Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 23 : Poincare Algebra

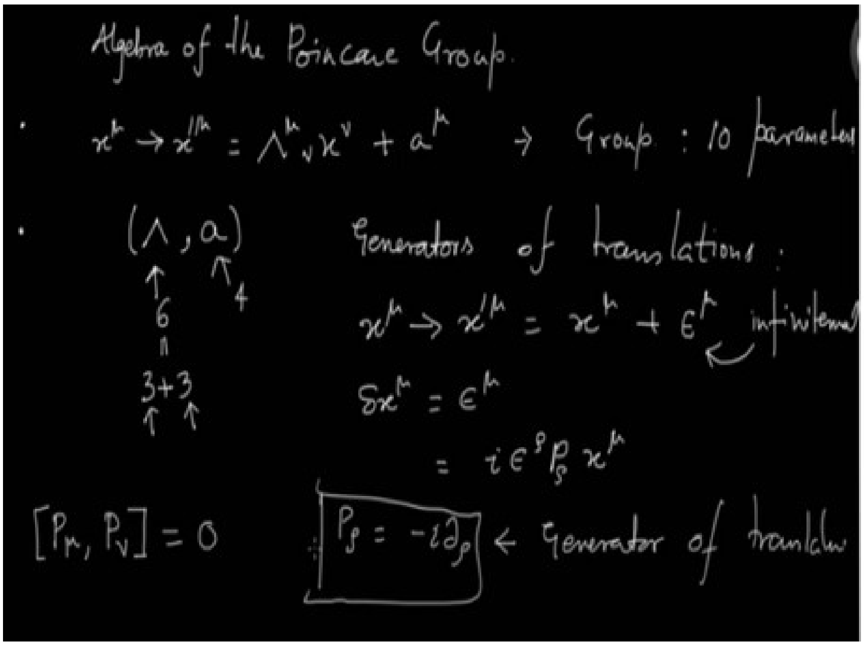

Figure 1: Refer Slide Time: 00:28

So we were looking at the Poincare group in the last video. And in this video we will look at the algebra of the Poincare group. Okay meaning that we want to find out the commutation relations of all the generators in this group, okay. So we saw in the last video that Poincare transformations are given by the following. So if you take x mu it goes to x prime mu where x prime mu is given by this. So this first term is a Lorentz transformation. And this is a translation, okay. These are the set of all Poincare transformations.

And we saw that this set of transformations form a group. And any element of the Poincare group you can parameterize by these objects. So you have four parameters which parameterize translations, so you have four parameters here, okay. I am just writing a mu is implied here, I am not putting the indices. So a, there four numbers here which parameterize how much displacement you are doing in space and time, okay.

And then you have lambda mu nu's sorry yeah, lambda mu nu's which are parameterized by the omega mu nu's. The parameters are omega mu nu's and these are the transformation matrices. So your omega mu nu you remember it was antisymmetric and that gave you six parameters, okay. And further this six were basically three plus three were three parameters parameterized rotations.

- $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

These set of transformation forms a group
• $(\Lambda, a) \rightarrow 6 + 4 = 10$ parameters
• Generators of translation

$$\begin{aligned} x^\mu &\rightarrow x'^\mu + \epsilon^\mu \quad , \quad \epsilon^\mu \rightarrow \text{infinitesimal} && (1) \\ \delta x^\mu &= \epsilon^\mu && (2) \\ &= i\epsilon^\rho P_\rho x^\mu && (3) \\ P_\rho &= -i\partial_\rho && (4) \end{aligned}$$

Three parameters parameterized boosts, okay boost meaning going from one frame to another frame or equivalently changing the velocities by some amount. So there are three boosts and three rotations and four translations. So you have in total 10 okay, 6 + 4 is 10. So this group is parameterized by 10 parameters. It is a 10 parameter group, okay. Now what we want to do is we want to find out the generators of translations, okay.

The generators of other transformation, the Lorentz transformation that we have, those we have already found, so there is nothing to be done there. Only this piece is new okay and this is what we want to find out now. So we will proceed as before, exactly the same way. So you have x mu going to x prime mu which is same as x mu plus a mu because I am right now switching off the Lorentz transformation.

So this lambda mu nu I have put equal to identity okay and only a mu's are non-trivial, so they are non-zero. So instead of a mu I would write epsilon mu to emphasize that these are infinitesimal, okay. So what we have is that the change in x is just epsilon, okay. So the difference x prime mu minus x mu is just epsilon mu, okay.

And we can write this as, this change as, so I am introducing an operator P rho okay and this object acts on x mu to give you the change delta x mu, okay. Now if you calculate this if you try to find out what P rho should be because epsilon is already given here. It is just a contraction of epsilon rho and P rho and this is acting on x mu. If you calculate you will find that P rho okay is minus i del rho.

So you see that this is the generator of translations, okay. So this is anyway the parameter, okay. And this operator is the one which acts when x on x mu gives you the translation. So that is the generator of it. And it is obvious that the, from the definition itself, not the definition from this expression P rho itself it is clear that the generators will commute, okay.

So if you take P mu and P nu two operators and or take their commutator that will vanish, okay. Because the partial derivatives commute. So del mu will commute with del nu. So that is why they commute. So let me go to next page now. Let me put a box around this, okay. What happened, something wrong? Okay I think I wrote on a wrong sheet. It does not matter I can change it later. So here we have the commutation relation of p mu and p nu. So let me now give you one exercise, but before that let me write this. P mu, okay they commute and then you have an exercise to do. So take the operators J mu nu which we already had from Lawrence transformations and take the P rho okay and find out what the commutator is. And what you will find is that this comes out to be the following g mu rho P nu plus i g nu rho P mu okay. And this is what you will find. Now see they are four generators for translations, the P mu's.

This is the generator of translation

$$[P_\mu, P_\nu] = 0 \tag{5}$$

Because partial derivatives commute

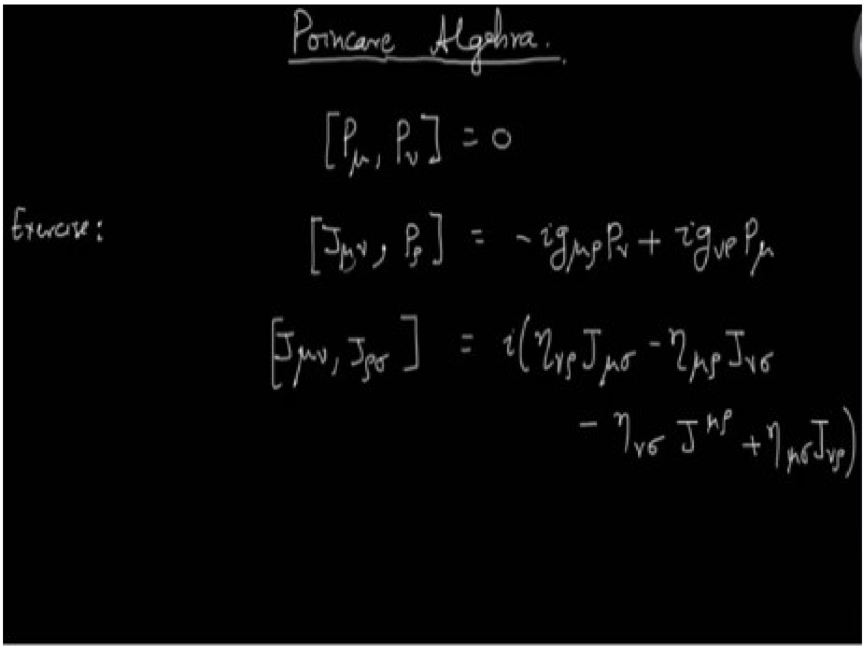

Figure 2: Refer Slide Time: 07:28

$$[P_\mu, P_\nu] = 0 \tag{6}$$
$$[J_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu \tag{7}$$

And we had six generators for the Lorentz transformations. So what I am doing right now is writing down all the commutation relations which can be written down between all these generators. So you can take the translation generator and find out how it commutes, what is its commutation relation with other translation operators or what is its commutation relation with the other generators of Lorentz transformations, okay.

Then you can find out the J mu's how they commute with P mu, which is the same here or J mu with g mu, okay. So the only thing that now remains to be listed down to know that full algebra of Poincare group is this, which also we have already done. So I will write down here for completeness, nu rho J mu sigma, mu rho J nu sigma nu sigma J mu rho, and the last one is J mu sigma J nu rho.

So these three commutation relations give you the algebra of the Poincare group. Or maybe I should just simply write Poincare algebra, okay. So be aware that we have found this commutation relations by looking at how the x mu transforms, okay. So you have already taken one specific representation, okay. Because you are asking how x mu's are transforming, okay.

And from that you have derived the commutation relations of these objects. But you can take a step back and say and pretend that we did not derive it from some by looking at the action on x mu, okay. And these were for example given to us. Let us say someone came and gave these commutation relations to us and then one could be interested in finding out all possible representations of this algebra or equivalently the group.

And of course, you will find one representation which will be equivalent to what you have now what you have been studying which is by looking at the action on the x mu's, okay. So that is coming from one particular representation. So all this, the way we have found is, we have been studying this using a particular representation, you can say it that way.

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\nu}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\nu\sigma}J^{\nu\rho}) \tag{8}$$

But we can think more abstractly about this and say that they have a meaning on their own. And then one could look at all possible representations of this algebra. Okay, so here I will stop this video and we will continue in the next one.