Introduction to Quantum Field Theory

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Lecture 25 : Theory of Scalar Fields

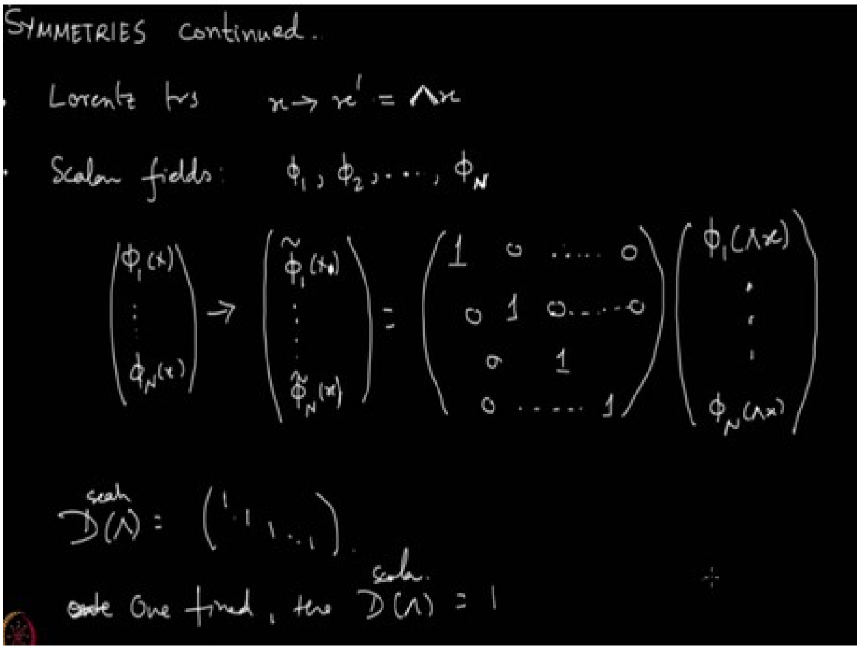

Figure 1: Refer Slide Time: 00:13

Let us continue our discussion on symmetries. We have learned about Lorentz transformations and other transformations which form a group, for example unitary transformations S U N or U N. So, let me begin by a quick recap and some more words on what we have already talked about. So, we learned about Lorentz transformations. So right now, I am not writing a transformation on the field.

I am just writing how they act on the space time points which are denoted by x here. So, you have lambda mu nu x nu here. And I was also mentioning about representations very briefly in one of the videos, meaning the elements of the group or the transformations, in this case the lambdas they can have different representations meaning you can find matrices which obey the same multiplication rule and they give a representation of Lorentz group or whatever group you are looking at.

And in this course, we have been very frequently looking at scalar fields. So, let me say something about that. By scalar, I meant always field more clearly. So, suppose instead of

having one field like you had in Klein-Gordon theory, suppose you have multiple scalar fields. So, suppose you have a field phi 1, another field phi 2 and many more and let us say we have total number N fields, total N fields and if they are all scalars then I mean the following.

I mean that if you do a Lorentz transformation, then these objects phi 1, phi 2, phi N they do not transform into each other. They do not mix among themselves, the transformation happens only on phi 1, only on phi 2 and so forth. Let me tell you what I mean here. So, if your theory has all these fields then let me write them as collect them into a column, then these fields will get transformed to phi tilde 1 x 1 and to phi tilde N x N, sorry not x 1 x where the phi tildes are related to phi by the following.

So, of course we have to have here phi 1 lambda x, phi N lambda x and as I said the there is no mixing that happens. So, phi 1 goes to phi 1, meaning you have an identity, a unity in the diagonal, similarly here, so these are all zeros. So, you have only diagonal entries, the diagonal entries are also same. So that is the matrix which you have. So, the representation matrix is identity and your phi tilde is just phi tilde of x 1, phi tilde of x is just phi 1 of lambda x, phi tilde N of x is phi N lambda x.

So, which means the phi is not changing. The argument is of course changing but that is beside the point. As far as the scalar term is concerned, it is just saying that the phi does not change into something else. So our representation matrices for lambdas if I am looking at scalar, then they are just okay. And if you have only one field like in real Klein-Gordon theory, then your D of lambda is just 1, just the number 1.

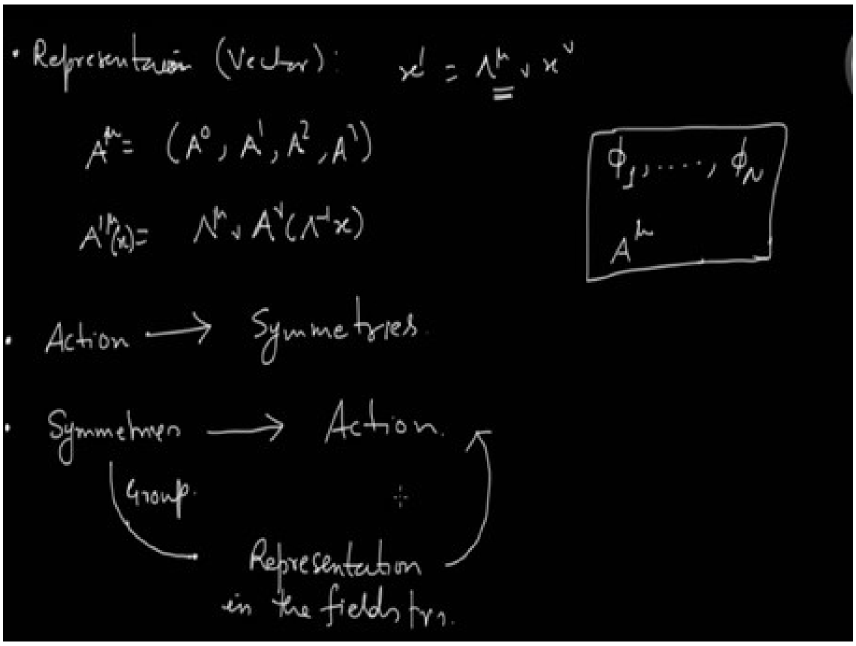

Figure 2: Refer Slide Time: 05:32

Lorentz transformation $x \to x' = \Lambda x$, Scalar fields: $\phi_1, \phi_2 \cdots \phi_N$

$$\begin{pmatrix} \phi_1(x) \\ . \\ . \\ . \\ \phi_N(x) \end{pmatrix} \to \begin{pmatrix} \tilde{\phi}_1(x) \\ . \\ . \\ . \\ \tilde{\phi}_N(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{N\times N} \begin{pmatrix} \phi_1(\Lambda x) \\ . \\ . \\ . \\ \phi_N(\Lambda x) \end{pmatrix} \tag{1}$$

No mixing happens

Representation matrices for λ

$$D^{scalar}(\Lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

if one field, then $D(\Lambda) = 1$

- Representation (Vector):

$$x' = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \tag{3}$$
$$A^{\mu} = (A^0, A^1, A^2, A^3) \tag{4}$$
$$A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(\Lambda^{-1}x) \tag{5}$$

- Action $\rightarrow$ symmetries

- Symmetries $\rightarrow$ action

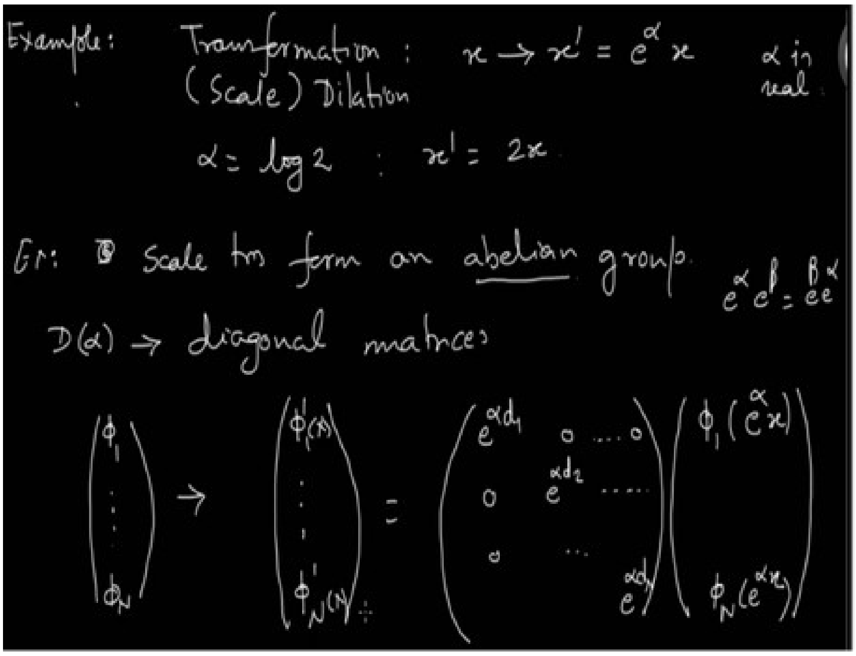

Figure 3: Refer Slide Time: 11:13

There is another representation that we have already been writing, in fact we have been mentioning and that is the representations formed by that is the vector representation which I am talking about now. So, when we wrote down the matrices these ones, these 4 cross 4 matrices, they form a representation of course, they were used to define the representation.

So, these 4 cross 4 matrices also form a representation and there can be many more representations, but now we have at least seen two of them, one scalar and another the vector one. And you might already be aware of electromagnetic fields and that you can write them as 4 vector potential which is a collection of these four fields. So, if you write down a theory with these fields, A mu's then the transformation log will be this.

So, the components will mix. Now, earlier in the previous example where we were talking about scalars A 0 just became A 0, A 1 just became A 1, I mean A 1 remained A 1 it did not change, but here there is a mixing. So, A 0 is lambda 0 nu A nu. So, you have a sum of nu's, so there is a mixing here and transformation is this. So, here you can imagine, I mean the way to understand this would be to imagine that you have this field A mu's.

And you boost to another frame or you boost them so they become a prime mu and these fields this index nu transforms according to this rule because this is a vector transformation you are looking at. And then because now you are looking at the space time point x and because you have done a boost, whatever was at lambda inverse x has been boosted to x or has been transformed to x. So, that is why you have lambda inverse x here, but what is more important to understand is that the fields are now mixing into each other.

So, the way we have been doing till now is that if you are given an action many of you are given a theory you start searching for its symmetries. You are interested in knowing what symmetries it has, so you try to figure them out but often you might be interested in doing the opposite. So, suppose you are constructing a theory of something and you have been told about what symmetries are there, so symmetries are known to you not the theory.

But some symmetries have been told to you, and then you construct the action. But to do so, not only you should know the symmetries which is meaning you should know the group, this means knowing the group, but let us say you have been told you have the symmetry which is Lorentz symmetry, so you choose the Lorentz group, but that is not sufficient.

You have to also decide on what fields you have to have in your theory, which means you have to decide on the representation in which the fields transform and then you are able to write down, then you can write down the action. So you write down all possible terms that are consistent with that symmetry. And once you have chosen your representation or representations.

So you might have multiple fields transforming differently under different representations of the same group, and then you write down the action that is consistent with the symmetry and including all the representations. For example here the representation was vector representation, so you could construct a theory which has scalar fields.

Let us say N number of them or one if you wish and together with that you also have let us say a vector field in your problem. So, you will try to write down a theory with these fields and construct an action which will be invariant under Lorentz transformations. So, that is also often done when you are trying to make a theory. I have planned to give you one more example of symmetry transformation, so let me do that. So, here is the transformation. Right now, I am not talking about the field but I am just telling about what happens to space time points. So, take a space time point x okay and let it get transform to x prime where x prime is related to x by a factor here, e to the alpha where alpha is real, it is a real continuous parameter.

So you can choose alpha whatever real number you would like it to be and it can be varied continuously. This is called a scaling transformation or scale transformation. It is also called dilation. For example, if you take alpha to be log of 2 then your x prime is 2 x, that is all your coordinates x 0, x 1, x 2, and x 3, they are all being scaled up by a factor of 2 so that is why it is called scaling, you are just scaling it up or you are dilating the coordinates.

Now, you can readily see that this transformation that forms a group, so you can check that this scale transformations form an abelian group. So, they form a group, so all the four properties they will be satisfied that form of group and also the order in which you do the transformation that does not matter. So, you can do transformation number one first followed by two or the second one first and then the first one.

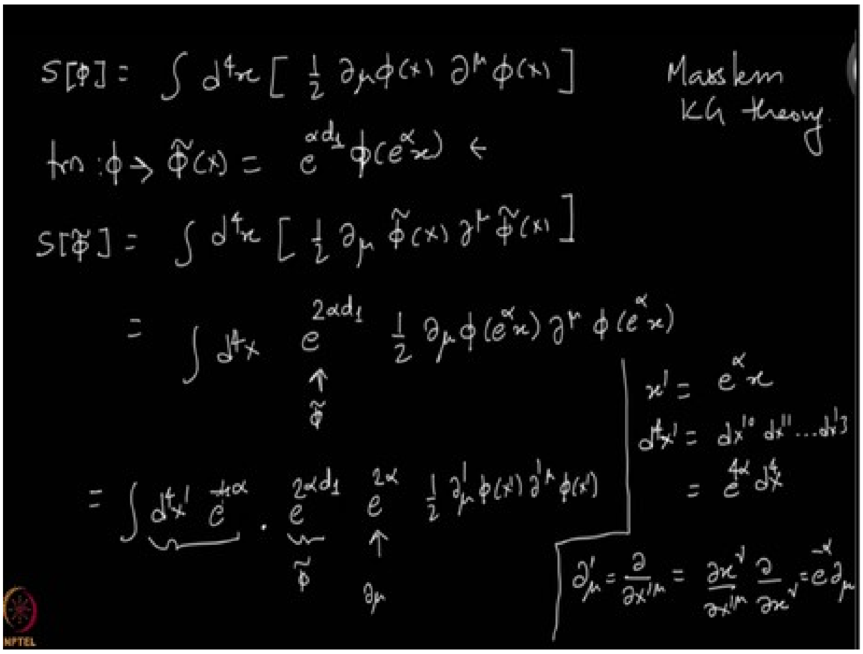

Figure 4: Refer Slide Time: 16:53

Constructing theory from known symmetries and their field representation

$$\begin{pmatrix} \phi_1(x) \cdots \phi_N(x) \\ A^\mu(x) \end{pmatrix} \tag{6}$$

Example:

Transformation $x \to x' = e^\alpha x$, α is real, scale transformation/Dilation, if we take $\alpha = \log 2$ then $x' = 2x$
Exercise: Scale transformation form an abelian group

$$e^\alpha e^\beta = e^\beta e^\alpha \tag{7}$$

So, if I were to write e to the alpha times e to the beta, this is same as e to the beta times e to the alpha, because this commute we say that this is an abelian. Now, if I am writing down a representation of this group, then the matrices, remember the matrix is D, let us call them D and because they are parameterized by this parameter alpha let me write it alpha, they would be diagonal matrices. So, it forms an abelian group right.

So if your group is an abelian group, then this has to be true, order should not matter. But now when you are looking at the representation, then the matrices have to commute because it is an abelian group. And matrices will commute if they are diagonal. So diagonal matrices they commute, so we expect the representation matrices to be diagonal. So, now you can ask how will fields transform under this group?

So, imagine we have N number of fields phi 1, phi 2, phi N, then under this transformation under scaling transformation they will go to something to phi 1 prime, phi 2 prime and phi N prime and they have to be related by this kind of transformation So, you should have the representation matrix here and the fields here, so the phi 1, phi N and what will be this matrix, it will be diagonal as we said and we can choose the diagonal elements all of them to be this side.

So, I write e to the alpha that is a parameter which parameterizes the transformation matrix but I keep a number d 1 here. So, d 1 is a number here and of course these are all zeros and 0, d 2 and so forth d N. So, these d 1, d 2, d 3; they will depend on what kind of fields these are. I will give you one example and it will be clear, the d's in general will be different and they will depend on what fields you have in the problem, but this is clear that it has to have this structure.

$D(\alpha) \to$ Diagonal matrices. how fields transform

$$\begin{pmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi_N \end{pmatrix} \to \begin{pmatrix} \phi'_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi'_N \end{pmatrix} = \begin{pmatrix} e^{\alpha d_1} & 0 & 0 & 0 \\ 0 & e^{\alpha d_2} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & e^{\alpha d_3} & 0 \\ 0 & 0 & 0 & e^{\alpha d_N} \end{pmatrix}_{N \times N} \begin{pmatrix} \phi_1(e^{\alpha}x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N(e^{\alpha}x) \end{pmatrix} \tag{8}$$

So, let me write down a theory which has a scaling symmetry and this is already known to you. So let us look at the action of real Klein-Gordon theory. Actually, I will take massless Klein-Gordon theory. Let me emphasize I am looking at massless Klein-Gordon theory. Now we consider the transformation phi tilde e to the alpha x, so that what happens to x.

But the field itself changes because we have seen what the representations are right, we know that phi 1 prime is e to the alpha d 1 phi 1. So, here I write e to the alpha d 1, in fact I will determine what d 1 would be. So, I hope this is not appearing difficult. This is just like the situation here, this one for example A prime 0 is lambda 0 0 A, 0 + lambda 0 1 A 1 and so forth. So, this matrix was your 4 cross 4 matrix.

But here it was mixing all the components, in this case it is simpler it does not mix phi 1 with phi 2, but there is an overall thing here just like you had in this case., so let us do this simple exercise and check whether this is a symmetry, this transformation is symmetry. So let us calculate as before, what is S phi of tilde? So, you just write d 4 x half del mu instead of phi we put phi tilde of x del mu phi tilde of x that is good.

And then it becomes the phi. So, phi tilde I substitute this one, so I get e to the 2 alpha d 1 because there are two phi tildes here. So, I get e to the 2 alpha d 1 and this is because of phi tilde. Then you have these argument changes, so you get half del mu phi of e to the alpha x del mu phi e to the alpha x and integral d 4 x. As before, we will change the variables. We will write x prime as e to the alpha x.

So, we can immediately write that d 4 x prime which is same as dx prime 0 dx prime 1 dx prime 3. This is e to the 4 alpha d 4 x, each of this brings a factor of e to the alpha. So, the d 4 x is e to the –4 alpha d 4 x prime. So, now that I am changing variables, I write this one is d 4 x prime, this one e to the –4 alpha. Then we have e to the 2 alpha d 1 and these will become phi of x prime, right.

Now, I should change the derivative piece that is easy. So, here let me write, remember that it is del over del x prime mu which is del mu prime basically. So del mu prime this is del over, I think I have done this before, so I could have skipped or something similar I have done before right. This is what I have done by using the chain rule and this is a summation over nu.

And this piece you can read from here. So, del x nu or del x prime nu will give you, see from here it will give you e to the – alpha okay times the Kronecker delta. So, this will give you e to the – alpha del mu. So that one we can substitute, because you have two such factors each of them brings out e to the alpha because I am writing del mu in terms of del mu prime. So, I get e to the 2 alpha these are coming from derivative pieces.

So del mu's and then this was coming from phi tilde and this is anyway coming from the Jacobian. So, you get half del mu prime phi x prime del mu del mu prime phi x prime. So, this together with these is your original action if I rename prime to n prime. So that is your S of phi but then you have these additional pieces okay. And you see that if d 1 is 1, if d 1 = 1 and these two together forming to the 4 alpha and this one is e to the –4 alpha and you get the same action back.

Theory with a scaling symmetry

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi(x)\, \partial^\mu \phi(x) \right) \quad \text{; massless KG field} \tag{9}$$

Transformation : $\phi(x) \to \hat{\phi}(x) = e^{\alpha d_1} \phi(e^{\alpha x})$

$$S[\tilde{\phi}] = \int d^4x \left(\frac{1}{2} \partial_\mu \tilde{\phi(x)}\, \partial^\mu \tilde{\phi}(x) \right) \tag{10}$$

$$S[\tilde{\phi}] = \left(e^{2\alpha d_1} \frac{1}{2} \partial_\mu \tilde{\phi}(e^\alpha x)\, \partial^\mu \tilde{\phi}(e^\alpha x) \right) \tag{11}$$

change of variables

$$x' = e^\alpha x \tag{12}$$

$$d^4x' = dx'^0 dx'^1 dx'^2 dx'^3 \tag{13}$$

$$= e^{4\alpha} d^4x \tag{14}$$

$$S[\tilde{\phi}] = \int d^4x'\, e^{-4\alpha} \cdot e^{2\alpha d_1} \cdot e^{2\alpha} \left(\frac{1}{2} \partial'_\mu \phi(x')\, \partial'^\mu \phi(x') \right) \tag{15}$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = e^{-\alpha} \partial_\mu \tag{16}$$

if $d_1 = 1$

$$S[\tilde{\phi}] = S[\phi] \tag{17}$$

$$\phi \to \tilde{\phi} = e^\alpha \phi(e^{\alpha x}) \tag{18}$$

So, if d 1 is 1, actually you have to take d 1 to be 1. So, if you choose this representation, then your S of phi tilde is same as S of phi, meaning phi goes to phi tilde e to the alpha phi e to the alpha x. This transformation is a symmetry of a massless Klein-Gordon theory. So that is one example I wanted to give. Now check that if you put mass term in this action, check that m square the mass term spoils the scaling symmetry that gets spoiled because there is a mass and mass is a scale.

So if there is an explicit mass sitting in the problem, the theory will not be scale invariant and that is why this will spoil the symmetry that we have found. So just do this, this is a fairly minor exercise because all other steps you have anyway done. Also check the following that if you have the following massless theory, again real fields, but still you do not have mass term because as I said mass did spoil the scaling symmetry.

But if you take this Lagrangian, this section this will still be scale invariant and that is easy to check. I think we can check it right now. So, this one anyway we have done, but let us look

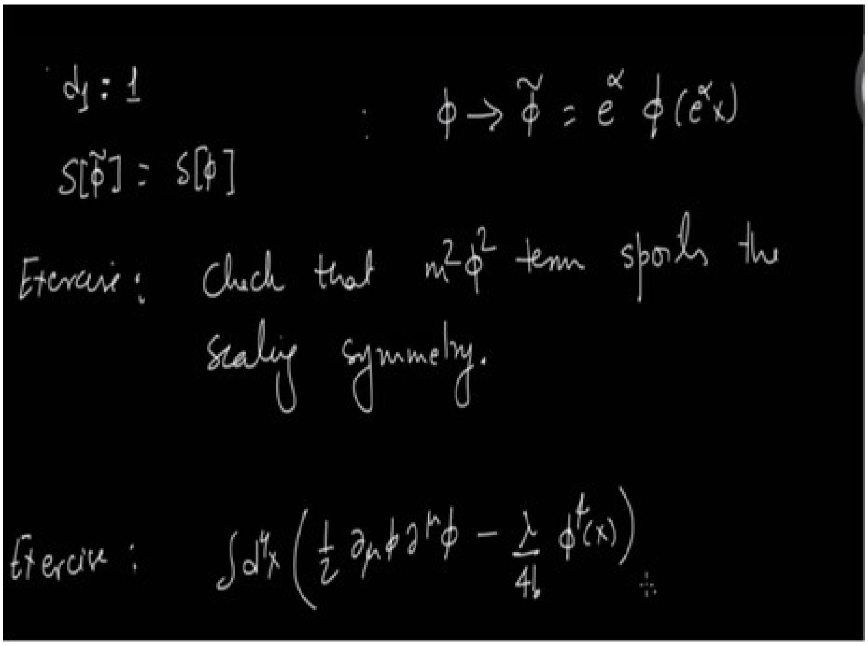

Figure 5: Refer Slide Time: 23:33

at this. This one phi 4 and remember each here, so there is no derivative term now because there are no derivatives here. So, the derivatives are not going to bring in any such piece, but you have phi tilde instead of 2 phi as you had in the kinetic term you have 4.

So you get e to the 4 alpha d 1 where d 1 is 1. So you get e to the 4 alpha and that will cancel against this piece. So, you see that this one will be invariant under scaling transformations.

Exercise: Check that $m^2\phi$ term (mass term) spoils the scaling symmetry.

Exercise: Show that,

$$\int d^4x \left(\frac{1}{2}\partial_\mu\phi(x)\,\partial^\mu\phi(x) - \frac{\lambda}{4!}\phi^4(x)\right) \tag{19}$$

this one will have scaling symmetry

So that is one example I wanted to give and we will continue the discussion on symmetries in the next video.