2 or equivalently psi 1 psi 2 whatever you write it just symbols. So, how does the spinors transform? The spinors transform like this, I was using tilde. So, you will have a 2 cross 2 matrix here, let me call it lambda half, do not worry about the half here.

This entire thing is a 2 cross 2 matrix times and the matrix will have indices; two indices, alpha beta and then you have psi beta there is a summation over beta. And what is lambda half? That you already know, lambda half is this. So, take this expression for example and put K's and J's as the 2 cross 2 representation these ones and then you construct the lambda half.

So, given a boost and rotation parameters you can construct the lambda half. So, this will be a two-dimensional representation. So, here in this case the matrix which was multiplying was 1, here you had 4 cross 4 matrices, here you have 2 cross 2 matrices.

- Scalar $\phi \to \tilde{\phi} = \phi$
- Vector $\{\phi_1, \phi_2, \phi_3, \phi_4\} = \{A^0, A^1, A^2, A^3, \} = A^\mu$

$$A^\mu \to \tilde{A}^\mu = \Lambda^\mu{}_\nu A^\nu \tag{14}$$

- Spinor $\{\phi_1, \phi_2\}, \quad \phi_\alpha \cdot \xi^\alpha \quad ; \quad \alpha = 1, 2$

$$\xi^\alpha \to \tilde{\xi}^\alpha = (\Lambda_{1/2})_{\alpha\beta}\, \xi^\beta \tag{15}$$

- n-dimensional representation $\phi_1 \cdots \phi_n$

$$\phi_a \to \tilde{\phi}_a = S_{ab}(\Lambda)\phi_b \quad ; \quad \begin{array}{rcl} a & = & 1, \cdots n \\ b & = & 1, \cdots n \end{array} \tag{16}$$

And if you had instead of these any other n dimensional representation, then of course you will have to have n number of fields first if you want to have a field transforming under that representation. So, you have phi, I am still suppressing the arguments. So, phi of what I was writing a under this group, it will go to phi tilde of a to love a which will be S ab.

So, given an element of the group Lorentz group lambda you will have its corresponding matrix S which is the element of which I am writing as S sab and a and b run from 1 to n. So, S multiplying the phi is the same thing, if you prefer you can write capital phi and define capital phi to be the column phi 1 to phi n, then your phi tilde is S lambda phi. It is this equation that I have written using this.

So, this will be the general transformation of fields under a general n dimensional representation. So, for our scalar case what is S? S is 1, it is identity matrix. What is it 4 vector? Your S instead of adding ab I will write mu nu and this is lambda mu nu. Do not worry that mu nu are down here but they are one up one down here, it is just that you know this S is given by these 4 cross 4 matrices and here you will have a new contracting, so this is fine.

And then similarly, for your spinor representation your S alpha beta where alpha and beta run from 1 to 2, you will have the corresponding matrix which I denoted by lambda half, whose elements you can find out using the poly matrices as I discussed in the previous slide. Now, let me come to I will still keep suppressing the argument and let us write down what it will be for a general n dimensional representation when we are looking at an infinitesimal transformation.

So for infinitesimal transformation, I will write the matrix A ab, let me write here instead of lambda, I mean lambda I am writing as 1 + omega, k omega is a small parameter, infinitesimal parameter, so your lambda this entire thing is capital lambda. And this is the matrix representing it. This will be of course delta an. If you are writing it as matrix, this will be an n cross n matrix plus i over 2 omega mu nu.

Let me go before I write this, let me show you the thing which we had before here. We had this right. So this is exactly the same thing I am writing, here it was 4 cross 4, everything was 4 cross 4, the J's were 4 cross 4, but instead of this 4 cross 4 thing, I will have some n cross n thing, right. And instead of this rho sigma, you will have these indices which run from 1 one to n, here they ran from 1 to 4.

So, I am basically writing this thing in the more general case, it is exactly this thing being written. Here it was a four-dimensional presentation, so I have now omega mu nu these are the parameters, 6 of them and let me denote by capital sigma, this is not fully standard but semi-standard notation, many authors use it mu nu. So, up to now I have just said that I have 6 parameters and 6 generators.

So, this is a sum of 6 terms, sum of 6 terms and containing these 6 generators. But I should now have also the matrix elements, right, because I am writing here on the left hand side the matrix elements of S and that I indicate by this. So sigma mu nu ab; these a, b are going to give you the elements of this matrix, that they label the elements of the matrix.

And of course, you will have order omega squared terms, which we will not consider here because we need only this. So this is fine, this will be something we will keep in mind. Let us see if I can make a box.

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}; \quad \tilde{\Phi} = S(\Lambda)\Phi \tag{17}$$

I think now I want to talk about arguments of the fields. So, arguments, so here when you are looking at scalars that is fairly easy. So, you say phi tilde of x is just phi of lambda x. So, you can view it actively. So you think of a field given in space time, so let us say it is only one dimensional space x; your field configuration. So, these points tell you what the field value is at a given point, this is this much, this is this much.

Now you can view it actively. You can think of taking the field phi and transforming it such that whatever is at x goes to lambda x. So, whatever the value of the field is at this point x becomes the value of the field at point lambda x that is the transformation. If you do so, then of course you are not going to write it like this, then you will write like this because what is at x gets transported to lambda x, then what was that lambda inverse x gets transported to x.

So, you can write it this way or you can also choose to view it passively and you can do this way, but both are fine there is no problem, they both will work. You can take both of these and put into the Klein-Gordon action and you will see that this transformation leads to action unchanged that is what really matters to you, right, that the transformation you are looking at is the symmetry of the action.

So even if you are not looking at any active or passive or whatever and you are doing it mechanically and just trying which one will work, you will see that both works. Now let us look at vector fields. Let us again keep the active transformation in view so now let us say my fields are A case, let us look at one of them A 0. Again, I do the same thing. I imagine transforming field such that whatever value the field had at this point x that gets transformed to point lambda x.

So again, what was at lambda inverse x gets transformed to x. So, whatever was the value at lambda inverse x that will go to x or equivalently whatever was at lambda x will go to x. So, either this way you want to do or that way you want to do, so let us do it this way activity. But then you know that field is not going to just as in the scalar case it will not just remain like this, there will be a matrix multiplying it.

So, your A tilde mu of x will be lambda mu nu A nu lambda inverse x, just like this. So, here this got transported to the point x, so here whatever the value of A was at lambda inverse x got transported to the point x. But then because it is a vector field, it transforms like this. If you were to do it this way, then you will see that if you write this, maybe I will give it to you as an exercise what you do is do not put lambda inverse, just put lambda x.

So, A tilde mu x is lambda mu nu A nu lambda x and put it in the action where you have –1 over 4 F mu nu F mu nu this in this section. Try to see whether you get an invariance, whether your action remains unchanged and see if you should do something here should you change it in some way so that you get an invariance. So, in that way you will be able to fix this matrix here in the case where here you want to put lambda x.

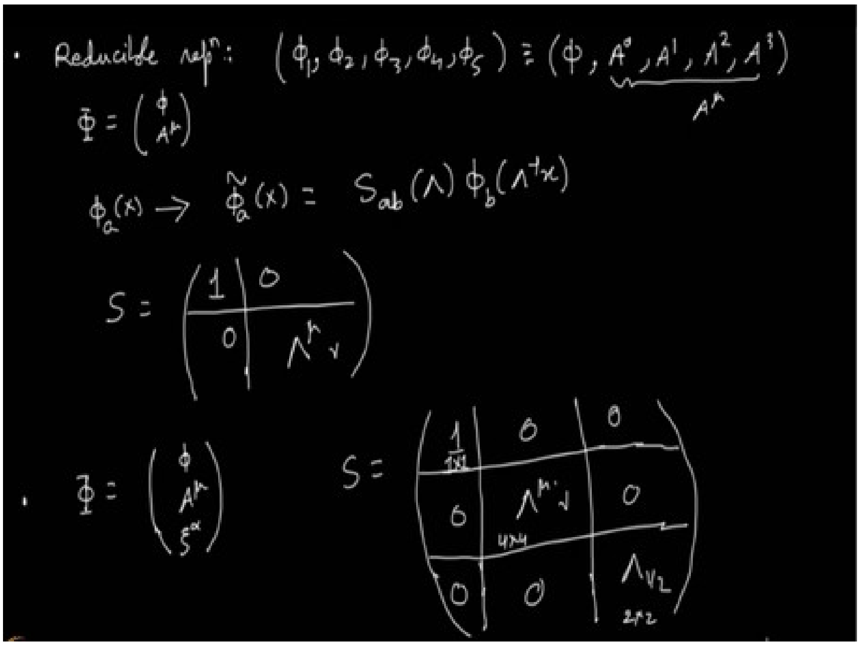
But anyway, we will use this one, we will not put lambda x here and continue with this. So, we have also talked about the arguments now. So, here go back and say, so now here you have this one more appropriately I should write phi tilde A = S ab lambda phi b lambda inverse x that is how I want to see it.

- In case of scalar

$$S_{ab} = \delta_{ab} \tag{18}$$

- In case of vector

$$S_{\mu\nu} = \Lambda^{\mu}{}_{\nu} \tag{19}$$



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- In case of scalar

$$S_{\alpha\beta} = (\Lambda_{1/2})_{\alpha\beta} \quad ; \quad \begin{aligned} \alpha &= 1,2 \\ \beta &= 1,2 \end{aligned} \tag{20}$$

So, I just said before that I have been talking about irreducible representations till now, but we can look at reducible representations also. Let me remind you what reducible representation is, I think I can go back and see where it was written first, here. So, this was an example of reducible representation, So, if your matrix D corresponding to each element gi can be written in a block diagonal form, then it is called a reducible representation.

Because then this block acts only on the fields, the first few fields and that number depends on what is the dimension of this representation, and this one acts only on the remaining ones and they never bother each other. So, that is what I want to do. I want to construct some irreducible representations. So, all I have to do is put side by side in the blocks irreducible representations and then I can form a reducible representation.

I am not sure I said it correctly. In blocks I should put irreducible representations and then it will give you reducible representation. So, let me show you a representation reducible representation and how fields transform under that reducible representation and more specifically I will choose two fields which are the following. So, I take the several fields which are phi 1, phi 2, phi 3, phi 4, phi 5 and these phi fields transform like this.

They transform like the following. The first one transforms like a scalar, meaning that guy is not going to mix with any of these. Then the remaining four mix among themselves and the Lorentz transformation like a vector field. So let me just rename phi 2 to A 0, phi 3 to A 1, phi 4 to A 2, phi 5 to A 3. It is just relabeling. I am relabeling them this way because I have said that this guy transforms like a scalar and these four transform like a vector together.

So, this is basically your A mu. Now if I define as a column capital phi, this is like this is five entries in the column. Then phi a is, maybe I can instead of putting capital here I will write small which is just means the ath entry of this capital phi. So, these ones I am calling as phi a, these entries. So, phi a at point x is transformed to phi tilde a of x and how does that transform, it transforms like the following.

You have S ab corresponding to the matrix lambda that matrix S times phi b lambda inverse x right, this is just the general form for all the cases that I have listed before and all other representations. Now, what is S ab in this case the one I am looking at? So, let us clear that S, I am just writing the matrix directly instead of elements will be this 1 and then here you will have lambda mu nu because this mu nu transforms according to lambda mu nu 4 cross 4.

So, this is 4 cross 4 piece and this here, right. So, this one acts on phi, this piece again gives you phi and this will just transform the remaining A's, the remaining these four according to the 4 vector. Let me give you one more example of reducible representation. So, now in this case I am imagining that I have a theory in which I have a scalar field, one scalar field only, you could have imagined several but let us say one.

You also have an A mu, so 4 vectors, 4 fields here which correspond to a vector field and a psi a spinor field, two basically alpha takes values 1 and 2, but I have written like this. So, these are in total 4 2 6 and 1, 7; these are 7 fields.. And for them what will be S? Your S will be 1, then you have lambda mu nu. This is a 4 cross 4 block and so here they are all 0 and here you will have lambda half, this is a 2 cross 2 block.

Infinitesimal transformation

$$S_{ab}(1+\omega) = \left(\delta_{ab} + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} + \mathcal{O}(\omega^2)\right) \tag{21}$$

Arguments of the fields

- Scalar

$$\begin{aligned} \tilde{\phi}(x) &= \phi(\Lambda x) \\ &= \phi(\Lambda^{-1}x) \end{aligned} \tag{22}$$

- Vector

$$\tilde{A}^{\mu}(x) = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(\Lambda^{-1}x) \tag{23}$$

Exercise

$$\int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{24}$$

Check the invariance

Reducible representation :$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\phi, A^0, A^1, A^2, A^3) \tag{25}$$

$$\Phi = \begin{pmatrix} \phi \\ A^{\mu} \end{pmatrix} \tag{26}$$

$$\phi_a(x) \rightarrow \tilde{\phi}_a(x) = S_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \tag{27}$$

$$S = \begin{pmatrix} 1 & \\ & \Lambda^{\mu}{}_{\nu} \end{pmatrix} \tag{28}$$

$$\Phi = \begin{pmatrix} p \\ A^\mu \\ \xi^\alpha \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\Lambda^\mu{}_\nu)_{4\times 4} & 0 \\ 0 & 0 & (\Lambda_{1/2})_{2\times 2} \end{pmatrix} \tag{29}$$

So, let me write. This is a 4 cross 4 block, this is a 2 cross 2 block and this is of course 1 cross 1 and others all of diagonal entries are 0. So that is what your S is. So this is what I wanted to talk about the representations and I will continue our discussion on symmetries.