Introduction to Quantum Field Theory

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Lecture W1L2 : Schrodinger field

Let us continue our discussion further. So, let us start with a quick recap. So, we started treating psi which was the field obeying Schrodinger equation as a classical field and the Schrodinger equation as a classical equation of motion. So, that is what we did. So, we regard psi as a classical field and regard your Schrodinger equation as a classical equation of motion and then we wrote down the action of the system. So, Schrodinger equation as the equation of motion and our action was this.

So you get s is equal to dt d cube x psi star and we had in the in this round brackets the entire Schrodinger equation the left hand side of the Schrodinger equation t over delta t minus the Hamiltonian of single particle quantum mechanics t that is what we did. And remember the Hamiltonian was this minus h bar square over 2m where m is the mass of well let us not say anything now because we have changed the description.

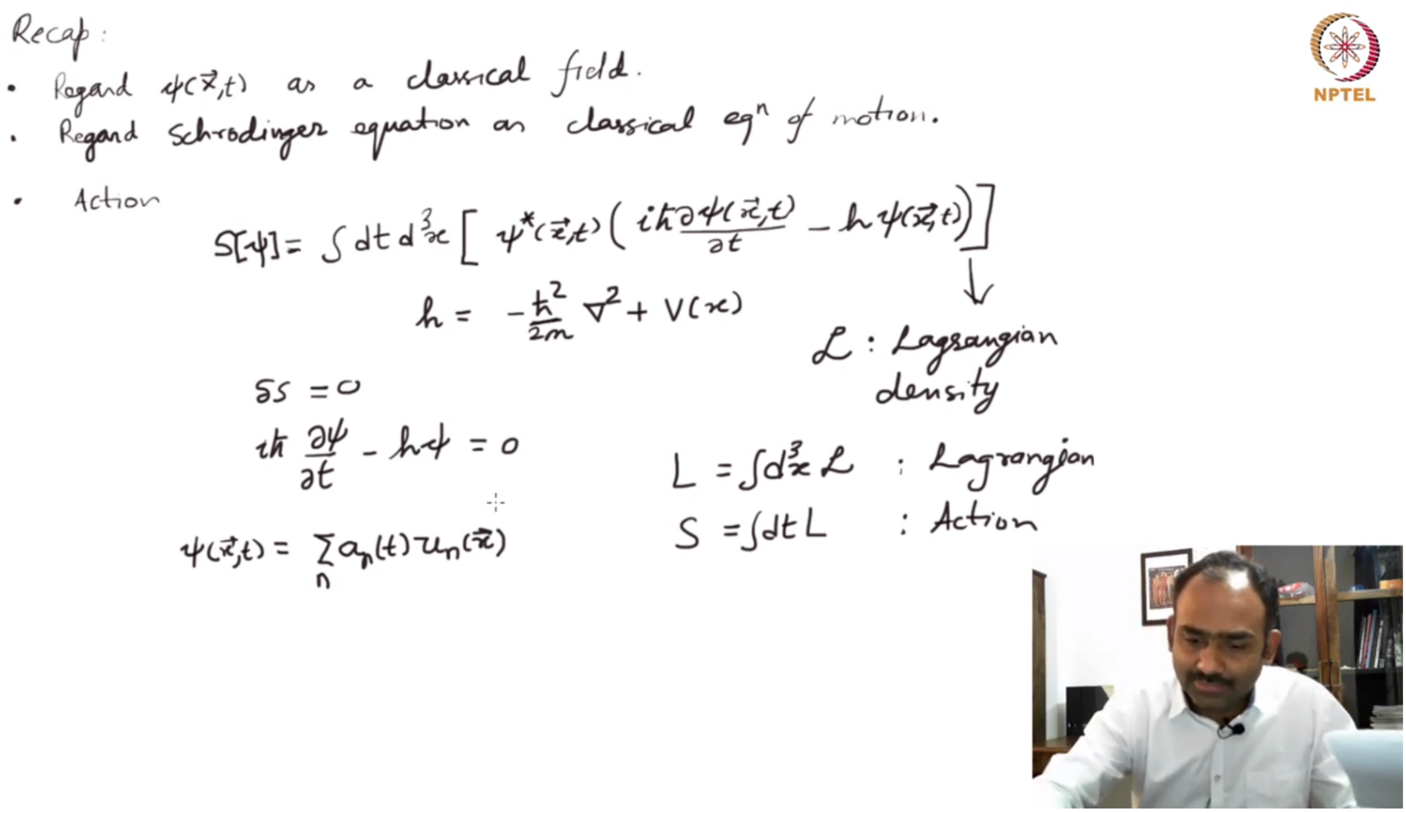


Figure 1: Refer Slide Time: 00:19

- Regard $\psi(\vec{x}, t)$ as a clasical field

- Regard Schrodinger equation as classical equation of motion

The action for this field is,

$$S[\psi(\vec{x},t)] = \int dt d^3x \left[\psi^*(\vec{x},t)\left(i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t)\right)\right] \tag{1}$$

Where the lagrangian density $\mathcal{L}$ is,

$$\mathcal{L} = i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t) \tag{2}$$

$$h = \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}) \tag{3}$$

The relation between lagrangian, action and lagrangian density is

$$L = \int d^3x\mathcal{L} \quad ; \quad S = \int dtL \tag{4}$$

Doing the variations in Eq.(1) will give us,

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = h\psi(\vec{x},t) \tag{5}$$

$$\psi(\vec{x},t) = \sum_n a_n(t)u_n(\vec{x}) \tag{6}$$

So, we had a potential in the problem and then we derived equations of motion using delta s equal to 0 and that treating psi star and psi as independent variables. We found that the equation of motion was what we wanted and that is not surprising that is how we constructed things. And let me say here this quantity in square brackets I forgot to put a bracket here the quantity within square brackets is called Lagrangian density it is denoted by curlier, Lagrangian density that is what you have within the square brackets.

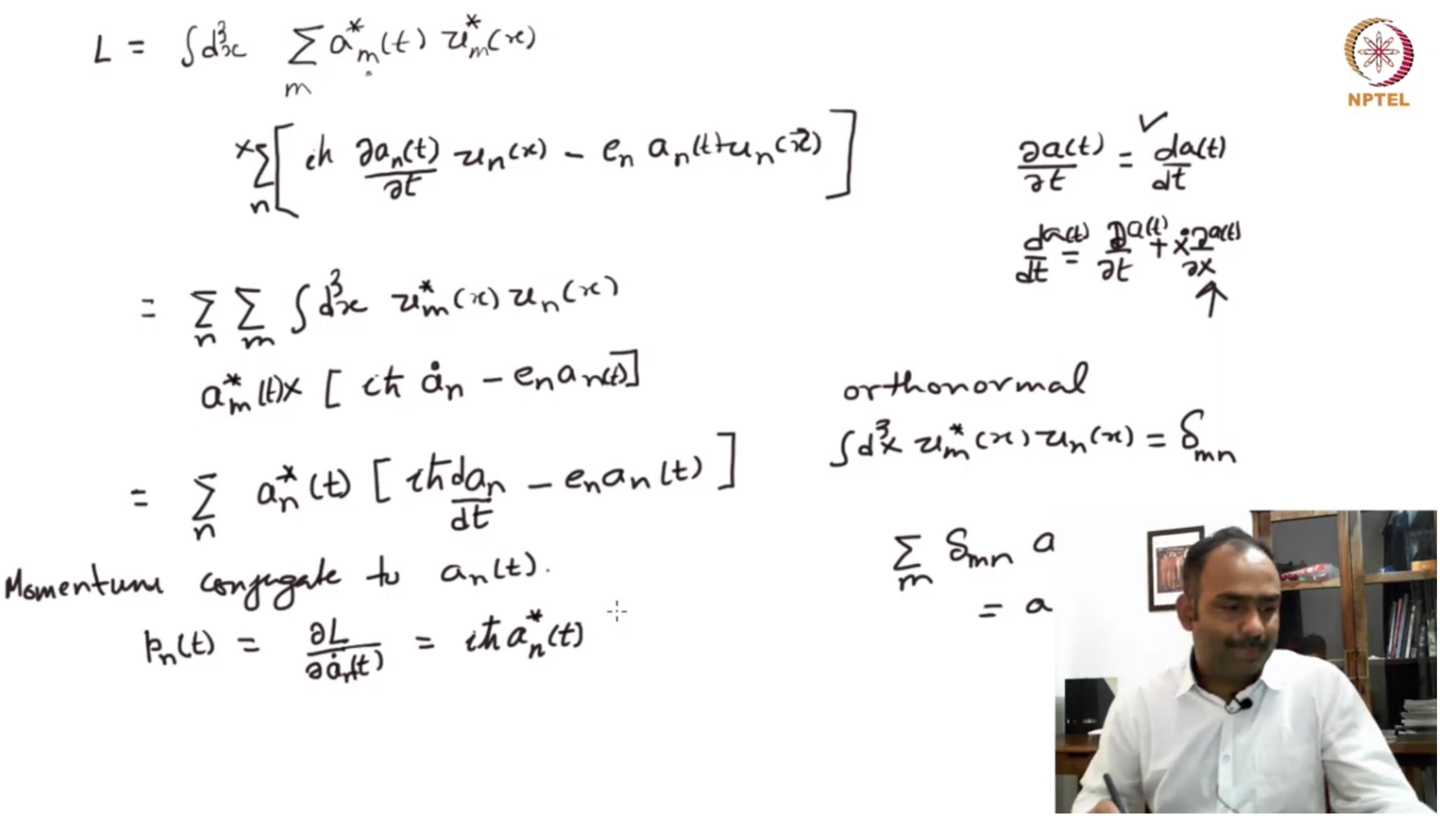
Then if you integrate the Lagrangian density over all of space then you get the Lagrangian that is your Lagrangian. And when you integrate the Lagrangian over time you get the action and let me write it a little bit nicely as action. Now you see here s is actually a functional of psi you see these are integration variables. So, dt and d cube x. So, you are integrating over the entire all times and all space.

So, that is gone anyway s does not depend on what t is and what x is because that is integration variables. It depends on psi and psi star. So, if you specify a field configuration meaning if you take the entire all space and now tell what is the value of psi at each point of time. So, at each point in space and time used to say how big psi is and in this case it is complex. So, you have to tell both sides real part of psi and complex part of psi but you have to tell at each point and at all times.

If you specify that then you can calculate the action if this changes, you get a different value of the action. So, s is dependent on this entire specification of psi what is psi throughout action depends on it. So, s action is a functional of psi. So, that is what we are making more specific here we will come back to functionals in more detail later but for now we make this note. And we also wrote psi as a linear sum over the wave functions of the Hamiltonian this small h because the wave functions form a complete set so we can write it.

So, I wrote earlier a n of t u n of x and of course as I mentioned last time this is a system of infinite degrees of freedom because you have n and it runs from one to infinity you have to specify infinite number of a's to specify psi that is good. And we also said that a and a dagger the generalized coordinates that describe this system. So, they are the dynamical variables of your problem. Now if a's are the dynamical variables of variables of my problem I would like to write down my Lagrangian, my action, hamiltonian everything in terms of these a's and that is what i'm going to do now, So, I will use this decomposition of psi and express everything in terms of the dynamical variables a.

So, let us go to the next slide yes good. So, let me write it down here again. So, your psi, let us go to Lagrangian the Lagrangian is integral d cube x psi star here. So, d cube x psi star and this piece so, I am going to substitute in this here in this part this decomposition of psi that is what I am going to do now. So, here is psi star but psi star is instead of n I am writing m sorry. So, what is psi stars I star is m a m of t u m of x but it is a psi star.



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Using the decomposition of $\psi(\vec{x},t)$ Eq.(1) will become,

$$L = \int d^3x \sum_m a_m^*(t) u_m^*(\vec{x}) \sum_n \left[i\hbar \frac{\partial a_n(t)}{\partial t} u_n(\vec{x}) - e_n a_n(t) u_n(\vec{x}) \right] \tag{7}$$

$$L = \sum_n \sum_m \int d^3x u_m^*(\vec{x}) u_n(\vec{x}) a_m^*(t) \left[i\hbar \dot{a}_n - e_n a_n(t) \right] \tag{8}$$

$$L = \sum_n a_n^*(t) \left[i\hbar \frac{\partial a_n(t)}{\partial t} - e_n a_n(t) \right] \tag{9}$$

So, there has to be complex conjugation here times so that part I have taken care of psi star and then in the bracket we have ih bar del psi over del t time dependencies only in the a piece now I am writing psi naught psi del psi over del t. So, u m will remain outside and you will have del a n t over del t let me try to write it a little neatly it looks horrible. I will use a different index n I will not use m del n t over del t and you have u n of x that is correct and I should sum over n that is ih bar del psi o del t then I have minus h bar square over 2m did I make a mistake somewhere no minus h minus h.

So, let me put a bracket here it will be easier minus h by minus h bar square over 2m then we have sorry I do not need to do that sorry. So, minus we have let us go back h acting on psi but psi I have expressed using a and u, u are the wave functions of the Hamiltonian h with eigenvalues e n. So, when h acts on psi it will act on u n it will give us e n u n and of course a is already there. So, what I will get is this.

Let me write down e n you will have u n and you already had a n of t that is what we should get and everything looks fine to me let us collect some terms. So, first of all note that del at over del t is same as da over dt there is no distinction because a is a function of t only there it does not have some other dependence I mean you can think you can look at this in more detail it is del over del t + del over del x into dx over dt.

So, that is what it is but if you put a function a which depends only on time this piece is 0 right because it is a partial derivative of a thing which depends only on t. So, this is gone and then it is clear that what I am saying is correct. So, you get the following. You get summation over n summation over m, let me collect u n and u m's. So, you get integral d cube x star of x u n of x times ih bar d over dt which is a n dot I will use a shorthand notation minus e n a n that is correct.

Let me not suppress time here good now I can use the fact that these wave functions are orthonormal. I can choose them to be orthonormal because they are eigen functions of hermitian operator the Hamiltonian of the single particle quantum mechanics. So, which means that if they are orthonormal it means that if you write this x u n of x then this is just delta m n so, when m and n are same equal meaning you have the same function in here then you get one if m and n are different then because they are orthogonal you get 0 delta m n will be zero.

So, that is the condition we have which if I put here. So, if I replace this piece with delta m and something is missing this piece is missing yeah this one is missing. I should write here a m star of t and this crosses yeah it is there I can remove it but it is so this gives you a delta m n now when you are summing over all m then it will. So, what you get here is summation over m delta m n and you have a m star of t another factor does not depend on m. So, it it is fine up to here and when I sum it it will pick up a n star.

So, I get summation over n a and star and then ih bar in let me write da n over dt there is no need for putting a dot but it is fine e n a n t. So, that is what we get for the Lagrangian and that is good. So, we wanted to write our Lagrangian using only the dynamical variables and which we have achieved now we have only the a n's and a n stars. So, that is fine now I would like to construct the Hamiltonian and if you recall what how hamilton is related to the Lagrangian it is p q dot - l.

So, I have to construct further first the momentum conjugate to the dynamical variables a. So, that is what I am going to do. So, let us construct momentum conjugate to a n of t and how do you get that. So, if I define that conjugate momentum to be p n then p n will be del l the partial derivative of the Lagrangian with respect to q dot and what is q here it is a n. So, it is a n dot of t good and what is that if I take this Lagrangian here and take a partial derivative with respect to a dot and this is the a dot I get I h bar a n star and this term anyway does not contribute because there is no a dot involved in here. So I get ih bar a n star of t. So, that is the momentum conjugate to a n.

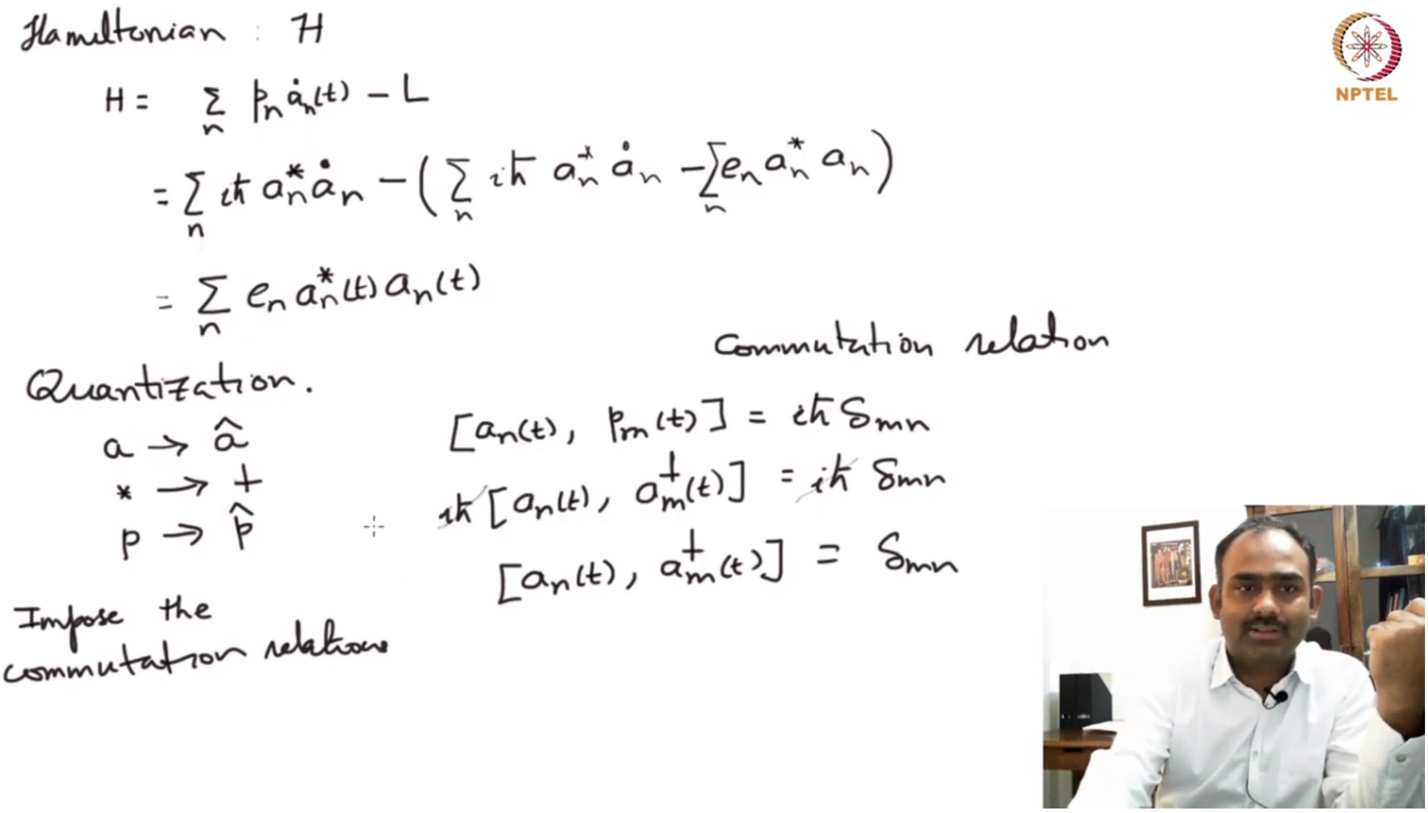


Figure 3: Refer Slide Time: 19:47

We have used

$$\frac{\partial a(t)}{\partial t} = \frac{da(t)}{dt} \quad ; \quad \frac{da(t)}{dt} = \frac{\partial a(t)}{\partial t} + \dot{x}\frac{\partial a(t)}{\partial x} \tag{10}$$

Momentum conjugate of $a_n(t)$.

$$p_n(t) = \frac{\partial L}{\partial \dot{a}_n(t)} = i\hbar a_n^*(t) \tag{11}$$

The hamiltonian will be

$$H = \sum_n p_n \dot{a}_n(t) - L \tag{12}$$

$$H = \sum_n i\hbar a_n^*(t)\dot{a}_n(t) - \left(\sum_n i\hbar a_n^*(t)\dot{a}_n(t) - e_n a_n^*(t) a_n(t)\right) \tag{13}$$

$$H = \sum_n e_n a_n^*(t) a_n(t) \tag{14}$$

Now I can construct the Hamiltonian that is straightforward. So, the Hamiltonian of this system now I am not talking about the Hamiltonian of single particle quantum mechanics I am talking about the Hamiltonian of this system and that I will write as capital H. So, how do I construct the hamiltonian hamiltonian is pq dot - l and you sum over, all the coordinates. So, you have summation over n p n and what is q? q is your a looking very ugly p q dot minus the Lagrangian and what did we find p n to be p n was ih bar n star.

So, let us write that ih bar a n star a and dot I am suppressing t now summation l minus let us go back and see what the Lagrangian was um. So, this is ih bar n star n dot this is the same thing here let us write anyway ih bar a and star a n dot and then go back then you have e n a n star a n the minus sign minus e n a n star a n that is nice. So, these two terms clearly cancel this minus sign cancels and leaves us with summation over n e n a n star t a n of t that is the Hamiltonian.

It is a sum of infinite number of terms and each term is e n a n star n that is the Hamiltonian we have found and as we wanted we have expressed everything now in terms of the dynamical variables a n we are still at the classical level everything is classical my action is classical Hamiltonian is classical these are not operators. Now I want to quantize this theory I want to make a quantum theory out of it.

So, let us quantize and recall what we need to do quantization I promote a to an operator of course a star or complex conjugation to hermitian conjugate and that is it. There is nothing of course we have to impose the computation relation but of course the p also becomes an operator and then I impose maybe not here impose the computational relations which we talked in the previous lecture.

So, that is what we have to do and let us see. So, if I look at the computation relation let us see what we get commutation relation. So, a and p if I take a n of t and p m of t and calculate its computation relation this will be ih bar times the Poisson bracket and Poisson bracket has to be 1 because these are or delta m n because these are canonically conjugate pairs. So, I get I h bar that is the prescription right that is the prescription of making quantum theory.

So, this is what I have this delta m n is coming from the Poisson bracket. Now something let us see let us see what pm was. So, I go back pm is ih bar a n star or p m is ih bar am star which becomes a m dagger. So, ih bar a m dagger that ih bar factor I can pull out it comes out of the commutator. So, you have ih bar a n of t and you have a m dagger of t is ih bar delta m n and the factors of ih bar cancel on both the sides this you can cancel.

So, I can cancel them and you get the commutator between a and a m dagger to be delta m n that is good and what is our Hamiltonian now in the quantum theory. So, the Hamiltonian of this system this quantum mechanical system which we have found by quantising the classical action whose classical equations of motion are Schrondinger equation that Hamiltonian h is now summation over n e n a and dagger am right. So, the Hamiltonian is let me write on the next page

Quantization

$$a \to \hat{a} \quad , \quad * \to \dagger \quad , \quad p \to \hat{p} \tag{15}$$

Commutation relations

$$[a_n(t), p_m(t)] = i\hbar\delta_{mn} \tag{16}$$

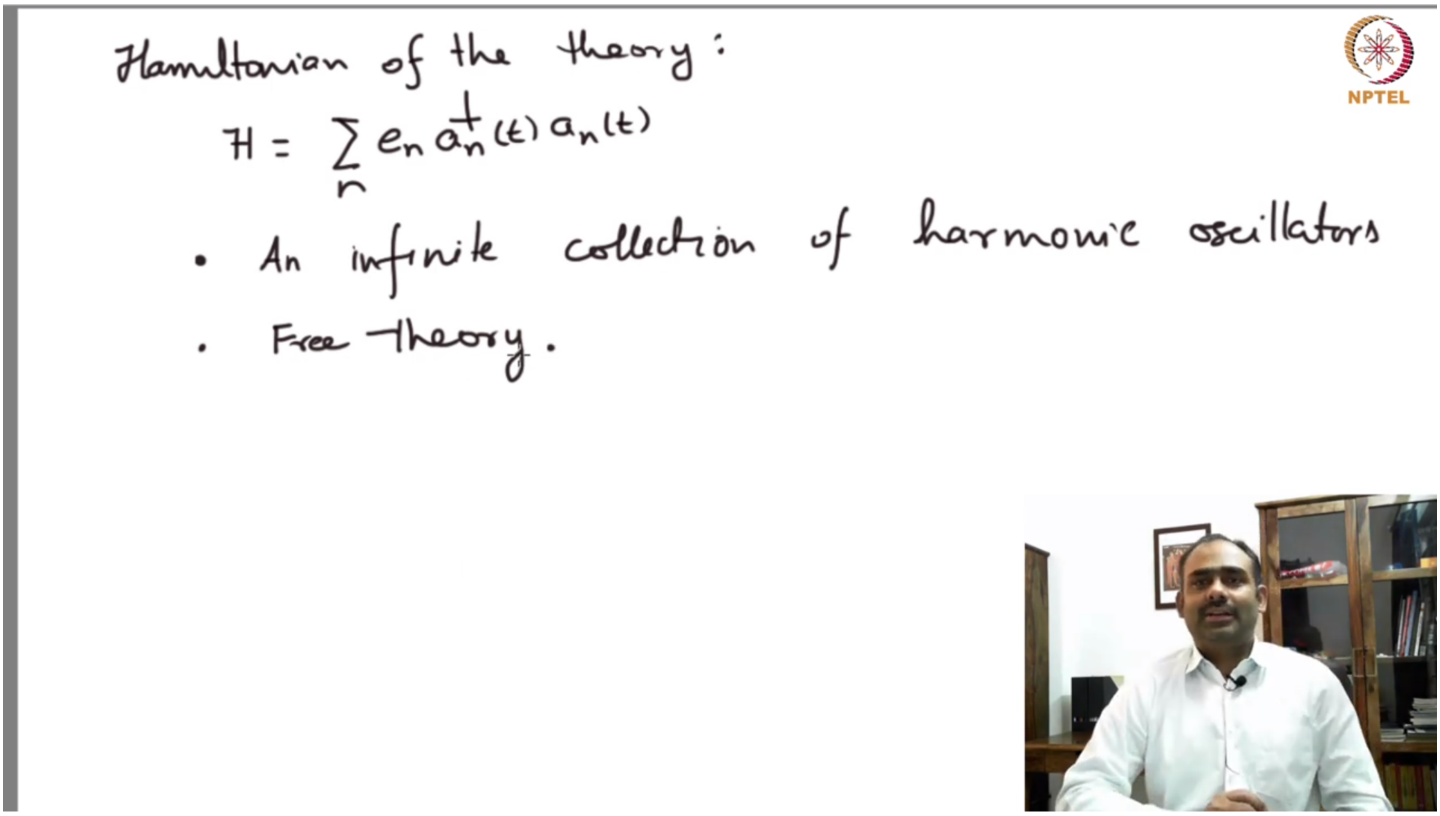
$$[a_n(t), a_m^\dagger(t)] = \delta_{mn} \tag{17}$$

$$[a_n(t), a_m^\dagger(t)] = i\hbar\delta_{mn} \tag{18}$$

Thus the hamiltonian of theory is,

$$H = \sum_n e_n a_n^\dagger(t) a_n(t) \tag{19}$$

So, the Hamiltonian of the theory is e n a n dagger p n of t let us check, perfect. And remember what is what is e n e n are the eigenvalues of small h that is what e n is. So, that is the Hamiltonian of the theory and we see that this is a this is something familiar right this is if you look at the each



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term like let us take the first term e 1 a 1 dagger a 1 that is the Hamiltonian alpha, Harmonic oscillator which has frequency as e 1.

So, that is the harmonic oscillator that you have and what we see here is that your system is a sum of infinite number of harmonic oscillators because n runs from 1 to infinity. So, the quantum mechanical system that we have got is a collection of harmonic oscillators an infinite number of them and these harmonic oscillators are not talking to each other. You see there is no coupling between a 1 and a 2 there are no cross terms in this.

So, it is always it is completely diagonal. So, there is no interaction between one oscillator and another oscillator. So, all these oscillators are completely free and this is a free theory the the system that I am showing to you is a free theory despite the fact that we started with the potential term in the original action or the Schrodinger equation had a potential. So, let us stop here let me just write down an infinite collection of harmonic oscillators and then there is no interaction. So, this is a free theory we will continue further in the next video.