Introduction to Quantum Field Theory

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Lecture 20 : Groups and Generators

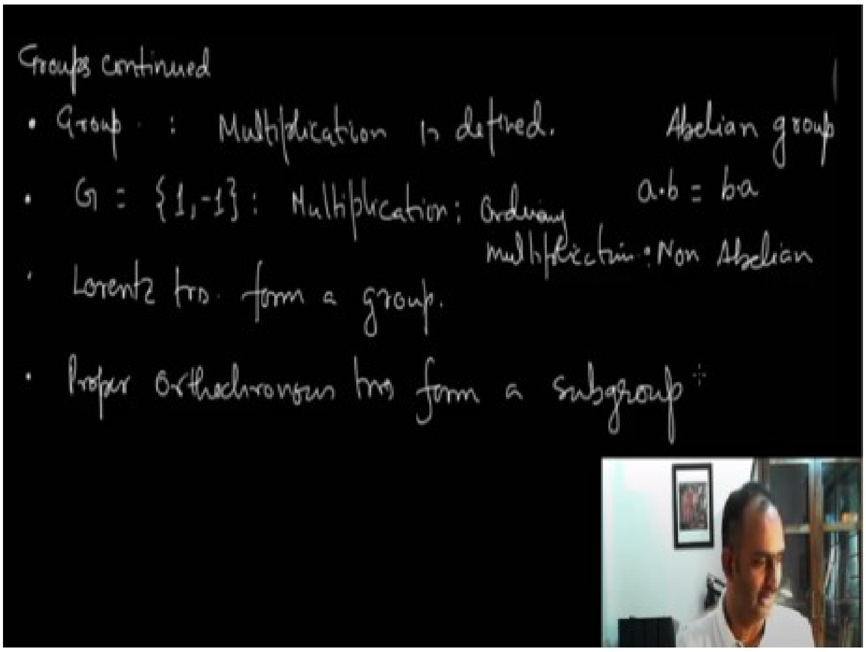


Figure 1: Refer Slide Time: 00:50

Welcome back. We were discussing about groups in the last video. And although we do not have a lot of time in this course to discuss about groups in detail, but avoiding it causes lot of trouble. So what I will do is make our discussion enough so that things are a bit easier, okay. So that is the plan. So I will continue with the discussion of groups and if you recall, we have already defined what a group is, okay.

- Group multiplication is defined, for abelian group

$$a \cdot b = b \cdot a$$

- $G = \{-1, 1\}$, multiplication: ordinary multiplication.
- Lorentz transformation forms a group.
- Proper orthochronous Lorentz transformation forms a subgroup.

And in that definition, there was only one operation that was defined okay, that the set has, there is an, there is a multiplication that is defined on that set, okay. So you can multiply two elements okay, just a second. There was some noise from outside, still is. Anyhow, so there is only one operation that is defined which is what we call multiplication, okay. And in general, if you multiply an element A times element B okay, it will give you some element C.

But multiplying in the reverse order B times A may not give you C okay. In some special cases it can happen, but in general it will not happen. So when it happens that A times B is B times A for all the elements that is called an Abelian group, okay. Meaning for all elements A and B this is true, okay. And in general, the groups are not Abelian. So they are called non Abelian groups, okay.

Recall that I gave a simple example, where a group G was containing just two elements 1 and -1 and the multiplication was defined to be an ordinary multiplication right, between real numbers. And this formed a group because all the four properties of a group were satisfied by these two elements, okay. And we have also seen that Lorentz transformations form a group.

We also noted that if you take a subset of Lorentz transformations, which is proper orthochronous transformations, remember that is the one which determinant of a matrix equal to 1 and lambda 00 term to be positive, okay. Those set of elements also formed a group under the same multiplication law, under the same multiplication. And that is it so happens you say that the subset forms a subgroup of the bigger group.

So proper orthochronous transformations form a subgroup of, form a subgroup okay of the full set of Lorentz transformations, okay. And remember, you have to have the identity element. So this subgroup should have the identity element of the full Lorentz group, right. So this identity element has to be shared with all the subgroups if there are more than one subgroups in the full group, okay.

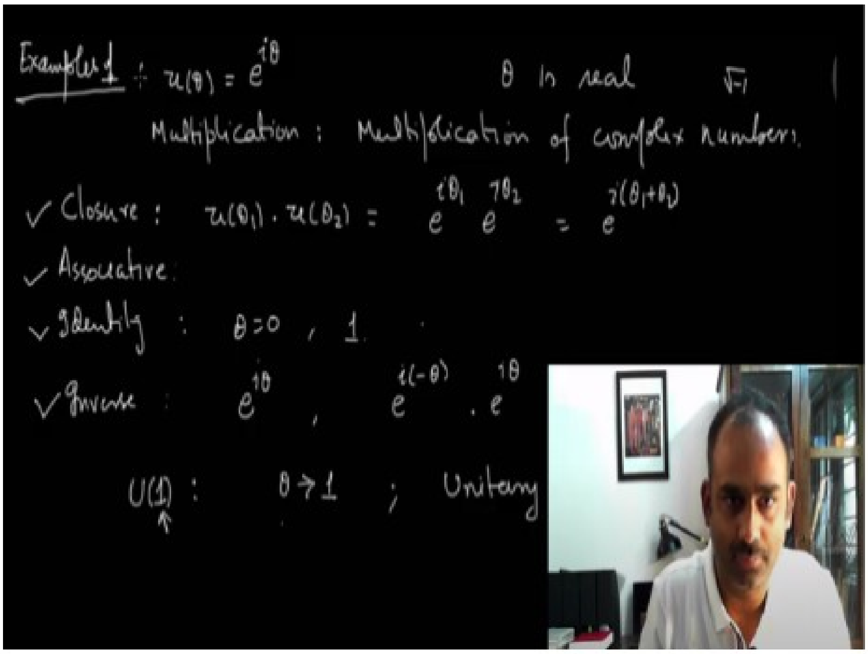
So that is what we discussed. Now I will do a few more examples so that we understand better about groups and groups is a very useful mathematical tool or language that we will use or we use in quantum field theory, okay. So even though I may not, in fact I will not be talking about all the groups that are discussed now in this course.

But it is nevertheless useful to have them you know learn in the beginning of the course on quantum field theory. So when we have another course on quantum field theory, which I plan to have, these things we can directly use, okay? So let me give you a few examples. So think of the following. Suppose you take all the numbers which are given by this okay, where theta is a parameter, real parameter. Theta is real so it is a parameter; i is the square root of -1, okay. So let us call these numbers as u of theta. So I am looking at the set of all numbers, which are written as e to the i theta, okay, where theta takes real values. Now I claim that this forms, this set forms a group. All these elements form a group, okay. So let us check about closure.

So and by the way, you should ask what is the multiplication I have defined. Unless I define that there is no meaning to talking about a group. And multiplication is ordinary multiplication between complex numbers. So that is the multiplication law that I have, okay. So let us see whether closure is satisfied. So I take an element u of theta 1, multiply with element u of theta 2.

Then what I get should be again an element of, again an element within the same set, okay. So let us check that. It is obvious because this is e to the i theta 1, sorry, times e to the i theta 2. This I can write as e to the i theta 1 plus theta 2. And theta 1 plus theta 2 is again a real number, which means e to the i theta 1 plus theta 2 does belong to this set, okay. So closure is true. Associativity is true, because multiplication of complex numbers is associative multiplication, so that is also true. Then identity, do we have an identity element? Yes of course, if I put theta equal to 0, then I have 1, okay. And 1 times e to the i theta gives you again e to the i theta. So theta equal to 0 corresponds to the identity element, so there is an identity element. Associativity is true, closure is true.

Do I have an inverse for each element? And that is also true that we have. So suppose I am given an element e to the i theta, okay? Then e to the i minus theta. This guy when you multiply with e to the i theta gives you identity, okay? And that is true for any theta. So we do have an inverse for all the elements, okay. So this forms a group, this set of numbers form a group and this has a name, it is called U(1), okay.



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Example of groups
Example:1

$$u(\theta) = e^{i\theta} \quad ; \quad \theta \text{ is real and } i = \sqrt{-1} \tag{1}$$

Multiplication:Multiplication of complex numbers

- Closure: $u(\theta_1) \cdot u(\theta_2) = e^{i(\theta_1+\theta_2)}$
- Associative
- Identity: $\theta = 0$ will correspond to identity element which is $e^{i0} = 1$
- Inverse: $e^{-i\theta}$, $e^{-i\theta} \cdot e^{i\theta} = 1$

This group is called U(1). 1 meaning you have only one independent parameter which is theta here. So it is one dimensional basically, okay. So there is only one parameter theta, that is why you have 1 here. And u means unitary. By that I just mean that if you take a u dagger multiply with u, you get 1. But here, this dagger is superficial because it is just complex conjugation, okay.

So the name u comes from unitary. So you can call it a unitary group of dimension one. I have not defined what is a dimension, but unitary group of 1x1 matrices you can call it like this, okay? So that is one example, which is okay. Now let us move on to something else. Now I want to talk about another group. So let me say this is example number 1. Now let us look at example 2. Now show that, which is an exercise, this example 2 is an exercise. Show that all 2x2 unitary matrices form a group, and multiplication I should specify under matrix multiplication, all okay. So that is what you should do. So take all 2x2 matrices and show that their set forms a group. So what you have to do is check whether if you multiply two matrices which are unitary and 2x2, whether they give you again a 2x2 matrix, which is unitary.

That should be easy to check. Then second is associativity that is automatically satisfied because matrix multiplication is associative, so that is done. Third is identity and of course you know identity matrix 2x2 will serve you, will serve as an identity in this in the set. And fourth is inverse. So you should be able to convince yourself that for each 2x2 unitary matrix there exists an inverse, okay.

And if this is true then this set will form a group, okay. Another exercise which is an example and also an exercise. So by the way this one is called U(2) okay, set off all 2x2 unitary matrices. So it is a group of 2x2 unitary matrices, okay. Now show that if you take a subset of this and which is the following.

So you take all the 2x2 matrices, 2x2 unitary matrices whose determinant is 1, okay. Then this also forms a group under matrix multiplication, so it means it forms a subgroup of U(2), okay. And that is what you should be able to do, not difficult. So here the only additional thing is you have to check that if you multiply two matrices which have determinant 1, what you get is again a unitary matrix with determinant 1, okay.So that should be easy and if I remember that was yeah, that also happened in the case of Lorentz transformations when you were looking at the proper orthochronous part of it, okay. So it is the same kind of same exercise here. Now let us look at SU, a more generalized version of this. In fact, there is not much to prove. If you have done the two and three, this one is automatic

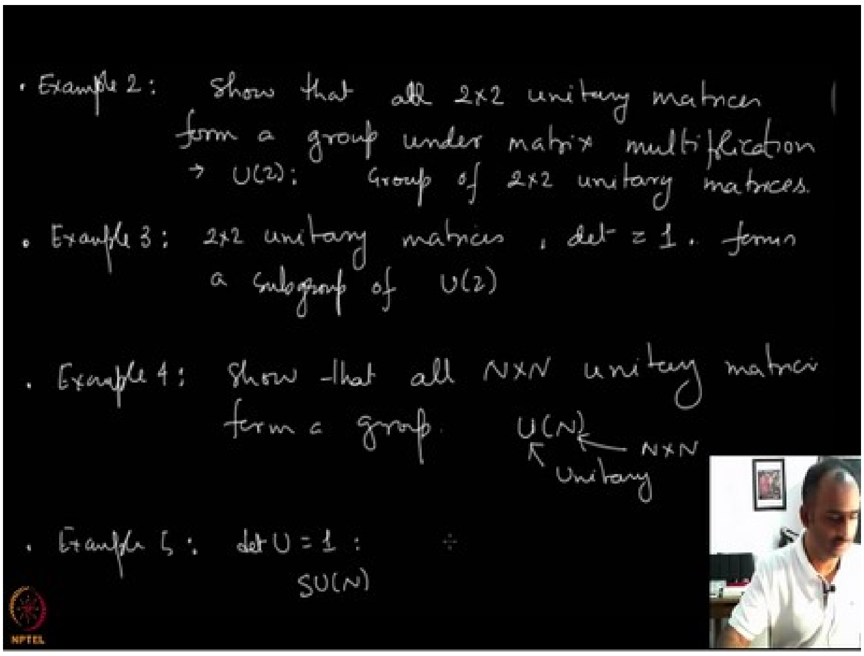

Figure 3: Refer Slide Time: 10:01

This forms a group called U(1), U $\rightarrow$ Unitary group.

- Exercie:1 Show that all 2x2 unitary matrices form a group under matrix multiplication named U(2)

- Exercise:2 All 2x2 unitary matrices with det = 1, forms a subgroup of U(2) names SU(2).
- Exercise:3 Show that all NxN matrices form a group named U(N)
- Exercise:4 All NxN unitary matrices with det = 1, forms a subgroup of U(N) named SU(N)

SU(N)

Show that all NxN unitary matrices form a group, okay. Same reasoning will give you. And this is called U(N). U stands for unitary and N for NxN matrices, okay. And obviously, you can expect that if I look at a subset of this, where the determinant is 1 for all the matrices it will also form a group. So all the elements in the group which have determinant U equal to 1 will also form a group.

And that is a subgroup and this one is called SU(N). So U is for unitary. N is for NxN and S means special, okay. And why it is special? It is special because determinant U is 1 for those elements, okay. So it is called special unitary matrices, okay. So please show that this SU(N) is a subgroup of U(N), that you should be able to show. Now I will talk a little more about SU(N). So right now we are doing pure mathematics. Physics will come after some time, but this is important. So let us look at SU(N). They are unitary matrices, which means U dagger U is equal to U U dagger and that is equal to 1, okay. But on the top of it, we said special unitary, which means determinant of U is also equal to 1, okay.

Right now when I am writing U, U is an arbitrary element of these groups okay, of SU(N), okay. So I am denoting the elements of the group by this capital U, okay. So now what I want to do is I want to see how to generate all the possible elements of this group, okay. That is what I want to do. I want to understand how to get all the elements of this subgroup or group. So let us see. What we will do is the following.

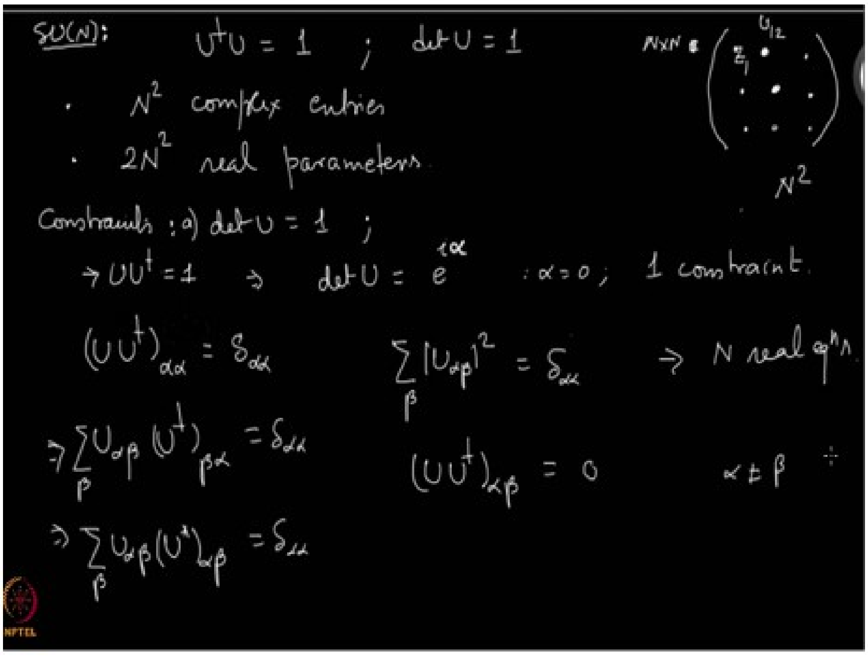

Figure 4: Refer Slide Time: 15:20

$$U^\dagger U = 1 \quad ; \quad \det U = 1 \tag{2}$$

How to get the elements of this group, N^2 complex entries that will give $2N^2$ real entries

Constraints

So if you have NxN matrix, how many entries you have? You have, so if you count all of these, this is NxN, so it will be N square entries. Now each N square entry is complex, okay. So when I am looking at an arbitrary complex NxN matrix, see I am starting with the most general thing, what is the most general NxN matrix, which I can write, which is complex? It is this.

It will have N square complex entries, okay. So the most general, right now this is not unitary, I have not imposed any constraint, I am just writing NxN complex matrices. So you have in total N square complex entries. But each complex entry is a sum of two real entries, right? So if this one let us say I call z1, and z1 is x1 plus iy1. So x1 and y1 they are in your hands right, that you choose.

So these are 2 real numbers. So when you are looking at NxN complex matrix, you have freedom to choose 2N square real numbers or parameters, okay. So these are, there are 2N square real parameters that parameterize this matrix, okay. That is good. Now I want to know about SU(N). So what I will do is I will count how many constraints there are, okay.

So given total number of parameters that I have, minus the total number of constraints, that will give me what are the total number of independent parameters t will parameterize a SU(N) matrix, that is what I am doing. So let us see constraints. So first constraint is easy, determinant U is equal to 1. So as I said, we have to count the constraints that we have.

So we are going to look at U dagger U equal to 1, what constraint it gives and what constraints you get from determinant U = 1. So determinant U = 1 let us do that one first. So U dagger U equal to 1 implies that if you take the determinant of the matrix U, it will be a phase. It will be e to the 1i alpha, okay? Because when you are taking a determinant of this, it will become determinant of U times determinant of U dagger, okay.

And on the right hand side, it is a unit identity matrix or determinant of identity will be 1. And this gives you determinant of U modulus square is equal to 1 which means determinant of U is a phase okay, e to the alpha. So that is true in general for unitary matrices. But now we have on the top of it put a constraint that determinant of U should be 1, which means that this phase is not arbitrary anymore, and you have to restrict it, okay?

And let us take it to be alpha equal to 0. You are allowed, you cannot continuously change it. This parameter alpha you cannot continuously change, it has to be phased. So let us take it to be alpha equal to 0. So that is one constraint, okay. Out of all the different parameters that parameterize this matrix U, one freedom is lost, okay. So there is one constraint. Now let us see what else we get from this piece.

Let me also remove the timeline. So let us see. Now I am going to do, I am going to find the constraints in two parts. One I will look at what happens on the diagonal, okay, and then I will look at what happens on to the remaining entries. So let us look at the diagonal entries. So I take UU dagger and take, see an arbitrary element will be alpha, beta, or yeah let us call it alpha and beta.

But now I am not looking at alpha and beta. I am looking at alpha alpha. And this is your delta alpha alpha, okay whatever that number is. And that number you know it is 1 because it is unity, okay. So I am looking at only the diagonal entries. Now UU dagger, you can write as U

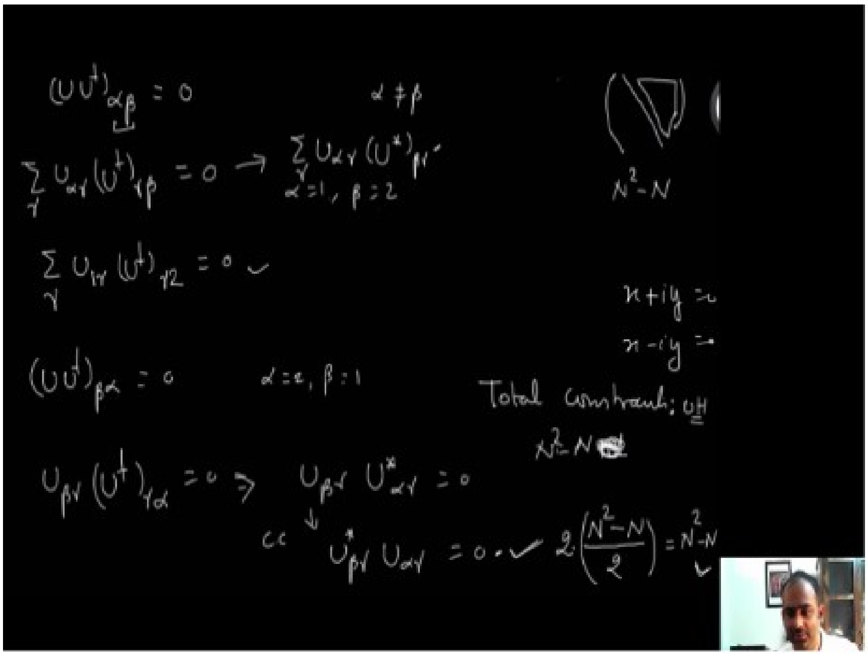

Figure 5: Refer Slide Time: 24:15

alpha beta U dagger beta alpha. Okay, there is no summation over alpha. I am not implying any summation, okay.

$\det U = 1$ and

$$UU^{\dagger} = 1 \Rightarrow \det U = e^{i\alpha} : \alpha = 1, \text{hence 1 constraint} \tag{3}$$

Let's look at diagonal entries

$$\begin{aligned}
(UU^{\dagger})_{\alpha\alpha} &= \delta_{\alpha\alpha} && (4)\\
\sum_{\beta} U_{\alpha\beta}(U^{\dagger})_{\beta\alpha} &= \delta_{\alpha\alpha} && (5)\\
\sum_{\beta} U_{\alpha\beta}(U^{\star})_{\alpha\alpha} &= \delta_{\alpha\alpha} && (6)\\
\sum |U_{\alpha\beta}|^2 &= \delta_{\alpha\alpha} \rightarrow \text{N real equations} && (7)\\
(UU^{\dagger})_{\alpha\beta} &= 0 \quad ; \alpha \neq \beta && (8)
\end{aligned}$$

This is same as U alpha beta. Now U dagger is a star and a transpose. So if I put a star here, and then I am only left with transpose and I do transpose by interchanging these two indices. It becomes alpha beta. Okay, let me make it explicit here. This is a summation over beta but not over alpha, okay. It is matrix multiplication so I have this, okay.

Now U alpha beta times U star alpha beta is just, and this is a real number. It is a modular square of a complex entry. So U alpha beta are these entries, right? This entry, so this entry is U12. So here we are saying that it is a sum of all these that has to be 1, okay. But each of this is a real number, because it is modulus square of a complex number, okay.

So how many equations are these, how many constraint equations are these? I should not remove, okay. So how many are these? You have n values of alpha, so alpha takes value 1, 2, 3 and so forth. Beta is anyway some lower. So these are N real equations, okay. So you have N constraints. So this is important because, unless I know that these equations are made up of real numbers, I cannot say that these are N equation.

Then I will have to say 2N equations, because equation could be complex, right. So this is what I have showed that there are N real equations as constraint coming from here. Now let us look at off diagonal entries. Do I need more space? Okay, maybe here itself. So if I look at off diagonal entries, so let us go back to this equation. You have UU dagger, alpha beta is 0 because delta alpha beta is 0. And I am saying alpha is not equal to beta. Maybe I should go to the next one. UU dagger alpha beta element of this is delta alpha beta and I am saying alpha is not equal to beta that is 0, okay. Now this I write is U alpha gamma U dagger gamma beta is equal to 0, okay. And so that is one equation. That is an equation and this is summation over gamma. So how many alpha and beta I have, how many questions are these? See total there are alpha and beta take values from N to N.

So we have already removed these entries right? So this out of N square these are gone. So these are N of those. But then you realize that see let us say alpha is 1 and beta is 2 and I have such an equation. So you have U 1 gamma U dagger gamma 2 is equal to 0, okay. Let me write it down. U 1 gamma U dagger gamma 2 is equal to 0 and this is a sum over gamma. Now these are complex numbers right, these ones.

So this is a complex equation, okay. So these are two real equations, one for the real part, one for the imaginary part. So these are two real equations. But then we realized something else. Let us look at not, so here right now here in this one I have taken alpha is 1, beta is 2, let us take other way around. You take beta alpha this one, where alpha is 2 and beta is 1. Let us see what happens.

Then you get U beta gamma U dagger gamma alpha equal to 0, okay. Now what I will do is I will take this equation and take the complex conjugate of it. if I take a complex conjugate, then I get U beta gamma star U. Okay, let me before I do a complex conjugation, let me write it slightly differently. And also this one, this one I want to write as first. U alpha gamma U star beta gamma, okay.

Because of the transpose I have taken care of already, I have interchanged the indices. And that is what I want to do here. So I first write it as U beta gamma and U star alpha, gamma. So I have taken care of conjugation transpose. Now I do a complex conjugate of this. So if x + iy is a constraint, then x - iy is also right. It does not change anything.

If this one is saying x is 0, y is 0, this one is saying same thing and these are related by complex conjugation. So I take complex conjugate of this and I get U beta gamma star U alpha gamma equal to 0. And you see this one is exactly the same as this one U star beta gamma U alpha gamma. So what has happened is when I instead of looking at one, two element here I looked at to an element okay, they have become equal, they are same, okay.

They are related by actually complex conjugation, not equal but they are related by complex conjugation. So that is not giving a new constraint to you. It is the same constraint, which is appearing from this one also. So which means that all the constraints that you are going to get from here is the same constraint you are going to get from this part. So this part does not give you a new constraint.

It just tells you repeats the same constraint. So our independent constraints come from this one, and this part, okay. And these are, that is fine, but also another point that these constraints are not real, these are complex, okay? So how many are these? See, total was N square, out of which if I remove the diagonal, I have -N okay. I am looking at only the upper half, because the lower half is fixed by that.

So I should be looking at only this part. This is all the elements in here, but these are complex equations. So each of them is two real equations, so I should multiply by 2 and that is N square minus N, okay. So how many constraint do I have? I have these ones N square minus N, total constraints. I have N square minus N. But then I had one more coming from the determinant part. So these are the total number of constraints. So N square minus N minus 1.

$$\sum_{\gamma} U_{\alpha\gamma}(U^{\dagger})_{\gamma\beta} = 0, \quad \text{take } \alpha = 1, \beta = 2 \tag{9}$$

$$\sum_{\gamma} U_{1\gamma}(U^{\dagger})_{\gamma 2} = 0 \tag{10}$$

Other way

$$(UU^{\dagger})_{\beta\alpha} = 0 \quad \text{take } \alpha = 2, \beta = 1 \tag{11}$$

$$U_{\beta\gamma}(U^{\dagger})_{\alpha\gamma} = 0 \quad \rightarrow U_{\beta\gamma}(U^{\star})_{\alpha\gamma} = 0 \tag{12}$$

Complex conjugate

$$U^{\star}_{\beta\gamma}(U)_{\alpha\gamma} = 0 \tag{13}$$

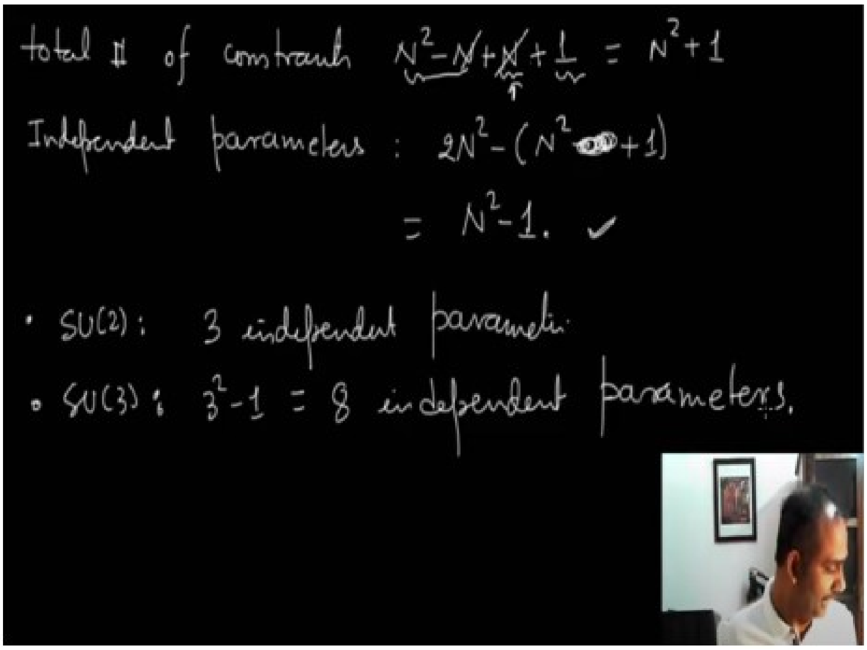

Figure 6: Refer Slide Time: 30:07

Let me write total number of constraints N square minus N sorry plus 1. I should not write -1. This is one additional constraint okay, plus one. Now let us see how many independent parameters we have to parameterize a unitary matrix. So independent parameters that parameterize the unitary matrix is to begin with we had N square complex parameters or 2N square real parameters.

Now I have these many constraints which I should remove and that gives you 2N square minus N square is N square. So something has gone wrong, let us see. Nothing went wrong I just missed a piece. So let me do this. Total constraints here these are from the upper half okay that is fine.

This is not total constraint this is N square minus N. This is from the upper half part, which I calculated here okay.

So now let us do it again. So total number of constraint is N square minus N. This is coming from upper half plus you had N real constraints coming from the diagonal entries. Let us go back right, this one. These were N constraints coming from the diagonal. So that is correct. Then you have one more constraint because you have coming from determinant U = 1 from the phase part, okay. That is what is here. So this is the one which I had missed, okay. So now I should write 2N square minus these two cancel, so this is N square plus 1 because this is gone and this is N square plus 1. So this should be removed. So you have 2N square. 2N square minus N square is N square minus 1, correct finally. Oh, good. So hope everything is fine. 2N square minus N square is N square minus 1, that is correct, okay.

After the scare we have the right answer okay, which means that if I am looking at SU(2) group, how many independent parameters I have? It is 2 square which is 4 – 1, which is 3. So 3 independent parameters. Okay, so 3 independent parameters parameterize the elements of SU(2) group. How about SU(3). So let us see 3 square minus 1, 3 square is 9 minus 1, 8. So this is equal to 8 independent parameters, okay.

$N^2 - N$ constraints, total constraints will be

$$N^2 - N + N + 1 = N^2 + 1 \tag{14}$$

Independent parameters

$$\begin{aligned} &= 2N^2 - (N^2 + 1) && (15) \\ &= N^2 - 1 && (16) \end{aligned}$$

- SU(2) : 3 independent parameters
- SU(3) : 8 independent parameters

SU(N)

That is good. Now I want to talk about a little more about these groups. So let us go back and go here. So I am now going to look at, so let us say I am looking at a unitary group, SU(N) group.

And I have already shown that it is parameterized by alpha N square minus 1 parameter. So okay these are all the parameters that you have for SU(N). And I parameterize in such a way such that in such a way that if all the parameters are equal to 0, you are sitting at the identity matrix okay, if all the parameters are 0, right now not 0, so I am going to write more terms.

But you know you are at the identity element when all the parameters are 0. That is what I am choosing as the way of parameterizing, okay. Now if you are slightly away from identity, infinitesimally away meaning you choose the parameters alpha to be infinitesimals then you will be slightly away from there and let us say that infinitesimal deviation you have from identity is given by H.

And this is I am saying of order alpha okay. So this is linear in alpha. Plus all the higher order terms which I will generically denote by order alpha square okay. Now let me look at what U dagger would be of this. If I take a dagger I get minus iH dagger, okay. I am dropping the okay let us keep it for a while. Now you know that you have the condition that these are unitary, okay.

This implies that if you take the product you will get 1 + iH. Now I am dropping the order epsilon square terms, no sorry order alpha square terms and right now alpha infinitesimal, okay. So alpha is infinitesimal. So I have 1 - iH dagger and this should be 1 identity and these are all

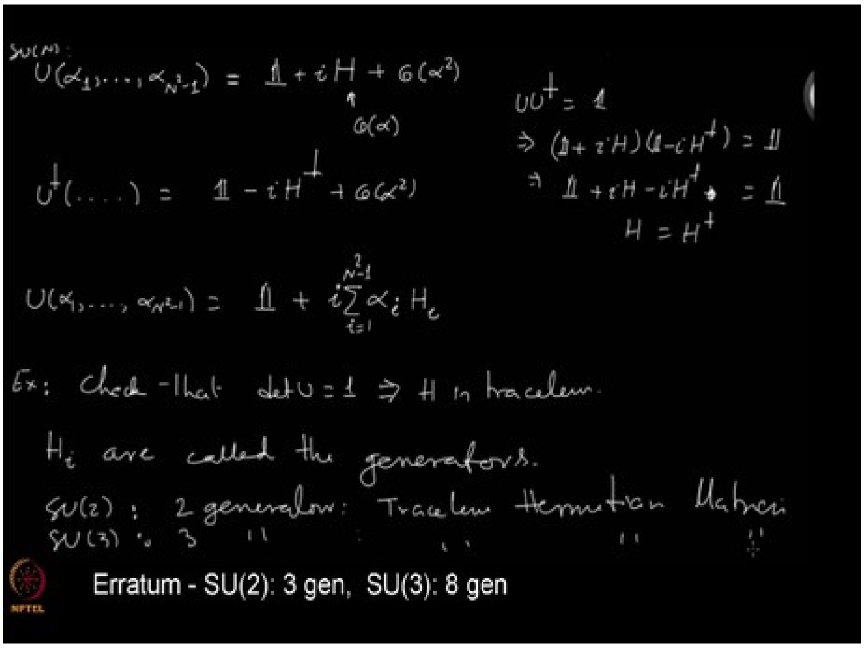

Figure 7: Refer Slide Time: 34:17

identities. This gives you identity minus plus iH minus iH dagger plus a term which is HH dagger, but that is of order first square, which I am dropping okay.

So I leave those pieces, this is 1. And this tells you that H is equal to x dagger, meaning H is Hermitian. So the H has to be Hermitian, okay. Now I will write U as the following. So U alpha 1 to alpha N square minus 1 is 1 plus i. Now H is a Hermitian matrix, N cross a Hermitian matrix. What I will do is I will write it as N square minus one independent Hermitian matrices.

So I will write it as a linear sum of Hermitian matrices. So now this H are constants, because I have pulled out the parameter alpha. So this is the linear sum which will constitute the H for you, okay. So summation over i, I am sorry there is another i here, but this i is different from this. So this is running from 1 to N square minus 1 okay. Plus all higher order terms and alpha that is there, okay.

Now exercise which is fairly easy. Check that determinant U = 1 imposes a constraint on H. See U is 1 plus these numbers which you can put whatever you wish to okay, times H. So H is the object which should know that you know U is unitary. And it knows that by this property that H has to be Hermitian. So but then it should know more. It should know that determinant U is 1.

So there should be something else which H has to satisfy and that is the fact that H is traceless, okay. So check that determinant U = 1 implies that H is traceless. Please do this exercise and these H are called the generators. So H i's are called the generators, generators of SU(N), okay. These traceless Hermitian matrices are called generators of SU(N). So for SU(2) you have 3 generators.

For SU(3) you have 8 generator. So SU(2) you have 2 generators, which are traceless Hermitian matrices. And for SU(3) you have 3 generators, which are traceless Hermitian matrices. Okay SU(2) and SU(3) are important groups because your entire understanding of fundamental interactions okay, that of that is electroweak interactions and strong interactions.

They are based on SU(2), U 1 and SU(3), okay? So that is why I am discussing here. I mean, not because I am going to cover those interactions here. But it is useful to know these things, okay? And that makes also understanding Lorentz algebra Lorentz group easier. Or rather, it is needed to understand that group, okay. So now I have said that there are three Hermitian traceless matrices for SU(2), okay. I want to construct those generators for SU(2).

$$
\begin{aligned}
U(\alpha_1 \cdots \alpha_{N^2-1}) &= \mathbb{1} + iH + \mathcal{O}(\alpha^2) && (17)\\
U^\dagger(\alpha_1 \cdots \alpha_{N^2-1}) &= \mathbb{1} - iH^\dagger + \mathcal{O}(\alpha^2) && (18)\\
& && (19)
\end{aligned}
$$

$$
\begin{aligned}
UU^\dagger &= \mathbb{1} && (20)\\
(\mathbb{1} + iH)(\mathbb{1} - iH^\dagger) &= \mathbb{1} && (21)\\
H &= H^\dagger \quad , \quad \text{H is hermitian} && (22)
\end{aligned}
$$

Now

$$
U(\alpha_1 \cdots \alpha_{N^2-1}) = \mathbb{1} + i \sum_{j=1}^{N^2-1} \alpha_j H_j \qquad (23)
$$

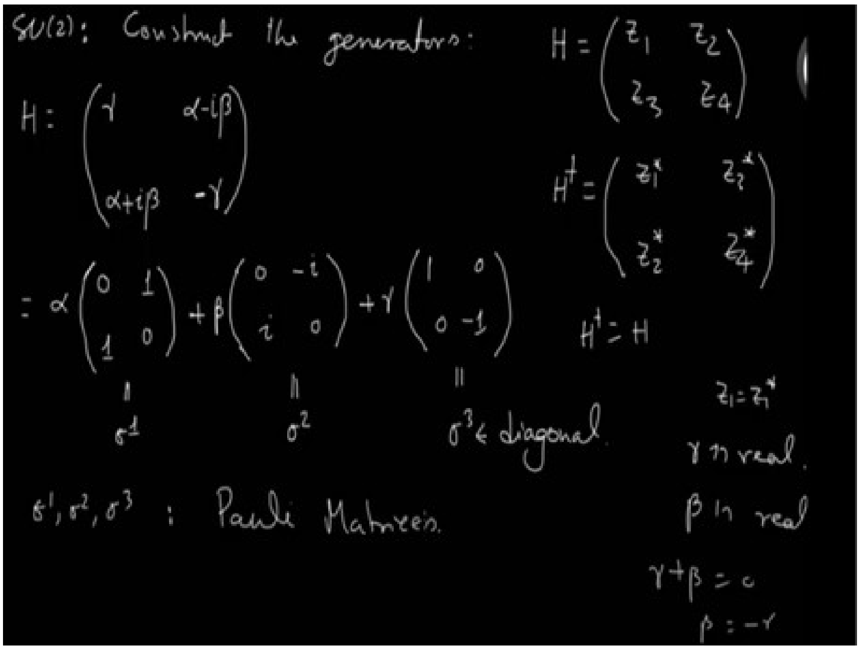

Figure 8: Refer Slide Time: 40:57

Exercise : Check that $\det U = 1 \Rightarrow$ H is traceless, H_i are called the generators of SU(N).

- SU(2) : 3 independent parameters
- SU(3) : 8 independent parameters

So I want to pursue SU(2) further and construct the generators. And you are going to see something nice if you have not already seen that before. So let us write arbitrary Hermitian matrix as this. So let us call it z 1, z 2, z 3, z 4. Okay, that is an arbitrary matrix, complex matrix, 2x2 matrix. But then I say that H dagger should be same as H. So H dagger is H 1 star, sorry z 1 star, z 2 star, z 3 star, z 4 star.

That is the dagger of it, Hermitian conjugate of it. And the fact that H dagger is equal to H gives you the following that your H is this. So z 1, this should be equal to z 1 star, which means just a second I should remove it. So z 1 is equal to z 1 star, which means this entry is real. And I want to call it a real number. I will call it gamma, okay. So gamma is real. Then I have z 4 star, which is, has to be same as z 4.

So that is also going to be real and I call it something, let us call it beta. So beta is located here okay, at this place, okay. But then I remember that my matrix has to be traceless. So gamma plus beta should be 0, which means beta is minus gamma. So I will put a minus gamma here, okay. And then you have a z 3 star or sorry z 2 here. So I write z 2. And here, instead of z 3 I should write z 2 star.

So z 2 here and z 2 star here, so they are complex conjugates of each other. So for z 2, I write alpha minus i beta, alpha minus i beta. And here it is complex conjugate. So I write alpha plus i beta, okay? If you had put a plus sign here and a minus sign there, and nothing changes, okay? It is just the parameters, you can change the sign later. So there is no nothing big going on here in choosing the signs.

So that is my Hermitian matrix. I have taken care of H, it be Hermitian and also tracelessness. So I can write it as the following, gamma. Okay maybe, let me write it slightly better. So I take the alpha part, so alpha, and I put everything else to be 0. So I put all the parameters to be 0 and I have only alpha. So I get 0 1 1 0. Then I have beta. So I keep only beta non zero, and everything else 0, then I get 0 -i i 0 plus gamma.

That is only on the diagonal. And that is 1 -1 0 0. Okay, does that appear familiar to you, these matrices? So these are the Pauli matrices. So this one is sigma 1. This one is sigma 2. This one is sigma 3. So note that sigma 3 is the one which is diagonal, okay. And you already know that they do not commute. So you can multiply these two and different orders and check the difference, that will not be 0.

So matrices do not commute in general. So let me write. So there is sigma 1, sigma 2 and sigma 3 these are Pauli matrices. So you see why you have gotten these matrices. These are basically the, apart from some factors these are the generators of Pauli matrices, sorry generators of SU(2). Okay good. Now check that if you take sigma i over 2, I am dividing by half, you will see immediately why. Sigma j over 2 and look at this commutator. You get i epsilon i j k sigma k. This I believe you have seen in your quantum mechanics course, okay. If I do not put a 2 here, it will start showing up here on the right hand side. So that is why there is a factor of half. So this is the commutation relation with which these matrices obey. And you have already seen this in a previous class, let us see. I think somewhere here. The same commutation relations you have seen. You see the commutation relations of j i and j j they are exactly the same right? It is again you have i epsilon i j k okay. And so you can understand that these commutation relations are, of SU(2) are important even for Lorentz group, okay. So now any element of SU(2) can be parameterized using Pauli matrices.

I will jump a bit okay I will again come back to that later, but I will just say here without telling you why I am saying this. That any matrix of SU(2) can be written as e to the i, some parameter let us call alpha i sigma I can put i up there is maybe a k, sigma k over 2. So there is a summation over k okay. So you have alpha 1 sigma 1 over 2 plus alpha 2 sigma over 2 plus alpha 3 sigma over 3 okay, And these are the generators. These are the generators of SU(2). See H was generator here. You can always multiply a half here right, you can overall and absorb the 2 in alpha, beta and gamma. So you take these as the generators because then commutation relations are nice like this. Otherwise, there will be a factor of 2 coming. Also check the following.

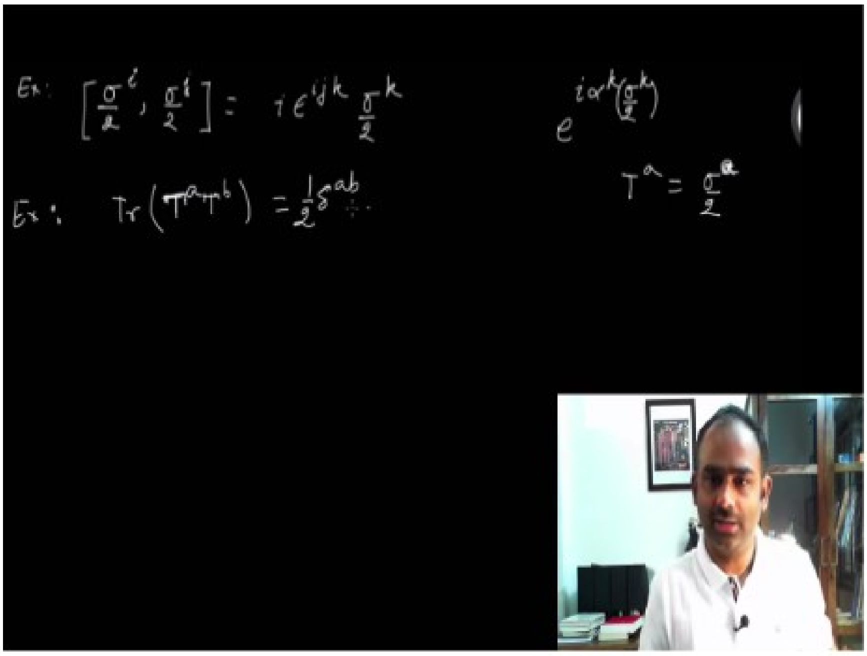

Figure 9: Refer Slide Time: 45:57

Constructing the generators of SU(2)

$$H = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \quad H^\dagger = \begin{pmatrix} z_1^\star & z_2^\star \\ z_3^\star & z_4^\star \end{pmatrix} \tag{24}$$

$H = H^\dagger$, real entries

$$\begin{pmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{pmatrix}, \quad \alpha + \beta = 0 \tag{25}$$

$$H = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{26}$$

$$H = \alpha\sigma^1 + \beta\sigma^2 + \gamma\sigma^3 \tag{27}$$

$(\sigma^1, \sigma^2, \sigma^3)$: Pauli matrices and these Pauli matrices are the generators of SU(2).
Exercise:

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{28}$$

Any SU(2) matrix can be written as

$$e^{i\alpha^k \frac{\sigma^k}{2}} \quad \longrightarrow \quad \text{generators of SU(2)} \tag{29}$$

$$T^a = \frac{\sigma^a}{2} \quad \longrightarrow \quad \text{generators of SU(2)} \tag{30}$$

Exercise:

$$tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{31}$$

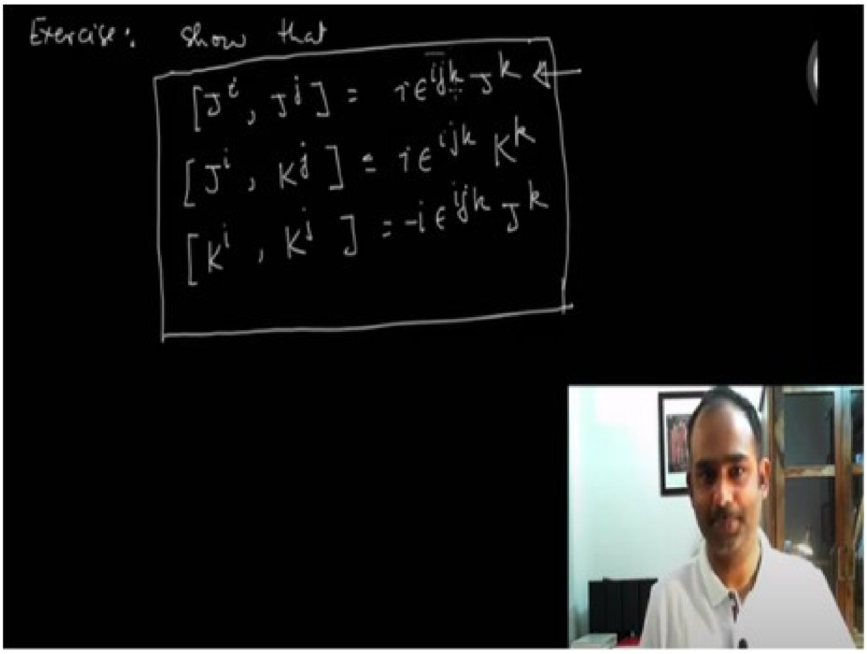

Figure 10: Refer Slide Time: 46:47

Check that if you take the trace of let me call the generators to be T a and which is sigma a, sorry sigma a over 2, okay. So this Pauli matrix is divided by half, they are called the generators and usually they are denoted by T a, okay generators of SU(N) groups are denoted by T a. That is a standard nomenclature. So show that if you take T a T b and if you take a trace of it, okay.

Then what you get is delta ab, is proportional to delta ab. In fact, it is half delta ab, okay. Check this property that this is indeed true, okay.