Introduction to Quantum Field Theory

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Lecture 24 : Theory of Scalar Fields

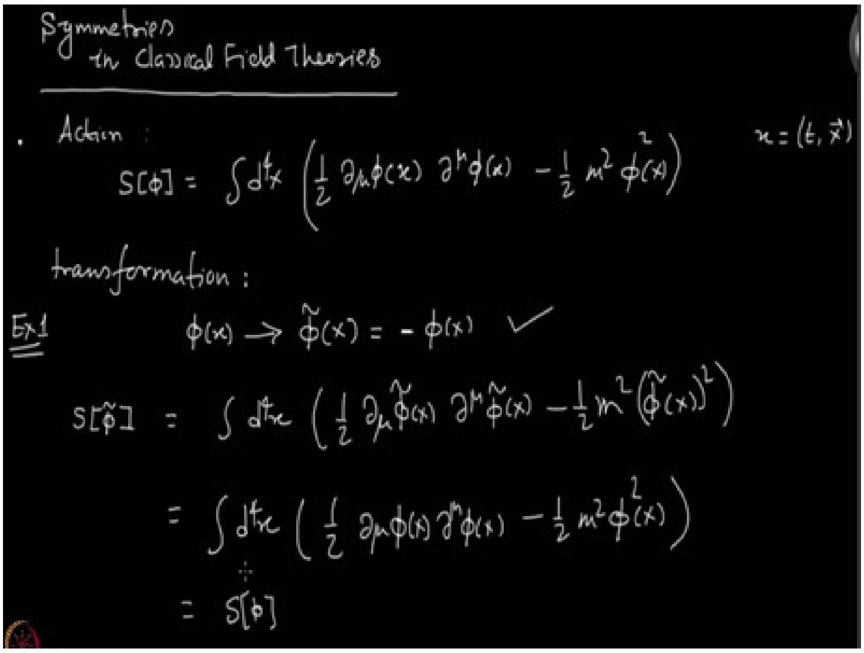

Figure 1: Refer Slide Time: 00:13

We will start a new topic today and that is symmetries in classical field theories. And whatever we have studied in the previous lectures on the groups of transformations and their representations, we did not discuss much about representation but just told what representation is. We will use those things that we learned in these lectures, so that is the plan.

Now symmetries are very important objects, to know about them is very useful because if you know what symmetries your theory has, then you can make many predictions about the kinds of solutions that you will get, what kind of properties that they will and they can also make your calculations easier. And also certain things that you are going to observe would not come as a surprise because then you would know that they are consequences of symmetry.

Let me give you an example which is not from field theory, but from mechanics. Think of a two-body system that you have solved. Imagine we take a sun and a planet which is moving around it. Once you solve this system, of course there are all kinds of orbits that are possible, but let us take an elliptic orbit. So, your planet is moving around the sun in an elliptic orbit. So, that is one solution.

Now, if you know that your system has rotational symmetry, you know that you will have angular momentum conservation, even if you are not aware of it that this system has rotational symmetry, you would still find that angular momentum is conserved but now you are aware that there is a rotational symmetry in the system. So you not only find it, but you also expect it.

And also, the other things which are responsible because of symmetries for example if this ellipse is a solution that is this ellipse satisfies the equations of motion, then if you were to rotate that ellipse and get another ellipse that would also satisfy the equations of motion. So, it is not that only this is a possible solution, only this ellipse is a possible solution, another ellipse which is rotated version of it will also be a solution.

So, there are such consequences of symmetries that evident if we are aware of what kind of symmetries are there in our system. So, we will now start looking at symmetries in quantum field theories or classical field theories and most of these symmetries will also be available in quantum field theories. So, let us start and try to be more precise about this. So, let us start by taking our familiar action.

So let us pretend that our universe is made up of only scalar fields, there is nothing else. Then the action of our world would be given by this. There is only one field phi which is a scalar field and its integral d 4 x half del mu phi del mu phi and square phi, maybe I should emphasize, not emphasize but I should make explicit x dependency phi x square given x is our t, N x.

Now let us look at a transformation and the transformation is on the field phi. See as far as this action is concerned here this subject, there is only phi which you have access to. If you look at x or t they are integrated over, so you cannot do anything to them. All you have is field, so if you are considering the transformation you have to consider transformations of the field.

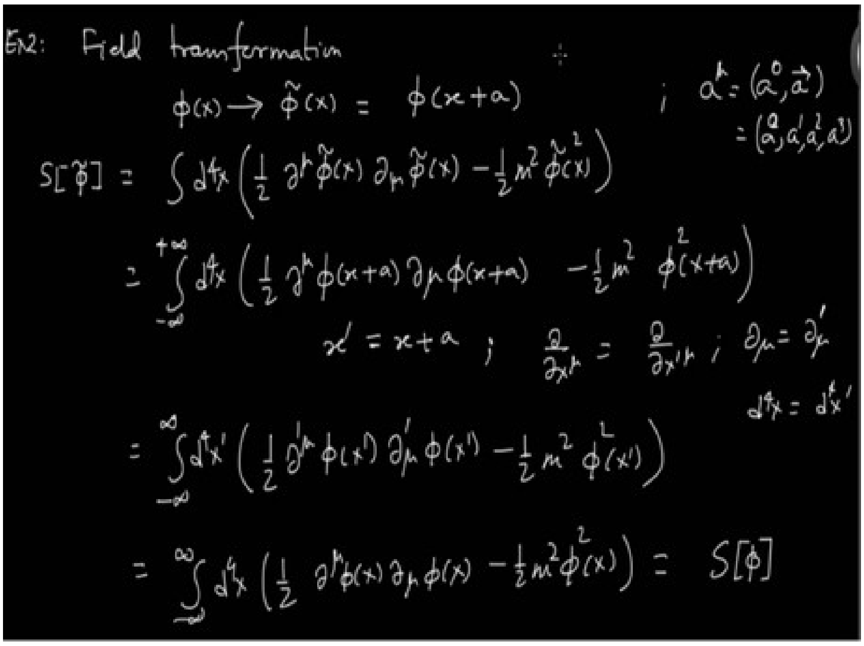

Figure 2: Refer Slide Time: 08:47

Action

$$S[\phi] = \int d^4x \left(-\frac{1}{2} m^2 \phi^2(x) \right) \quad ; \quad x = (t, \vec{x}) \tag{1}$$

Example(1)

You cannot say that I am looking at transformation effects, there is no x that you have handle on. They are all integrated over. So, you have only fields and let us look at transformation of fields. So, let us take this transformation. Let phi of x go to a new field configuration phi tilde of x which is related to the old field configuration phi of x by this relation.

So, suppose I look at this transformation and ask if I do this transformation, meaning I change phi to phi tilde of x what happens to the action? So, let us see what happens to the action. So, now I am asking what is S of phi tilde? So, S is the function of phi or phi tilde because once you specify the full field configuration phi, then you get a number S. So, let us write it down.

So, S of phi tilde is d 4 x half, so wherever I have phi of x I should put phi tilde of x, phi tilde of x del mu phi tilde of x minus half m square phi tilde of x square that is my action as a functional phi tilde. Now, I want to relate S phi tilde to S of phi. I want to see how they are related? So, let us see del mu phi tilde is – phi of x and because there are two factors, two sets of phi, the minuses will cancel and there will be no minus sign and I get phi of x, so phi tilde I have put – phi of x.

So phi tilde gives – phi and there is a phi here, minus sign cancels because there are two phi tildes and I get this. I get the same thing here and this term also. So, I see that S of phi tilde is S of phi if this is the transformation of the field I consider. Let me give you another transformation and let us check how it is related to down transformed field. So let us say this is our example 1. I have given you an example of a transformation here which keeps the action unchanged. Let us look at another example. So, consider the field transformation. So field, again I want my fields phi x to be transformed to some phi tilde and the relation between phi tilde and phi is that phi tilde of x is phi of x + a, where a is a vector because you see the x is a vector, so I can add to a vector only vector, you cannot add something else here. So a is basically a mu, a is a shorthand notation here, but it is a mu and it has four components; a 0, a 1, a 2 and a 3. Maybe I should write it more explicitly.

So, you have 4 numbers here and choose whatever you wish them to be. You can choose whatever value you wish. Now if I do this transformation let us see what happens to the action. So I want to know what is the action as a function of phi tilde. It is again d 4 x half del mu phi tilde of x del mu phi tilde of x – half m square phi tilde of x square. Now, I substitute the expression for phi tilde in this transformation and I get del mu phi of x + and similarly the other term.

Now note that the d 4 x, I mean x is integrated over, so it is a dummy variable and I can do a change of variables and go from x + a to some x prime. So, let us say I define or I change the variable from x to x prime like this. Then check that, you can easily check the del over del x mu is same as del over del x phi mu which is to say that del mu is same as del mu prime, where del mu prime is the subject.

Transformation:

$$\phi(x) \to \tilde{\phi}(x) = -\phi(x) \tag{2}$$

What happens to the action

$$\begin{aligned}
S[\tilde{\phi}] &= \int d^4x \left(\frac{1}{2}\partial_\mu \tilde{\phi}(x)\, \partial^\mu \tilde{\phi}(x) - \frac{1}{2} m^2 \tilde{\phi}^2(x) \right) && (3) \\
S[\tilde{\phi}] &= \int d^4x \left(\frac{1}{2}\partial_\mu \phi(x)\, \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \right) && (4) \\
S[\tilde{\phi}] &= S[\phi] && (5)
\end{aligned}$$

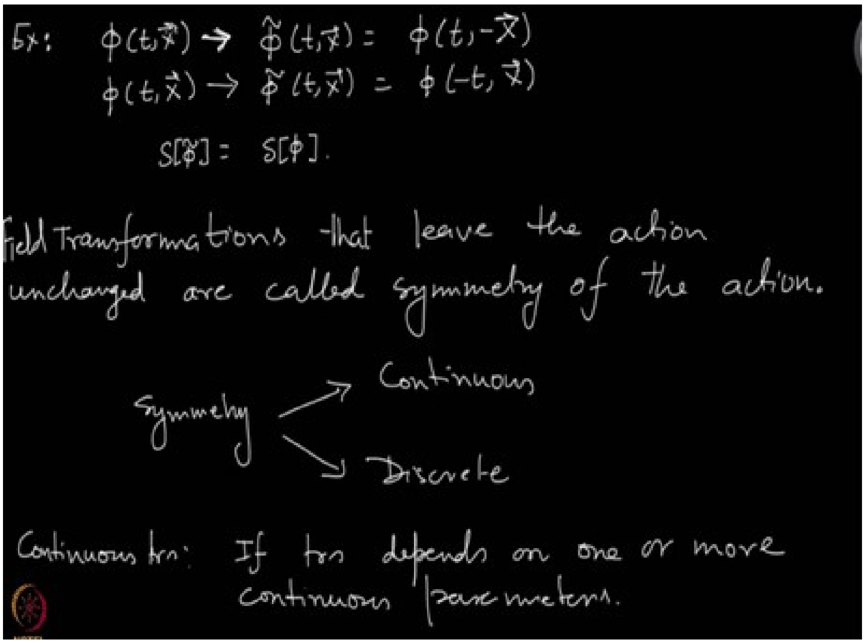

Figure 3: Refer Slide Time: 13:59

I hope you are already aware o **Example(2)** f the fact that if you have index up here in the denominator in the partial derivative here, then this corresponds to a down index in the derivative. So, with this change of variable, I can write this as and of course another thing that d 4 x will be same as d 4 x prime. And here the integration limits are from minus infinity to plus infinity.

So if I do that, I get d 4 x prime half, now I am training x to x prime, so this becomes del mu prime, so it should have been up here, then phi of x prime, then del mu prime phi of x prime – half m square phi of x prime squared. So that is the expression and this is the same expression as what you had for the Klein-Gordon theory with the field S phi. You can always of course write x prime as x, this just change the name and you can write it like this also.

It does not matter; we can stop here as well. There is no need to write this step, but let me write it which is same as your action as a functional of phi. So, you see that this field transformation where phi of x goes to phi of x + a that also leaves the action unchanged.

Example(2)

Field transformation

$$\phi(x) \rightarrow \tilde{\phi}(x) = \phi(x+a) \tag{6}$$

Where $a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3)$

$$S[\tilde{\phi}] = \int d^4x \left(\frac{1}{2}\partial_\mu\tilde{\phi}(x)\,\partial^\mu\tilde{\phi}(x) - \frac{1}{2}m^2\tilde{\phi}^2(x)\right) \tag{7}$$

$$= \int d^4x \left(\frac{1}{2}\partial_\mu\phi(x+a)\,\partial^\mu\phi(x+a) - \frac{1}{2}m^2\phi^2(x+a)\right) \tag{8}$$

Let me give you two trivial exercises to do. There is nothing to be really done here. But you can check that if you take the following transformation, now I am writing not as x but explicitly

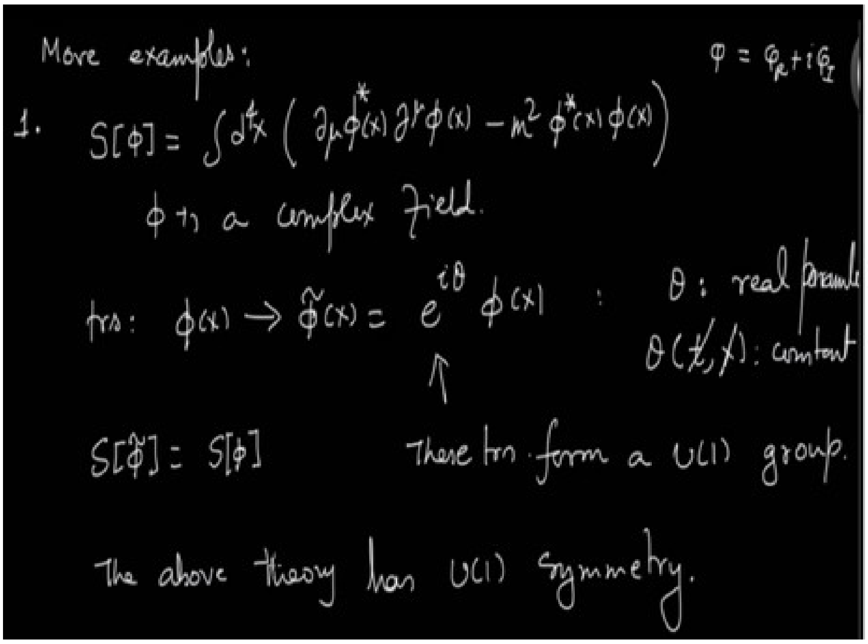

Figure 4: Refer Slide Time: 19:27

the time and space components because I want to do a field transformation which creates these different. So if you take this transformation or if you take that transform this one, both of these will keep the action unchanged.

So this is leave, check that this leave S of or they give you S of phi tilde is equal to S of phi. So, such transformations that leave the action unchanged after the transformation, they are called symmetry of the action. So, let me write it down. Transformations or field transformations that leave action unchanged are called symmetry of the action. These transformations are of course going to be very useful and that is why we are studying them.

Now, maybe I should give you, that is all and we already consider this one. So, the examples that I have given to you, you can see that they fall into two categories. One is a continuous symmetry. So, remember symmetry is what? It is a transformation. So, you have two varieties of; there are many varieties but let us start with these two varieties So, one is called continuous symmetry, another is called discrete

So, here for example this one, this transformation you have a parameter a which you can change continuously that you can take a to be 0, 0.1, 0.2 whatever values you wish you can take. So, this parameter a by saying 1.2, I was talking about only one of them for example this one. So, far there are four numbers and you can change each of these four numbers continuously.

So, your transformation is parameterized by continuous parameters a 0, a 1, a 2 and a 3 for this transformation and this transformation is called a continuous transformation because the transformation can be changed continuously. So, that is the continuous transformation and then of course there can be transformations like this where you do not have any continuous parameter that parameterize is a transformation. So, this transformation there is no parameter here, it just changes like this and that is a discrete transformation. So, your transformations come in two varieties, continuous and discrete. So, let me write what a continuous transformation is. If your transformation depends on one or more continuous parameters then it is called a continuous transformation.

$$x' = x + a \quad ; \quad \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\mu} \quad ; \quad d^4x = d^4x' \tag{9}$$

$$S[\tilde{\phi}] = \int_{-\infty}^{\infty} d^4x' \left(\frac{1}{2} \partial'_\mu \phi(x') \, \partial'^\mu \phi(x') - \frac{1}{2} m^2 \phi^2(x') \right) \tag{10}$$

$$S[\tilde{\phi}] = \int_{-\infty}^{\infty} d^4x \left(\frac{1}{2} \partial_\mu \phi(x) \, \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) \right) \tag{11}$$

$$S[\tilde{\phi}] = S[\phi] \tag{12}$$

Exercise:1

$$\phi(t, \vec{x}) \rightarrow \tilde{\phi}(t, \vec{x}) = \phi(t, -\vec{x}) \tag{13}$$

$$\phi(t, \vec{x}) \rightarrow \tilde{\phi}(t, \vec{x}) = \phi(-t, \vec{x}) \tag{14}$$

will give $S[\tilde{\phi}] = S[\phi]$

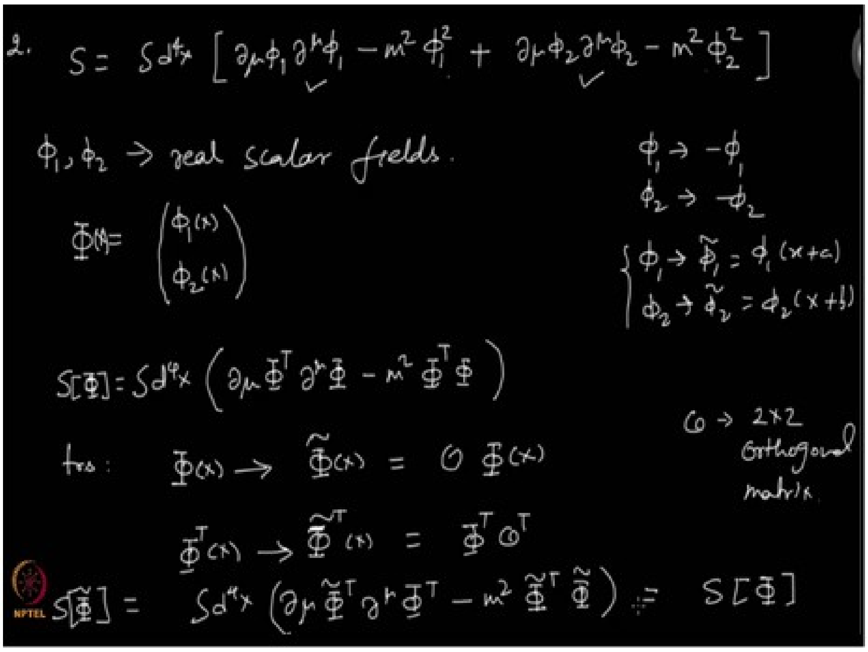

Figure 5: Refer Slide Time: 23:55

So, I want to take another theory, not the real Klein-Gordon theory, but I want to take a theory of complex scalar fields. So, imagine that we have a universe which is made up of complex scalar fields. So, I will denote by phi, a complex scalar field which has two parts; a real part and an imaginary part. So, it is basically two fields phi r and phi i and the action is given by the following.

Now phi is, so this is before d 4 x induction is this del mu phi of x del mu phi x and this is the complex conjugate here that is the action of a complex scalar field theory. Now I will show you transformation that is not going to change in this section. So, think of this transformation. So, take the transformation like this. Take phi of x going to phi tilde of x which is equal to e to the i theta phi of x.

Theta I am taking to be a real parameter and of course theta is continuous, we can change it continuously and also theta does not depend on t and x, it is neither a function of t nor a function of x, so it is a constant. It is a constant parameter. So, if you take this transformation and find out what is S of phi tilde, then you see that because if here you have a complex conjugation, so here you write del mu phi tilde star that is what you will write and here del mu phi.

So, phi tilde start would become e to the –i theta phi star and this one will give you e to the i theta. So, e to the –i theta from here and e to the i theta from here they will cancel and they will leave you with exactly the same thing what is written here and the same is true for this term also. So, you see that S of phi tilde will turn out to be S of phi.. So, now this transformation that your phi goes to e to the i theta phi.

These transformations you already know that they form a group and we have learned that they are called the group name is U 1, so we say that this theory has a U 1 symmetry. So, we say remember these transformations for U 1 group, so we say that the above theory has U 1 symmetry. So good. Let us take one more example.

Field transformation that leaves the action unchanged are called the symmetry of the action. Symmtries are of two types, continuous symetries and discrete symmetries

Continous transformation: If transformation depends on one or more continuous parameters. Now let us take a different kind of universe where the action is given by the following. Now I am going to have two fields and both the fields are real. So, I am going to introduce two fields phi 1 and phi 2 and both are real scalar fields. So, the action would be this del mu phi 1 m square phi 1 one phi 1 which is phi 1 square, let me write phi 1 square. Of course, there is x.

I am suppressing the x right now plus del mu phi 2 minus, I am taking the same m here square, these two m I have taken them to be the same. So, le us see this another symmetry which I want to show that is present here. Of course, you know this is just two Klein-Gordon, real Klein-Gordon fields put together and there is no term which couples these two parts right. So, these are just noninteracting fields.

Now, we have already seen that for this one for example. If you take phi 1 goes to – phi one that is a symmetry. So, if you take phi 1 goes to – phi 1 and keep phi 2 unchanged that is one symmetry. And if you take a transformation where phi 1 goes to – phi 1 and phi 2 goes to – phi 2 that is a symmetry of course. And of course, again if you take phi 1 goes to phi 1 tilde which is phi 1 x + a that will turn out to be symmetry as we have seen in the previous examples.

And similarly, if I take the full transformation to be this that will also turn out to be a symmetry, right. So, these symmetries which we have already seen in the case of real scalar field theory they are anywhere present here because this is just almost the same theory, but they are two fields which are present here but they both are obeying the same Klein-Gordon equation.

So, those are there anyway, but there is more that is present and let me try to show you what is that other thing which is present here. Let me slightly change the way I am writing it though it is not necessary to show, but let me do it anyway. Let me write a capital phi as a collection of these two fields. So, I have constructed a column and that column contains these two fields phi 1 and phi two 2, phi 1 phi 2 x.

Now, with this notation, I will write the action as the following d 4 x del mu phi transpose because this you have to see as a row now because row times column will give you this space. So, del mu phi, this term gives you these two terms right and then you have –m square and these two terms will appear like this phi transpose phi. I have just rewritten in this way.

Now, you see that if I consider this transformation where the field phi, the phi is not one field, it is two fields, but I will call it field. If the transformation is this that is capital phi goes to capital phi tilde where capital phi tilde is given by the following. So, this is a 2 cross 2 matrix, 2 cross 2 orthogonal matrix. So, a 2 cross 2 real orthogonal matrix. If this transformation I take then what happens to phi transpose?

Then phi transpose goes to phi tilde transpose which is transpose of this and that would be, right. Now you see that if I substitute this in my action, let me put here. So, let us substitute this in our action. So here phi transpose gets replaced by phi transpose O transpose and the phi here gets replaced by O phi. So let us see. Let us look at this term and this is exactly the same apart from the derivative, so let us look at this one. So, this term becomes, let me do it slightly slowly even though I am sure that everyone understands it. I first write down S of phi tilde and what is that S of phi tilde that will be d 4 x del mu phi tilde transpose del mu phi transpose –m square phi tilde transpose phi tilde. So, this is the action which we are looking at now and we want to see how it is related to S of phi.

And clearly phi tilde transpose phi tilde is same as phi transpose phi because O transpose O is one will give you the same thing back. So, this is going to be the same as this one, which means that the action is unchanged under this transformation. So, we say that this theory has an O 2 symmetry. So, let me write it down the above theory has and O 2 is the group of all orthogonal transformations, two-dimension orthogonal transformations. So that is another example.

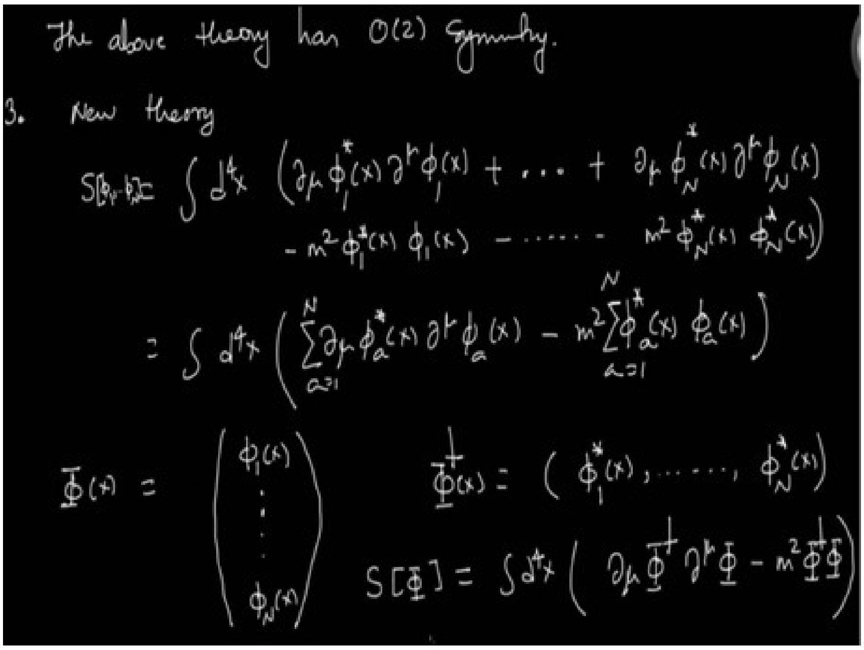

Figure 6: Refer Slide Time: 31:39

More examples

1. Complex scalar field theory

$$S[\phi] \;=\; \int_{-\infty}^{\infty} d^4x \left(\partial_\mu \phi^*(x)\, \partial^\mu \phi(x) - m^2 \phi^*(x)\phi(x)\right) \tag{15}$$

ϕ is a complex field,

So, this has this symmetry in addition to all the symmetries which I talked about and maybe there may be more symmetries which I have not said here. So, what all symmetries the theory has? That is not always easy to find. And in this case, it is not so difficult, most of those things are apparent, but you may have theories where things are not easy, the symmetries are not easy to figure out just by inspection.

Maybe I should take probably one more example, I think. at least one, maybe I am going to do more later. But in this video at least one more example I want to take. Now this I have done. So just like I looked at this theory which is just multiple copies of the same real scalar Klein-Gordon field theory, I have taken several copies and added them together and they do not interact. I am going to do the same thing for complex scalar fields.

So, I am going to take this action the real scalar field and I am going to just replicate it N times. So, consider a new action where the action is given by the following del mu phi star x del mu phi x or rather I will put it like this del mu phi N star x del mu phi N x, so I have introduced N fields, phi 1, phi 2, phi 3, so and so forth up to phi n and I should now write down the mass terms. I do not have to write this but let me write it anyway.

Now del mu x minus m square phi N x star phi N star so that is the action I want to look at. I could have written it down more neatly as this del mu phi a star and then del mu phi a and then you have –m square phi a star and phi a and of course there has to be a summation over a which I can make explicit here or we could have used Einstein summation as well, either way.

So, we are looking at this theory and of course it has many symmetries which you have already seen and so let me talk about one symmetry here which I am interested in right now. So, I will do again what I did in the previous case. I will define a capital phi as a collection of fields. So, you have phi 1 of x and up to phi N of x. So, you have N fields which have been put as a column vector.

Then if you look at phi x dagger that will be just phi 1 star and up to phi N star. So, the dagger puts a star and turns the column into row, so that is what you have and in this notation you can write the action as the following. So, you can write the action S as a functional of capital phi, here I should have written S as functional of phi 1 up to phi N. So, S of phi is integral d 4 x and what it would be? It would be del mu phi dagger del mu phi, I hope you agree and then of course I have m square phi dagger phi. So that is the action. Now let us write down the transformation which I am interested in, maybe let me write quickly the action again here d 4 x del mu phi dagger del mu phi. It does not matter whether it is up here or down there.

$$\phi = \phi_R + i\phi_I \tag{16}$$

the transformation

$$\phi(x) \to \tilde{\phi}(x) = e^{i\theta}\phi(x) \quad ;\theta \text{ is a real parameter and constant} \tag{17}$$

These transformation forms a $U(1)$ group, the above theory has $U(1)$ symmetry

2.

$$S = \; = \int_{-\infty}^{\infty} d^4x \left(\partial_\mu\phi_1(x)\,\partial^\mu\phi_1(x) - m^2\phi_1^2(x) + \partial_\mu\phi_2(x)\,\partial^\mu\phi_2(x) - m^2\phi_2^2(x)\right) \tag{18}$$

ϕ_1, ϕ_2 are real scalar fields

$$\phi_1 = -\phi_1 \quad ; \quad \phi_2 = -\phi_2 \tag{19}$$
$$\phi_1(x) \to \tilde{\phi}_1(x) = \phi_1(x+a) \tag{20}$$
$$\phi_2(x) \to \tilde{\phi}_2(x) = \phi_2(x+a) \tag{21}$$

$$\Phi(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \tag{22}$$

$$S = \int d^4x (\partial_\mu \Phi^T \partial^\mu \Phi - m^2 \Phi^T \Phi) \tag{23}$$

Transformation

$$\Phi(x) \to \tilde{\Phi}(x) = \hat{\mathcal{O}}\, \Phi(x) \quad ; \quad \hat{\mathcal{O}} \to \text{2x2 orthogonal matrix} \tag{24}$$

$$\Phi^T(x) \to \tilde{\Phi}^T(x) = \Phi^T(x)\, \hat{\mathcal{O}}^T \tag{25}$$

$$\begin{aligned} S[\tilde{\Phi}] &= \int d^4x (\partial_\mu \tilde{\Phi}^T \partial^\mu \tilde{\Phi} - m^2 \ \tilde{\Phi}^T \tilde{\Phi}) && (26) \\ S[\tilde{\Phi}] &= S[\Phi] \quad ; \qquad \hat{\mathcal{O}}^T \hat{\mathcal{O}} = \mathbb{1} && (27) \end{aligned}$$

Above theory has O(2) symmetry.

Example 3: New theory

$$\begin{aligned} S = \int d^4x\, \partial_\mu \phi_1(x) \partial^\mu \phi_1^*(x) + \cdots + \partial_\mu \phi_N(x) \partial^\mu \phi_N^*(x) && (28) \\ - m^2 \phi_1^*(x) \phi_1(x) \cdots - m^2 \phi_N^*(x) \phi_N(x) && (29) \end{aligned}$$

$$S = \int d^4x \left(\sum_{a=1}^{N} \partial_\mu \phi_a(x) \partial^\mu \phi_a^*(x) - m^2 \sum_{a=1}^{N} \phi_a^*(x) \phi_a(x) \right) \tag{30}$$

Where,

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_N(x) \end{pmatrix} \quad \Phi^T(x) = \begin{pmatrix} \phi_1(x) \cdots \phi_N(x) \end{pmatrix} \tag{31}$$

$$S[\Phi] = \int d^4x (\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi) \tag{32}$$

Transformation,

$$\Phi \to \tilde{\Phi} = U\Phi$$

$$S[\tilde{\Phi}] = \int d^4x (\partial_\mu \tilde{\Phi}^\dagger \partial^\mu \tilde{\Phi} - m^2 \ \tilde{\Phi}^\dagger \tilde{\Phi}) \tag{33}$$

$$\begin{aligned} \tilde{\Phi}^\dagger \tilde{\Phi} &= \Phi^\dagger U^\dagger U \Phi && (34) \\ &= \Phi^\dagger \Phi \quad ; \quad U^\dagger U = \mathbb{1} && (35) \end{aligned}$$

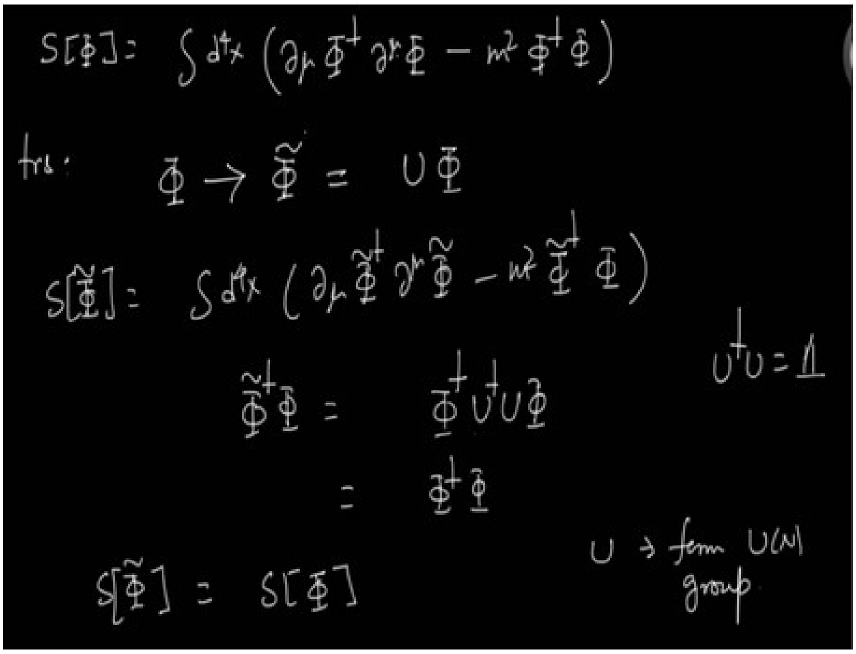

Figure 7: Refer Slide Time: 37:44

Hence,

$$S[\tilde{\Phi}] = S[\Phi] \tag{36}$$

U forms U(N) group

They have to be just one up, one down and then the contraction is fine minus m square phi dagger phi that is what we wrote in the previous slides. Now, look at the following transformation. I take this column, this column vector phi and transform it to a new field or the set of fields phi tilde which is given by the following. U, so I take our N cross N matrix, let me not write this. So, U is a N cross N unitary matrix and that multiplies this phi.

Now, you can see immediately that phi dagger phi will go to when you are writing S of phi tilde this you will have d 4 x del mu Phi tilde dagger, always you have to repeat the same step here phi tilde – m square phi dagger phi. And if you look at phi tilde dagger phi that you can write as phi is your phi and phi tilde dagger will give you U dagger phi dagger and because we are saying that U is unitary, I mean U dagger U is 1 that is what a unitary matrix is. Then we get this back as phi dagger phi.

So, from this you can easily conclude that the action of the transform field is the same as the action of the original field. So, clearly this transformation is a symmetry and because these U matrices these are n cross N, these form a U n group we say that this theory has a U N symmetry. So, U form U N group, so we say that the above action has U N symmetry. That is another example let us see. I think this is all what I wanted to say in this video. We will continue our discussion on symmetry in the next video.