Introduction to Quantum Field Theory

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Lecture 22 : Representation of groups. Poincare group

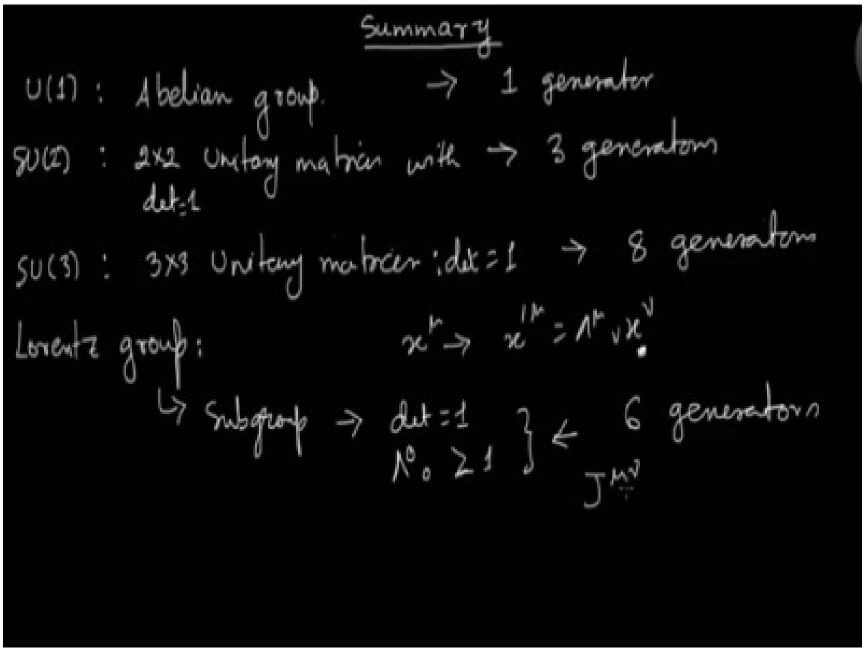


Figure 1: Refer Slide Time: 00:13

Summary

Okay, here is the summary of what we have learned so far. We introduced U(1) group okay, which you know it is an Abelian group. Let me go back and show you where it was, somewhere here I think we wrote, yeah here. So e to the i theta where theta is real, okay. And this satisfied all the properties. So this one was the U(1) group, okay. And it is Abelian because all the elements of this group, they commute with each other. So if you multiply element G 1 with element G 2, which are parameterized by the parameters theta, then the multiplication commutes. So this is an Abelian group.

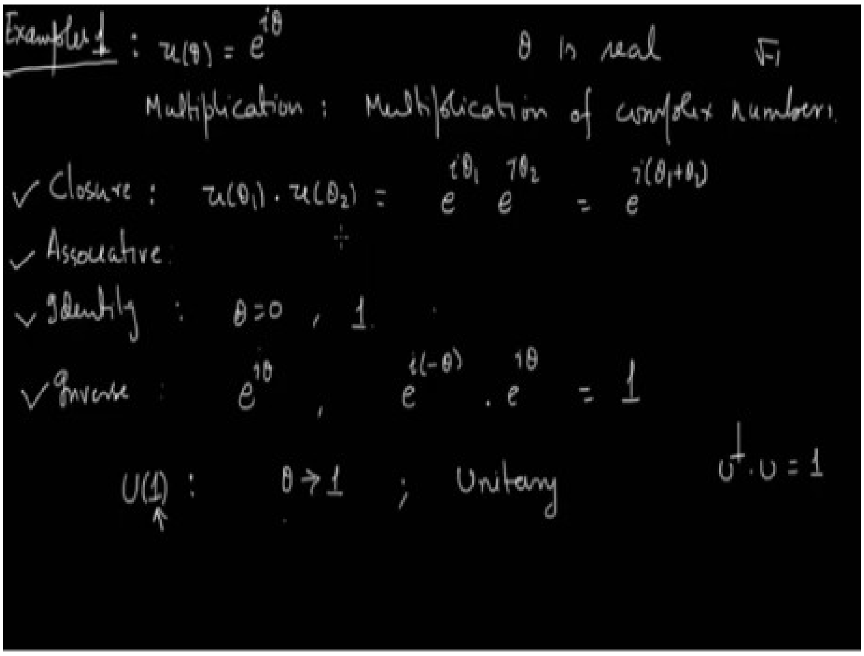
And we discussed other non Abelian groups such as SU(2), SU(3) in Lorentz group. Now when we were looking at SU(2) group, we introduced this group by studying 2x2 unitary matrices with determinant 1, okay. So our starting point was these matrices. And then we found that this is these matrices form a group and then we looked at the generators of the group by looking at the SU(2) matrices in the vicinity of identity, okay.

So we found that there are three generators T a, T b, T c or T 1, T 2, T 3, which are basically half of Pauli matrices, okay. Then we looked at the set of 3x3 unitary matrices with determinant

1, okay. And we found that that also forms a group and that is called SU(3) special unitary group, okay. And we found that it has 8 generators.

And also recall that we will, we started this entire thing by looking at Lorentz group okay, which is basically a set of transformations which are given by this. So if you start with x mu and go to x prime mu, the transformation is through these matrices lambda, okay. And we also learn that this group, which is Lorentz group has a subgroup which is the set of all elements which has determinant 1 and the lambda 00 element is greater than 1.

And you saw that it has 6 generators. So the generators were generically denoted by J mu nu okay. But you know that it is antisymmetric under mu nu interchange. So you have only 6 independent generators, okay. So that is what we had studied since till last time.



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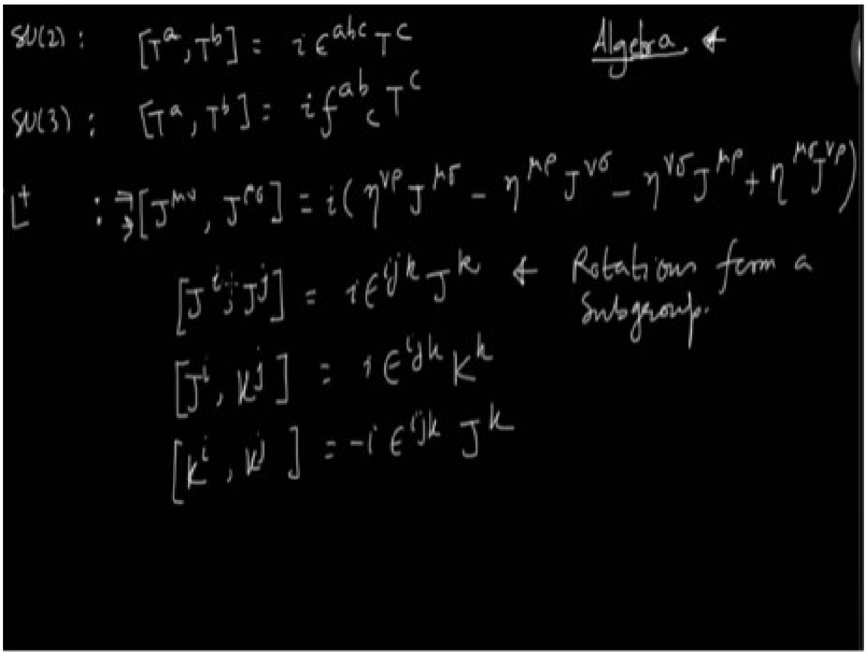
- U(1): Abelian group $\rightarrow$ 1 generator
- SU(2): 2x2 unitary matrix with det = 1, $\rightarrow$ 3 generators
- SU(3): 3x3 unitary matrix with det = 1, $\rightarrow$ 8 generators
- Lorentz group :$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $\rightarrow$ 6-generators ($J^{\mu\nu}$)

$$\begin{aligned} \text{Subgroup} &\rightarrow \det = 1 \\ &\rightarrow \Lambda^0{}_0 \geq 1 \end{aligned} \tag{1}$$

Algebra

$$SU(2) \quad : \quad [T^a, T^b] = i\epsilon^{abc} T^c \tag{2}$$

$$SU(2) \quad : \quad [T^a, T^b] = i f^{ab}{}_c T^c \tag{3}$$



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Then we had also calculated the commutation relations which these generators satisfy, okay. So if you look at SU(2) group then the commutators, if I look at commutator T a with T b, it comes out to be i times epsilon abc T c, okay. There is a summation over c implied here and your epsilon is Levi-Civita tensor, antisymmetric tensor and if you look at SU(3) similarly, you can find the commutation relations and I think I gave it as an exercise to find out all these constants, okay.

And note that I put the second index down here because this behaves differently compared to these two. So in the left hand side this object is antisymmetric under interchange of a and b indices. Because if you put T b followed by T a here okay, that is just negative of T a commutator. So it gives you a minus sign when you interchange a and b, which is what we imply here by putting these two on the top.

So if the two indices on the top are antisymmetric under interchange of a b, but there is no such property implied here between interchange of b and c. So that is why it has been put down to distinguish it from the status of these two. But of course if you choose the generators to be such that they satisfy this normalization property which we were discussing last time, this one okay.

So you can choose a linear combination of these Hermitian traceless generators to be such that this property is true. Then you can argue that this will be fully antisymmetric under abc interchange and in that case I will put the c up okay so that they all are having the same status. Also we looked at Lorentz group and here I have written down written down the commutation relation.

So as before you have 6 independent generators. So if I look at the commutation relations I get this object, okay. Now the same commutation relation we had written down by defining new objects which have only one index. So here J has two indices, here J has one index so clearly they are different objects, okay. So we introduced J i's and K i's. K i's were the generators of rotation, sorry boosts and J i's were the generators of rotation, okay.

And that these both objects are in J mu nu. So if you write down this commutation relation here in terms of boost and rotation generators you obtain the following, okay. So you see that if you look at J i and J j commutation relation it gives you something which involves again a rotation generator. But if you look at the commutation relation of J and K you have a K but if you look at K with K you get a J i rotation, okay.

So and this is happening because rotation forms a subgroup of Lorentz transformation and that is what you see here, okay. And these commutation relations are I mean the mathematical structure which is behind these commutation relations is what is called algebra. We will not go into giving the definitions of algebra. But for us it will suffice to say that these form the algebra of the Lorentz group or SU(2) group or SU(3) group. Okay, maybe I should say, make a remark here, okay. Anyway never mind. So here it is good.

$$L^{+} = [J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\nu}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\nu\sigma}J^{\nu\rho}) \tag{4}$$

$$\begin{aligned} [J^i, J^j] &= \iota\epsilon^{ijk}J^k \\ [J^i, K^j] &= \iota\epsilon^{ijk}K^k \\ [K^i, k^j] &= -\iota\epsilon^{ijk}J^k \end{aligned} \tag{5}$$

Rotation forms a sub-group

$$SU(2) \quad \to \quad 2\times 2 \quad \text{unitary matrix} \tag{6}$$

$$g_1 \cdot g_2 \quad = \quad g_3 \tag{7}$$

Okay. Actually what happened was I recorded this lecture and then I realized that the sound was not recorded. So now I am redoing it so that is why the content is already written down. Okay, and I am sorry that this has to be done this way this time. Anyway, so let us revisit SU(2). As I said, we were looking at 2x2 unitary matrices with determinant 1, okay. But if we were to forget about the fact that those matrices were 2x2, okay.

And just look at each matrix as an element okay, not really see the matrix there. Then given two elements, you get a third element by multiplying, okay. In that case, the multiplication is that of a 2x2 matrix, but let us forget how you get it but just see that given two elements you get a third element, okay. Now suppose someone comes and gives you another set of matrices which is not 2x2.

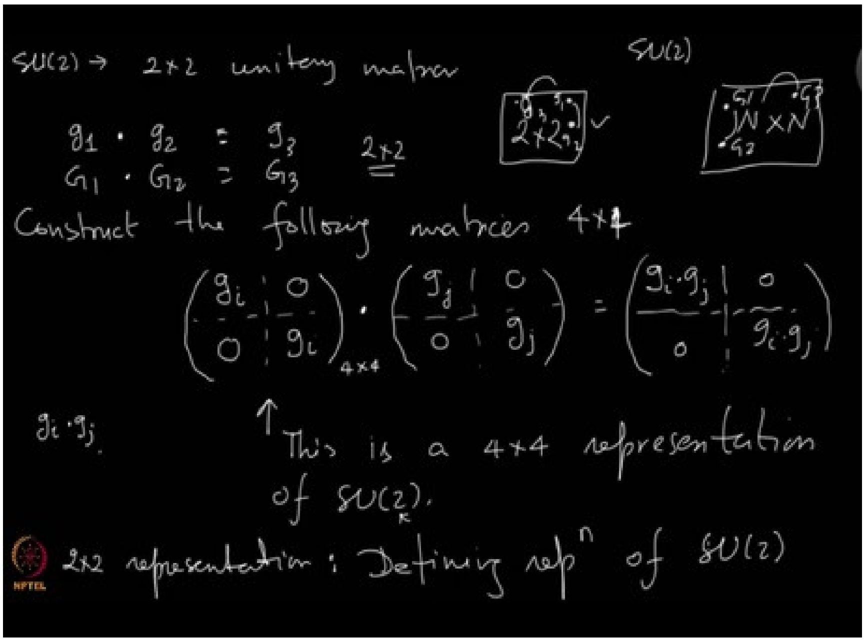
But it is such that if you associate so let us say that set has a the matrices in that set is NxN, okay and the elements are labeled as G 1, capital G 1, capital G 2 capital G 3 and so forth. And here I have written down the matrices which are 2x2 the SU(2) matrices and which I have written small g 1, small g 2 and small g 3. And if it so happens that the multiplication here is identical to the multiplication there, okay.

Construct the following 4×4 matrix

$$\begin{pmatrix} g_i & 0 \\ 0 & g_i \end{pmatrix}_{4\times 4} \cdot \begin{pmatrix} g_j & 0 \\ 0 & g_j \end{pmatrix}_{4\times 4} = \begin{pmatrix} g_i g_j & 0 \\ 0 & g_i g_j \end{pmatrix}_{4\times 4} \tag{8}$$

This is a 4×4 representation of SU(2)
2×2 representation: Defining representation of SU(2)

$$\begin{pmatrix} g_i & 0 & 0 \\ 0 & g_i & 0 \\ 0 & 0 & g_i \end{pmatrix} \quad : \quad \text{Representation} = R \qquad D^R(g_i) \tag{9}$$



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Meaning if I take the corresponding capital G 1, take the corresponding capital G 2 and what I get is the corresponding G 3 the one okay here. If it so happens then this group SU(2) group defined through a 2x2 matrices will be the same group here right, because the group is specified by telling what elements are there and what multiplying one element with another gives you okay.

There is you do not need to tell exactly how it appears as a matrix, what is its matrix form. That we do not have to specify. Even without that the group is specified. Which means that this would also be another way to represent the same SU(2) matrix, the same SU(2) group even though the matrices are NxN not 2x2 where n could be different from 2. But they are talking about the same object same group SU(2), okay.

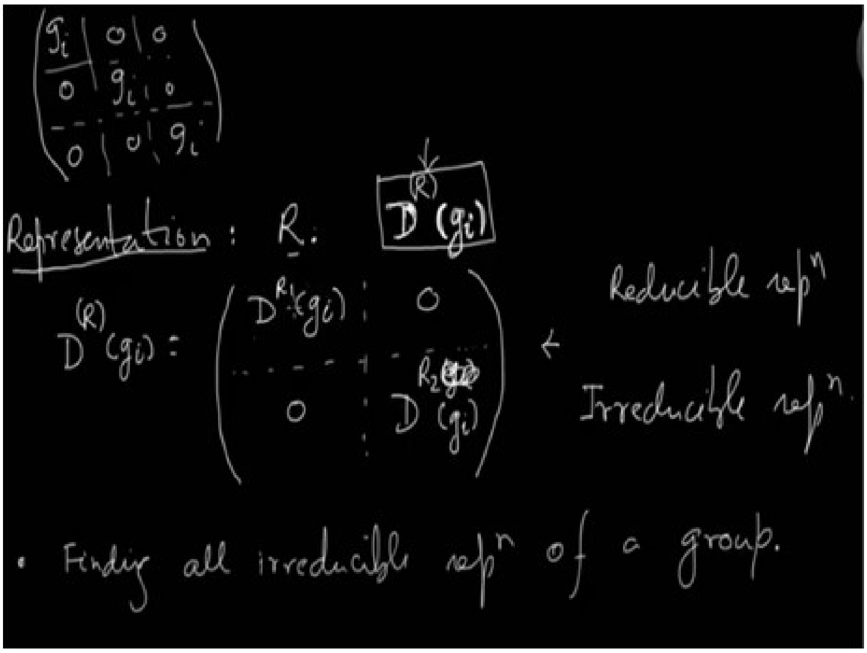
So right now we are not saying that 2x2 matrices form SU(2) group. We are saying 2x2 matrices form one representation of SU(2) group but you could have different representations, okay. Let me give you a specific example. So there was a mistake here. So construct the following matrices which are 4x4 matrices, okay. So right now g i small g i's, they are 2 cross 2 matrices, okay.

So I am imagining I put a 2x2 SU(2) matrix here, and the same 2x2 SU(2) matrix there and put a 00 here. So these zeros are basically 2x2 matrices, 2x2 null matrices. So this entire matrix is 4x4. And if I were to multiply this with this matrix, where again I have put 2x2 matrices here and 2x2 matrices here, then you can see that what you will get is this, okay.

And if you see that these these matrices, they satisfy exactly the same multiplication, as you had for SU(2), right? Because really, the multiplication is happening between these blocks, right? So it is not a surprise that what multiplication you are getting is identical to what you had for 2x2. So if you were to look at these matrices okay, this 4x4 matrices, they would be satisfying the exactly the same SU(2) group multiplication rule, okay. So we can say that these matrices form a representation of SU(2), okay. So as I was talking about here, this is a particular example of one representation of SU(2), which is not 2x2 but 4x4, okay.

That is what I have written here, this is a 4x4 representation of SU(2). So what is that 2 here in SU(2). Clearly it does not mean that you are looking at 2x2 matrices. What it means is that the way you have discovered this group is through, is by looking at 2x2 unitary matrices. So we call that 2x2 representation because now we can have representations of different dimensions.

So the 2x2 is also representation. And we call it defining representation because that is the representation that you use to define SU(2) group. So it is called a defining representation of SU(2), okay. And that is what I have written here, 2x2 representation is defining representation of SU(2).



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$$D^R(g_i) = \begin{pmatrix} D^{R_1}(g_i) & o \\ 0 & D^{R_2}(g_i) \end{pmatrix} \tag{10}$$

This is reducible representation.
Finding all irreducible representation of a group

$$dx^2 = dx'^2 \tag{11}$$
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \rightarrow \quad \text{Poincare group} \tag{12}$$
$$a^\mu \quad \rightarrow \text{translation} \tag{13}$$

You can think of many more. So here is another very simple example. You can put blocks like this, okay. And this will be obviously following the multiplication of SU(2). So in general, we say a representation R for SU(2) or any group, okay. So R is the symbol we use for specifying the representation.

So if you look an element g i of a given group okay, in representation R then it corresponds to the matrices which are represented by this, okay. So D (R) is the matrix representation in the representation R of the element g, okay. So this is sum matrix, okay. And this matrices will change, the dimensionality of them will change and elements of them will differ depending on the representation R you are looking at, okay.

So of course, there is one variety of representation that we have already talked about which is of this form. So if I look at a representation R and if I look at the elements g i and their corresponding matrices, so this is the matrix corresponding to the element g i in representations

R, then it would be, then it could be in this form where you have the matrix in representation R 1. So that is put in one block.

Then you have these null matrices here. And then you have the element g i and sorry it should not have been here, it should have been here. But in representation R 2 okay, where R 1 is different from R 2. You can take it to be different. So these matrices are from another representation and if you put them together like this in a block, in these two blocks in this matrix that will also form a representation.

Because when you multiply such matrices the multiplication will happen in the respective blocks, okay. And because these form a representation, this entire thing will also form a representation. Okay, and if you are able to write any representation in this form where you have such blocks it is called reducible representation, okay. It is reducible because the real thing is in here or in here, okay.

And if you cannot write a representation in this form as blocks here, then it is called irreducible okay. And of course, it would be interesting to know all the irreducible representations of a group. Because once you know all the irreducible representations of a group, you can construct all other remaining reducible ones because you just have to put blocks like this okay whichever way you like. And of course, the 2x2 matrices of SU(2), they are irreducible representations because they do not have this structure, okay.

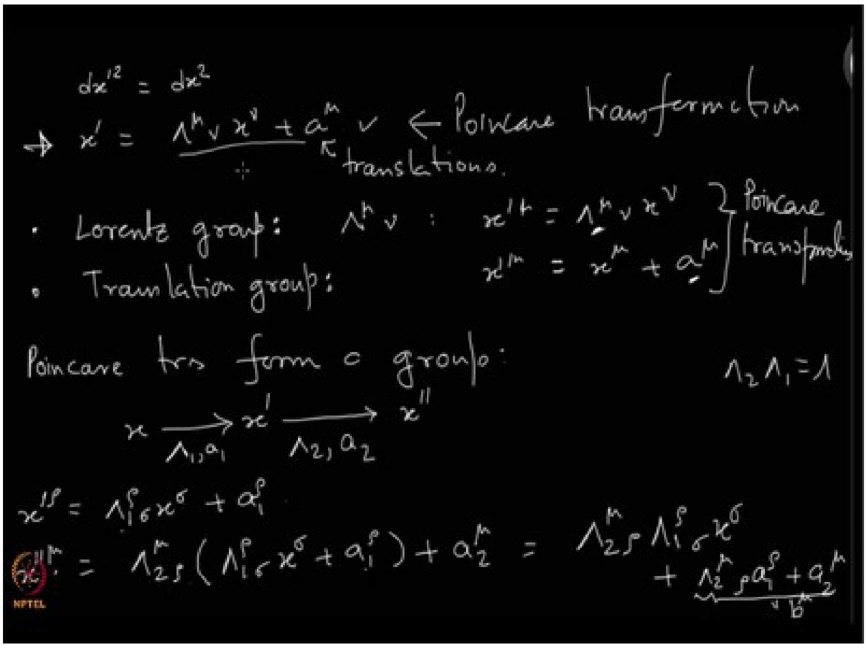

Figure 6: Refer Slide Time: 17:35

- Lorentz group :$\Lambda^{\mu}{}_{\nu}$; $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$
- Translation : $x'^{\mu} = x^{\mu} + a^{\mu}$

Both transformation combined together gives Poincare group, Poincare transformation forms a group.

$$x \longrightarrow x' \longrightarrow x'' \tag{14}$$

$$x'^{\rho} = (\Lambda_1)^{\rho}{}_{\sigma}\, x^{\sigma} + a_1^{\rho} \tag{15}$$
$$x''^{\mu} = (\Lambda_2)^{\mu}{}_{\rho}(\Lambda_1^{\rho}{}_{\sigma}\, x^{\sigma} + a_1^{\rho}) + a_2^{\mu} \tag{16}$$
$$x''^{\mu} = \Lambda_2^{\mu}{}_{\rho}\Lambda_1^{\rho}{}_{\sigma}\, x^{\sigma} + \Lambda_2^{\mu}{}_{\rho}a_1^{\rho} + a_2^{\mu} \tag{17}$$
$$x''^{\mu} = \Lambda_2^{\mu}{}_{\rho}\Lambda_1^{\rho}{}_{\sigma}x^{\sigma} + b^{\mu} \tag{18}$$
$$\Lambda_2\Lambda_1 = \Lambda \tag{19}$$

Thus Poincare transformation forms a group, elements of Poincare group (Λ, a)

Okay, now I want to talk about we go back to our this Lorentz transformations, okay. So we were looking at all the transformations which keep this unchanged, okay. And we found that the most general solution was this that x prime is lambda mu nu x nu plus a mu where a mu is constant.

And we also talked about the fact that this a mu's are the translations, so they generate time translations and space translations, okay. So if you took lambda to be identity meaning you are not doing any Lorentz transformation, then it is x prime is x plus a mu. So it is just shifting the values of x, okay. And we have already seen that Lorentz transformations form a group.

So if you look at all the matrices lambda mu nu and the corresponding transformations here then they form a group, okay. And then we have translation group. So if you take these transformations x prime mu x mu plus a mu you can easily verify that these transformations form a group, okay.

Because you have to check the closure, inverse and everything and identity and it will be very easy to verify that translations do form a group. And these two transformations together is what constitutes Poincare transformations. So here what you have is Poincare transformation, okay. So if you exclude the translations, then you have only Lorentz transformations.

If you include translations and have both Lorentz transformation and translation then you call Poincare transformations. Now you may wonder whether Poincare transformations form a group and indeed they do. And here is how to find it. So suppose you start with a spacetime point x and do a Poincare transformation which is parameterized by a lambda and a okay where a is constant and of course, this is also a constant matrix, okay.

So you do a transformation on x and go to x prime, okay through and the transformations are parameterized by lambda 1 and a 1 and this is exactly the transformation that you have. And once you have reached x prime you do another transformation which is now parameterised by lambda 2 and a 2 and you reach some x double prime. So how do I verify that these transformation form a group?

What I should do is do this followed by that and whatever I get I should be able to put it in this form. Whatever I get I should be able to say that it is a Lorentz transformation acting on x followed by a translation. If I can say so then I would have proved that Poncare transformations form a group, okay. So that is what we are doing here. So x goes to x prime, so x prime rho is lambda 1 rho sigma x sigma.

That is this part here plus a 1 rho okay, that is the translation part. Now once I have reached here, I do the second transformation on this object and now with lambda 2 a 2. So x double prime mu is what is there here lambda 2 lambda 2 mu sigma acting on x prime and what is x prime? x prime is this, right? So I put it here. Lambda 1 rho sigma x sigma plus a 1 rho plus a translation transformation plus a 2.

So I should write a 2 mu, okay. Now this is I can write as lambda 2 lambda 1x. So that is what I write here, lambda 2, lambda 1x and let us see whether all indices are fine. Sigma is contracted, so that is not an index free, not a free index. Rho is contracted, it is not a free index. Only mu is a free index, which is on the left hand side. So that is good. Here rho is contracted only mu is free. So that is also good and I have mu here, okay. So let us call this last two terms together as b mu. See these are constants, right? There is no x in this part, there is no x mu in this part; x is only in this part. So as far as a translation is concerned, I have a translation. You see the translation part is not multiplied with x. Translation part sits separately. So that is the part here that is the translation.

Now let us look at this part. This part has two lambdas unlike the lambda here, but that is not an issue because we know that Lorentz transformations form a group. Meaning one lambda followed by another, the multiplication of two lambdas is again another lambda because it is a group, right. So lambda 1 times lambda 2 will be some lambda within the group.

So these two together is some lambda, which means what we have here has exactly the same form here, okay. Thus, we have proved that Poincare transformations form a group, okay.

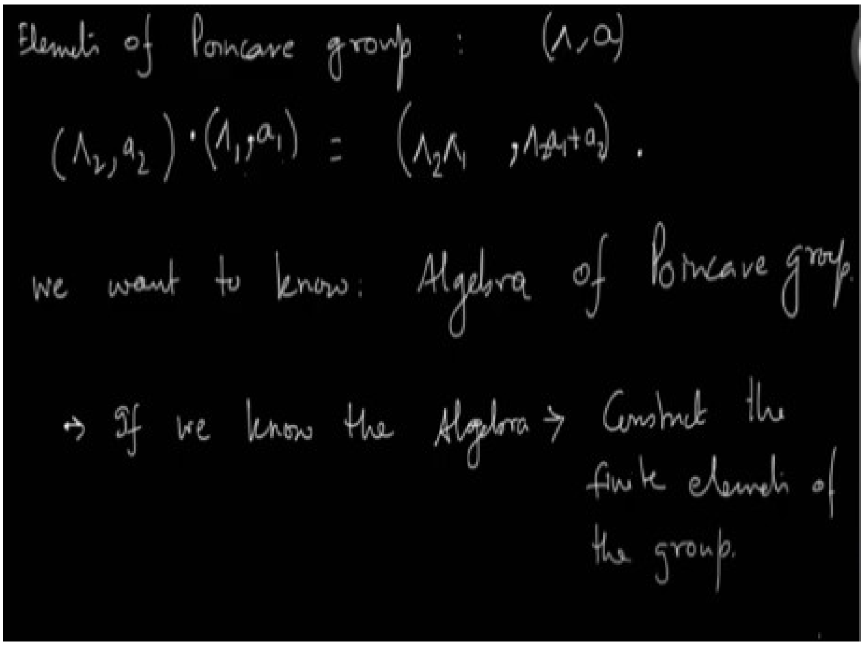

Figure 7: Refer Slide Time: 24:00

So if I write down the elements of Poincare group like this, so I say that there are two pieces to it, one is the translation part, the one element is one parameter is translation, actually these are four parameters not one and another is lambda, okay. Then if I take one transformation, which is parameterized by a 1 lambda 1, followed by the another one, which is a 2 lambda 2, so I am imagining I am doing a transformation on some x here okay, some x here.

First by this and then by that. Then what I get is, this piece, okay. Let us go back and check how do I get this. So here x double prime is exactly what I was writing on the next page. And the answer is that you get a 2 plus lambda 2 a 1, a 2 plus lambda 2 a 1, okay; a 2 lambda 2 a 1. So that is the translation you have. And this one was of course lambda 1 times lambda 2, okay.

And you would like to know the generators of Poincare group and what is the algebra of Poincare group. So we would want to know algebra of Poincare group, okay. So what are the generators of translations and how do they commute with themselves and Lorentz transformations, okay? So that is what we want to know and that is what we will take up in the next video.

But the reason why this commutation relations are important is because once you know this commutators you can construct finite transformations. And these commutators are the only thing that you know, that you need to know, okay. So if you know, if we know the commutators meaning the algebra, then we can construct the finite elements of the group oops okay.

And you do not need to know anything more than the commutators. So you do not have to know for example commutator with several, so you do not need to know something like this for example. Let us say K i, J i okay and something here. Let us say again K i. Do not need to know these kind of. Just knowing the commutators is sufficient.

So once you know the commutation relations with two generators, they will give you all the elements which are connected to the identity, okay. See there may be parts of the group which cannot be generated by doing knowing only the generators of the

algebra, right? Because for example, you remember in Lorentz transformations you also have parity and time reversal.

And those things you will not get by using you know the rotation and boost generators. They cannot produce parity transformation. But you will be able to produce all the elements of the group which are connected to identity by using the generators, okay. So that is why the algebra is important. And we will look at the Poincare group and its algebra in the next video.

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{20}$$

We want to know the algebra of Poincare group and their generator.

• If you know the algebra, then you can construct the finite element of the group we only need to know the simple commutator.