Introduction to Quantum Field Theory

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Lecture 21 : SU(3) Generators

SU(3)

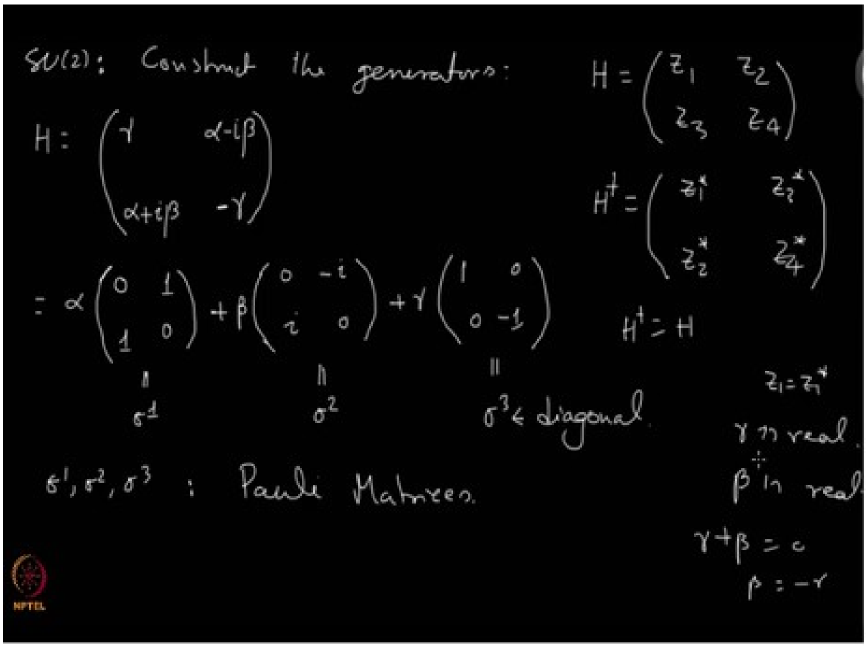


Figure 1: Refer Slide Time: 00:26

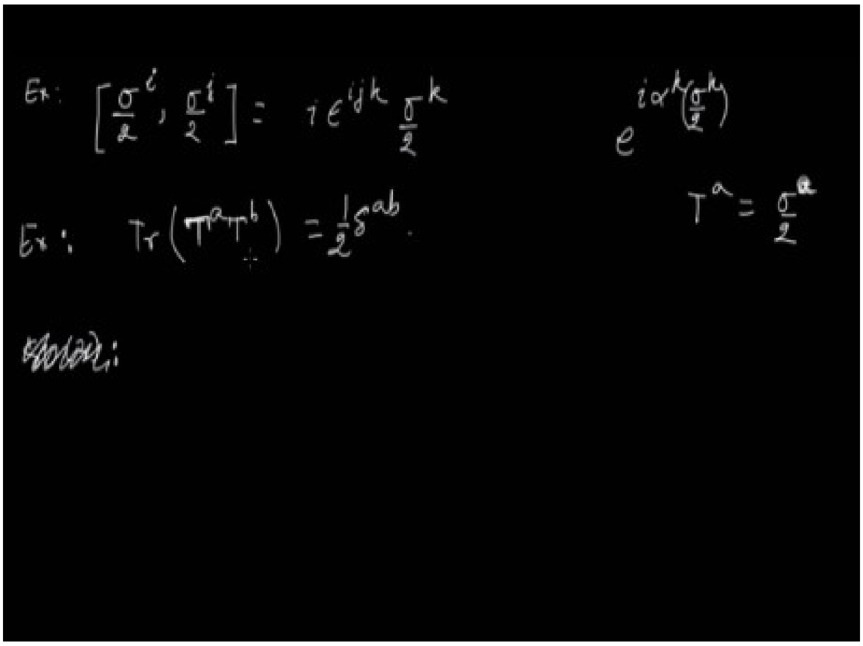
In the last lecture, we talked about SU(2) group and discussed in some detail that group. So here for example, we wrote down the generators of SU(2) which are half times the Pauli matrices and these Pauli matrices we found when we were looking at this quantity here, okay.

Generators : Repeat same steps

$$H = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} + \begin{pmatrix} 0 & z_2 & 0 \\ z_2^\star & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & z_3 \\ 0 & 0 & 0 \\ z_3^\star & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z_4 \\ 0 & z_4^\star & 0 \end{pmatrix} \tag{1}$$

$$x + y + z = 0 \tag{2}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3}$$



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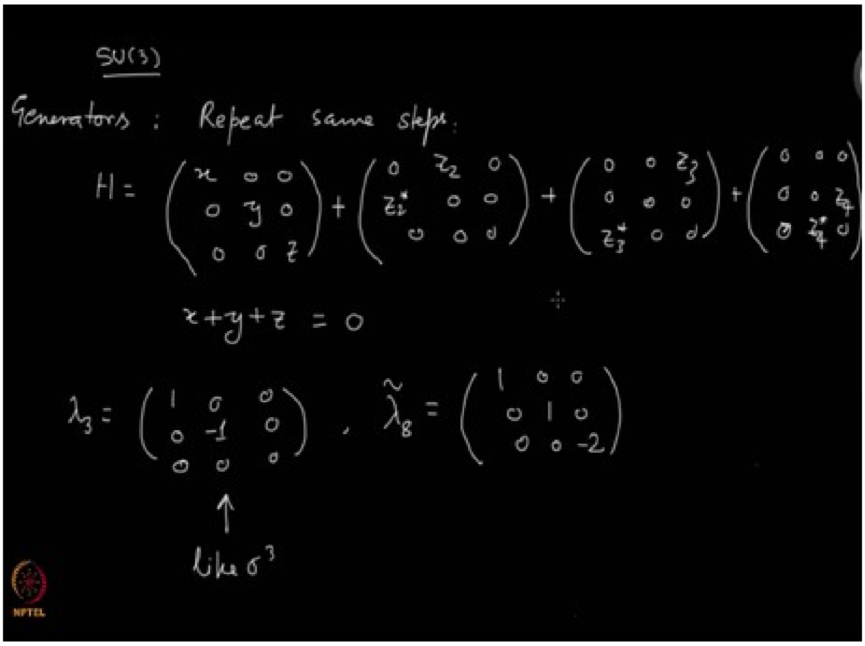


Figure 3: Refer Slide Time: 02:46

And then I gave you for homework to check that if you look at commutators of these generators, the generators are half of Pauli matrices then the generators of various commutation relations, okay where epsilon ijk you know that this is completely anti-symmetric matrix with epsilon 1 2 3 as 1 okay and other elements are determined by the anti-symmetry property. Also I gave another exercise that you check that the trace of this matrix, So trace of T 11 trace of T 12 and so forth. So this forms a matrix and this matrix has I mean the trace of this matrix is this much. Okay, let me say it more correctly. So the left hand side is a matrix and you were asked to show that the, that matrix is proportional to identity matrix and the proportionality constant is half. Okay, so that was the exercise.

Now we want to look at SU(3) group. Not that we needed in this course but because we have done SU(2) and exactly the same steps can tell you about SU(3) and give you some experience with working with groups, okay. So that is the only reason why I am including this. So if you wish you can omit this video without any consequences in later parts.

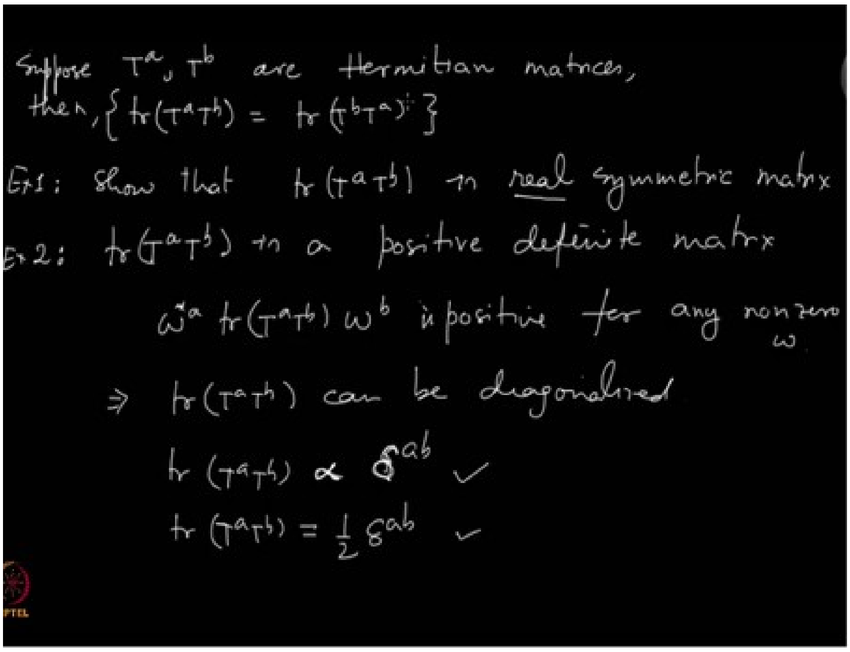


Figure 4: Refer Slide Time: 07:26

Suppose T^a, T^b are hermitian matrices then

$$\{Tr(T^aT^b) = Tr(T^bT^a)\} \tag{4}$$

Exercise (1): Show that $Tr(T^aT^b)$ is a real symmetrix matrix
Exercise (2): Show that $Tr(T^aT^b)$ is a positive definite matrix

Positive definite matrix:$Tr(T^aT^b)$ is postive definite if

$$\omega^{\star a}Tr(T^aT^b)\omega^b > 0 \quad , \quad \omega \neq 0 \tag{5}$$

Okay, so let us ask what are the generators of SU(3) group? Generators okay, that is what you want to know. So if you do repeat the same steps as before for SU(2) meaning you start with yeah, writing such a matrix with all these entries z 1, z 2 and so forth, okay. Here, so z 1, z 2, z 3, z 4. So you do the same thing for 3x3 matrix and then impose that H dagger should be H that the Hermitian conjugates are equal and write something of this sort, okay.

So that is what you are supposed to do. And when you do so you find the following. That your H will take the following form where x, y and z are real, okay. So let me go back and show you. Just like here on the diagonal entries, you got the diagonal entries, you got real numbers, rather than complex. That is what you are going to get here as well.

That these are complex and that is going to follow from the fact that H dagger should be H plus you will get 0, some complex number here, z 2 and the conjugate will appear here and 0 0 z 3 z star 0 0 and finally 0 0 0 okay. That is what you will get. And just as before your z 2 you write x plus i y or x minus i y and you will get these matrices, right. Just like what you got here, you will get 1 1 and –i i okay.

Exactly the same thing will get repeated. So this is what you will get and also the tracelessness condition tells you that this is 0. So this will correspond to two matrices, 2 here 2 here 6 and

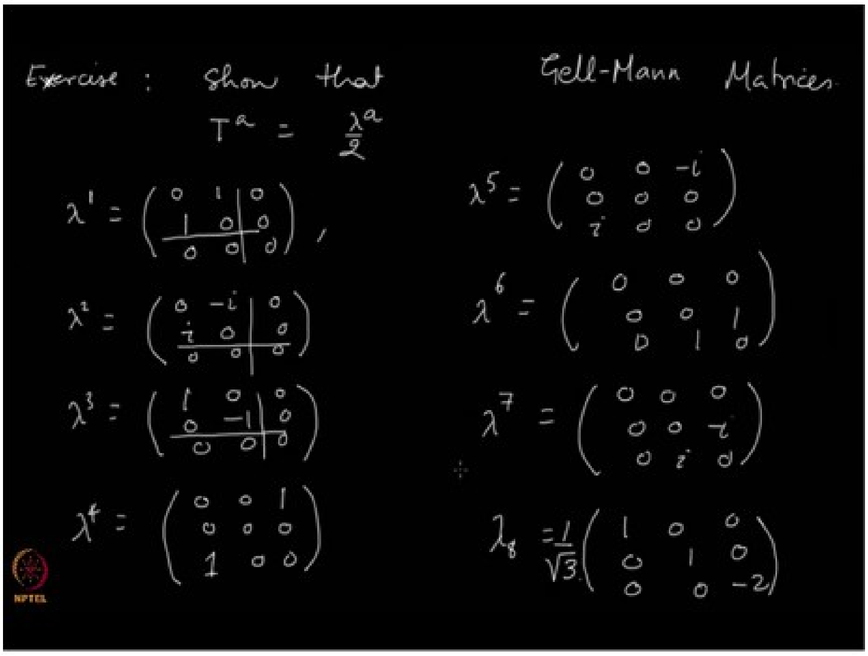


Figure 5: Refer Slide Time: 12:12

although they are 3 entries here, so would be 3 but then you have one constraint, so 2. So each one is giving 2. So it is 8, 4 times 2 is 8. So you get 8 generators, okay.

Now this ones the two matrices that you are going to get from here they both are diagonal, okay. And these diagonal matrices you can take to be lambda 3. I mean we call lambda 3 and lambda 8 the matrices that you are going to, the two independent matrices that you are going to get from here, okay. They are called lambda 3 and lambda 8. 0 and 0 -1 0 0 0 0 that you will get.

And I will call lambda 8 tilde and this matrix would be 1 0 0. Okay, I can choose this because these are linearly independent, okay. So you will note that this is like your sigma 3, okay. So these this is like sigma 3, which you had. And this one will have the other two generators of SU(2) okay. So these three together this sigma 3 and the two matrices coming from here they will have exactly the same form as the Pauli matrices, okay.

$Tr(T^aT^b)$ can be diagonalized

$$Tr(T^aT^b) \propto \delta^{ab} \tag{6}$$

$$Tr(T^aT^b) = \frac{\delta^{ab}}{2} \tag{7}$$

Exercise: Show that

$$T^a = \frac{\lambda^a}{2} \tag{8}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{10}$$

Now okay. So I will give you an exercise. Suppose T a and T b are Hermitian matrices. Then trace of T a T b okay right now Hermitian is not important, let me anyway continue writing that. So trace of T a T b is trace of T b T a. Okay, let me write it in bracket. So this is not the part of the statement, okay. This is anyway true.

So suppose T a T b are Hermitian matrices, then exercise number 1, show that trace of T a T b this matrix that you have is a real symmetric matrix. Because you can interchange a b and you do not get a different thing, you get the same thing so it is symmetric. But the thing that you have to show is that this is real okay, this matrix is real, that you can show.

And then second exercise is and the second exercise is that trace of T a T b is a positive definite matrix, okay. So what is a positive definite matrix? I think you are already aware of it. So if you take any matrix which is positive definite, in this case trace T a T b and you sandwich with, so take a complex column vector omega a okay and multiply this T a T b this matrix. Okay, so you have a row here and a column here and omega is complex.

So I have put a star on the left hand side. Okay, if you calculate this, this should give you a positive value. This is positive for any nonzero omega, okay. So if you are given any nonzero omega column vector, which is complex and you construct this object it should be positive and if it is true then this is a positive definite matrix. So show that this matrix trace of T a T b is a positive definite matrix.

Now once you have shown that then it follows that trace of T a T b can be diagonalized, okay. So because you have a real symmetric matrix which is positive definite and that can be diagonalized, which means that you can put it in the following form. It will be proportional to delta ab. So here what I mean to say is that you can choose to I mean you can choose different T a's okay for which this will be true, okay.

So you are allowed to, you will be able to find such T a's for which this will hold true. Okay, that is identity matrix. And you can choose the proportionality constant to be half. But you do not have to but you can choose it to be half. And then trace T a T b is half delta ab. And this is exactly what you saw in the case of generators of SU(2), okay. So this could be done because of this property that I have given as an exercise, okay.

Now another exercise. Show that T a are lambda a over half, okay. So the generators of SU(3) are given by the following matrices which I am going to write where lambda a. So let me write lambda, I think okay lambda 1 is, okay that is lambda 1. We write lambda 2 is 0 –i 0 i 0 0. So if you look at this one, this is really the sigma 2, I think it is called sigma 2. This is sigma 2 even though this is 3x3, but you can identify the sigma part here, okay.

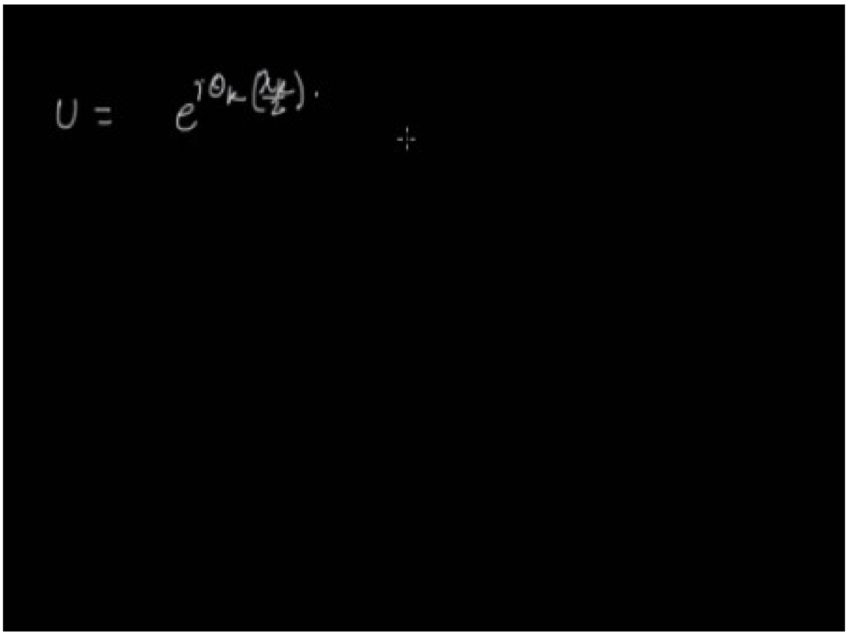
So let me remove this thing. Or let it be here no problem. And then lambda 3 is 0 sorry 1 0 0 0 -1 0 0 0 0. So if you look at sigma 1, sigma 2 and sorry lambda 1, lambda 2, and lambda 3 and if you look at these blocks, these are exactly the Pauli matrices that you have, okay. Then let us look at lambda 4. And that turns out to be and this you will get directly when you are doing the steps.

0 lambda 4, okay. And lambda 5 is, okay. Again the same thing. Of course it is not and that is just like this one, but it is exactly -i i i and –i i and that appears here. And lambda 6 let me write down that is 0 0 0 0 0 1 0 1 0 okay. Again you see this thing. 1 1 here. It was the same in lambda 1. Lambda 4 5 6 7. So 7 would be, okay. And lambda 8 is the following.

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{11}$$

These are Gell-Mann matrices, they are the generators of SU(3)

$$U = e^{i\theta_k \frac{\lambda_k}{2}} \tag{12}$$



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Okay you can choose them right to be this, but because we are going to impose this condition that trace T a T b is half delta ab that is what we want to impose, then you can check that you will have to put 1 over square root of 3. So if you use these as the matrices, these are called Gell-Mann matrices, then the generators will satisfy this property, okay. So this is a simple exercise. And then of course, any element of SU(3) you can write as the following. Okay, these are the generators. Okay, so please do these exercises so that you get some practice with playing withe matrices