Introduction to Quantum Field Theory

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Lecture 19 : Lorentz Group

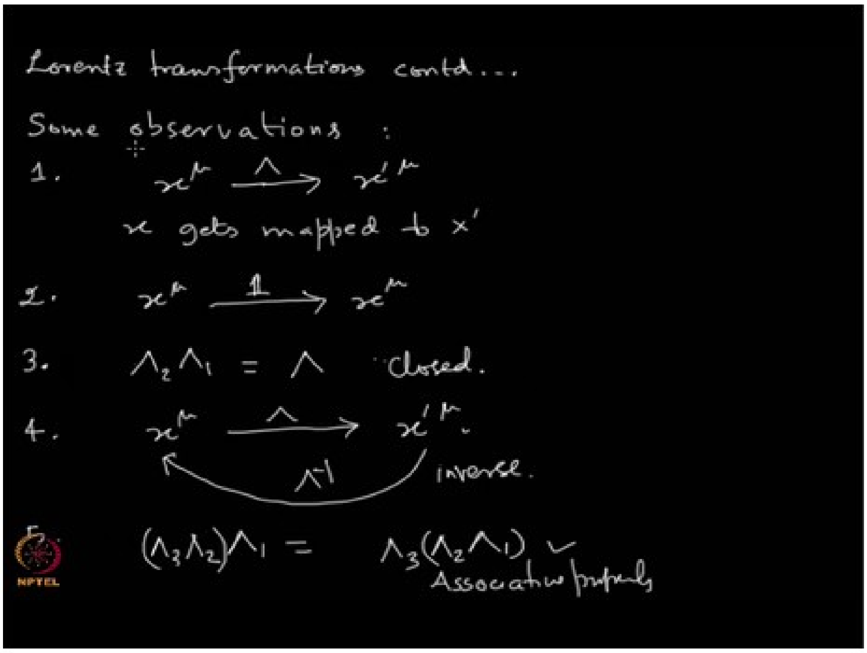


Figure 1: Refer Slide Time: 00:59

Structure of Lorentz transformation, some observation

We have been talking about Lorentz transformations for some time now. And we will continue to do so for next few videos. Actually, the set of all Lorentz transformations have a very nice mathematical structure. And that is what I want to talk about in this video, okay.

I am not so much right now interested in the transformation themselves, but rather I am interested in making some observations which will not only apply to Lorentz transformations, but to many more, to other sets of transformations also, okay. So let us begin by making some observations. Some observations, okay. So first is that what Lorentz transformations are doing is the following. They take some space time point x okay and they map it to another space time point x prime. So under an operation lambda, this point gets mapped to this point. Okay, so let me write x gets mapped to x prime okay, that is what is happening.

Also note that in the set of all Lorentz transformations, you have one transformation, which does nothing, which leaves things the way they are. So if you start with x mu, under that transformation, your x prime is again x mu, okay. And that is your no transformation at all which we have denoted by identity transformation, okay.

Also you have done in exercises I believe that if you do a Lorentz transformation by lambda 1 okay, some Lorentz transformation lambda 1 followed by another transformation lambda 2, what you get is still a Lorentz transformation. So what you get is again a Lorentz transformation which means that, if you take the set of all Lorentz transformations, then that set is closed under this multiplication.

Here the multiplication is the multiplication of these matrices. But if you are looking at Lorentz transform, if you are thinking in terms of transformations, then one transformation followed by another transformation still gives you an element which belongs to the same set, okay. So the set of all Lorentz transformations is closed under this multiplication, okay. We will say that this is closed, okay.

If this is not very clear what I am saying right now, it will become more clearer or clearer in a while, okay. Also if you do a transformation by lambda and go to another point x prime, then you can do a transformation from here to there, okay. Meaning you can do a transformation which is inverse of the transformation lambda and it will bring you back, okay.

So there exists inverse elements to every element lambda, in the set there is an inverse, okay for every lambda. And obviously, you also have if you have lambda 1 followed by lambda 2 and lambda 3, so I am thinking of doing first lambda 1, okay. Then doing a product of lambda 2 and lambda 3. This is same as first doing lambda 1 and lambda 2 this product followed by lambda 3, okay.

So this is true you know already and this is called associative property. Say these ones are true this one is true for example, and this one is true and this one is true. They are true because you already know that lambdas are matrices and the multiplication here is matrix multiplication and matrix multiplication is associative, okay. So these properties which I have listed down are mostly due to lambda being a matrix, okay.

Now I want to leave aside Lorentz transformations and talk in general about some of the things which we have noticed here. You see whatever I have talked about here, I have made no use of the fact that these are indeed Lorentz transformations. I am just talking about some transformation with some properties, but that use of them being Lorentz transformations has not been made, okay.

Now typically what mathematicians would, a mathematician wants to do is abstract out certain features, which are simple enough to be made as a basis of a new structure, okay. And based on that structure, the mathematician will proceed, and in finding out what is basic and really needed, one should be able to figure out what is not necessary and can be discarded. So here we have listed down the most elementary and basic things, and this is what we are going to proceed with, okay. So let us see.

- $x^\mu \xrightarrow{\Lambda} x'^\mu$ $\quad$ x gets mapped to x'
- $x^\mu \to x^\mu$ $\quad$ Identity transformation
- $\Lambda_2\Lambda_1 = \Lambda$ $\quad$ Closed
- $x^\mu \xleftarrow{\Lambda^{-1}} x'^\mu$ $\quad$ Inverse transformation
- $(\Lambda_3\Lambda_2)\Lambda_1 = \Lambda_3(\Lambda_2\Lambda_1)$ $\quad$ Associative property

Now what I will do is, I want to move away from Lorentz transformations and abstract out some of the features that we have listed down there, okay. So instead of x mu's, I look at a set of elements, okay. So maybe slightly differently, I will draw. So that is set S. These are the elements of this set. You can think of these being equivalent to the space time points, okay. But you do not have to.

But if you want to make a connection with what I have been talking about Lorentz transformations, then you think of these points as denoting space time points. But nevertheless, you just think of a set S and given some elements, so the set is not empty, there are some elements. Okay, let me, it is the same set as being drawn again, the same thing, okay. Now I think of a map from this set to itself.

So I am interested in looking at maps, which are defined on S and they are one-to-one maps and they are onto, meaning all the elements are covered when you map. So think of some map, which I will call sigma. So sigma is name of a map, and let us say that map is doing the following. It maps this one to this element. Say this is a map from S to S, okay.

This element gets mapped to this one, this element gets mapped to that one. This element gaps get mapped to this one, okay. So that is a map sigma. You could think of some another map, which we could call mu in which this element instead of getting mapped to this one goes to some other element and this one goes to another element and so forth, okay.

And clearly, the way I have defined this, this is one-to-one. So one element goes to only one element, and onto because all the elements are mapped. Okay, so that is one-to-one and onto map, okay. Now let us do a simple trivial exercise. I will ask how many maps are there on this set, which has only four elements here? Okay, I have fixed the elements to be four, which are one-to-one and onto.

How many elements are there? How many such maps are there? So that is easy to find out. Let us say so you want to know how many one-to-one onto maps are there for this one, for the set S okay, which has four elements. So you see first element, let me draw it again. So first element could go to any of the four elements, okay. So there are four possibilities. So let us say it goes to this one, okay. So it can go to four, it has four possibilities. So you do one of them, you take one of them. Then this guy could go to any of the remaining three, okay. So let us say it goes to this one. So there are three possibilities on the top of the first four. Then this guy could go to any of the remaining two. So let us say this goes to that one. And then there is only one possibility left, which is this. Okay?

So this one has only one possibility. And that makes four factorial. So there are four factorial maps, let us call them sigma 1, sigma 2, sigma 4 factorial. These are all 24 maps okay, 4 into 3 24, okay. So 4 into 3 into 2 24. So these are these many maps. And what I have shown here is one of them, okay. So that is one example of a map. Now I will define a product of maps, okay. So I am going to define a product of maps.

Exercise: How many one-to-one onto maps are there?

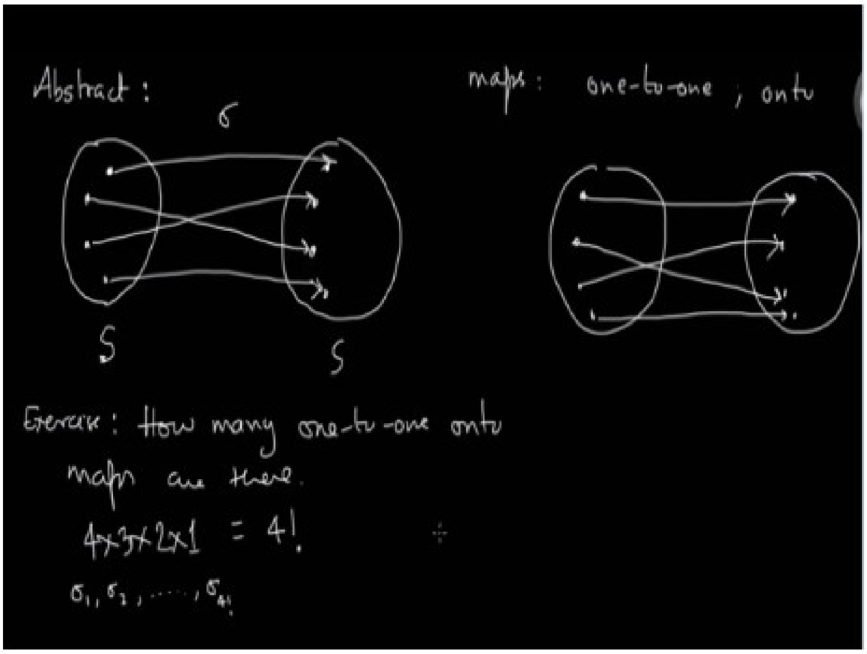
Define a product of maps $\sigma_1, \sigma_2, \sigma_3, \sigma_4$

$$\sigma_2 \cdot \sigma_1 = \sigma_i \tag{1}$$

$$\begin{array}{ccc} x_1 & \text{mapped} & x_2 \\ x_2 & \text{mapped} & x_1 \\ x_3 & \text{mapped} & x_3 \end{array}$$

Set of all maps

$$A(s) = \{\sigma_1 \cdots \sigma_{4!}\} \quad \text{Closed under multiplication} \tag{2}$$



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Inverse mapping

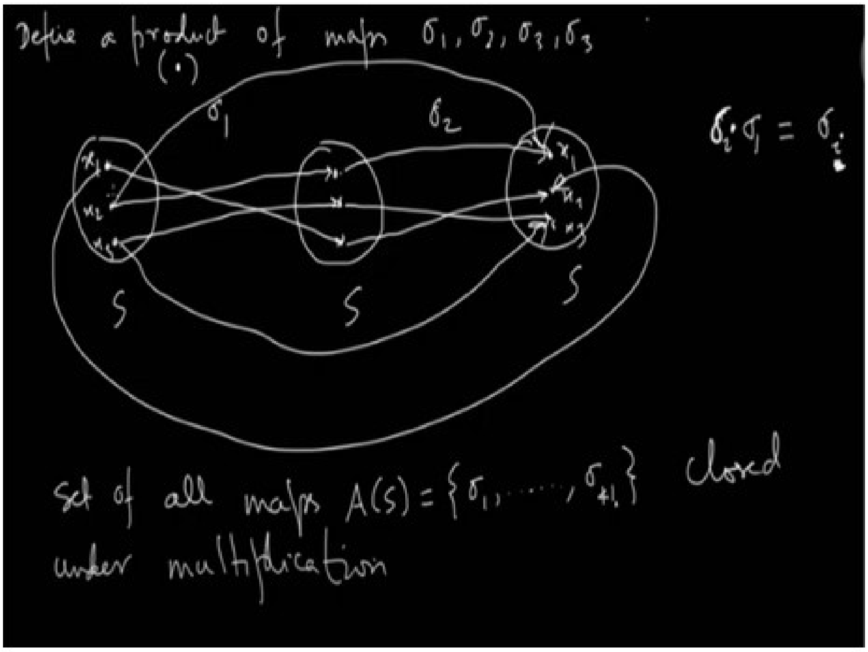
- $\sigma \cdot \sigma^{-1} = e$
- A(s) ; identity element
- σ, σ^{-1} Inverse
- $(\sigma_3 \cdot \sigma_2) \cdot \sigma_1 = \sigma_3(\sigma_2 \cdot \sigma_1)$

So sigma 1, sigma 2, and these are all elements of, I mean these are all the maps, and I want to define a product sigma 1 dot sigma 2. And the product I will denote by dot. So that is the product, okay. And it is, we have a natural candidate, which can be defined as a product. So let me, see these are all maps from, every time I am talking about a map, it is always from S to S.

So these are all the same things, all the same sets. So suppose we have a map sigma. Okay, it would be easier if I do not draw 4, but I only have 3 elements here. Then I do not have to do a lot of work, okay. So think of a map sigma. Let us call it sigma 1, which takes you, this elements gets mapped to this element, this element goes there, gets mapped to this. This element gets mapped to this and this entire map is sigma 1.

You could think of another map sigma 2, again from S to S. And just for ease, I will draw here between these two, okay. But you should understand all the maps are defined on the same set to the same set, okay. But I am drawing it here. So that it looks easier to see. Suppose that map is this. This one goes there. That one goes there. And that one goes there. Let us call this one as sigma 2.

Now we can define a composition. You have sigma 1, sigma 2 and that composition I will say that that is what I mean by product. So product of map sigma 1 and sigma 2. What do I mean by that? I mean by that, that I am looking at the composition of sigma 1 and sigma 2. So sigma 1 takes these elements there and then sigma 2 does this. So what does sigma 1 followed by sigma 2 does? It does this.



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It takes, sigma 1 takes this one here, and then sigma 2 takes that element there, right? So what has happened is if I call this as x 1, x 2 and x 3 these elements, this is also x 1, x 2 and x 3. This is also x 1, x 2 and x 3, okay. Then x 1 has gotten mapped to x 2, okay. And x 2 has been mapped to x 1 and x 3 has been mapped to, x 3 has been mapped to x 3. Okay it goes back.

You see this map, which we are seeing starting from x 1 and going to x 2. So look at this map. x 1 going to x 2. Then x 2 going to x 1, this one. And then x 3 going to x 3, okay. That map is one of the maps in the entire set of four factorial maps, right? So sigma 1 dot sigma 2 is some map which for some value of i. I mean sigma 1 dot sigma 2 gives you a map, which is one of the sigma i's, okay.

So clearly that is also a map. So you see that if you are looking at all the maps defined on this set, which are one-to-one and onto, then that set of maps is closed under this multiplication, okay. So now, I am not talking about the set S. I am saying the set of all maps okay sigma 1. So these are the elements of the set. Let me call it A. In fact, I will call it the set of all maps A defined on the set S which are one-to-one and onto okay. And this set has all this maps. This is closed under the multiplication which I have defined, okay. So this is, these elements are now like your Lorentz transformations, okay. These x 1, x 2 are like your space time points.

These elements are like your Lorentz transformations and the product there was a matrix multiplication, but here is just composition, right. So see now what we are talking is something even more basic and elementary. We are just talking about sets and maps. So whatever structure we are going to find here would be would have wide applicability in maths and physics okay, just because it is based on something very elementary.

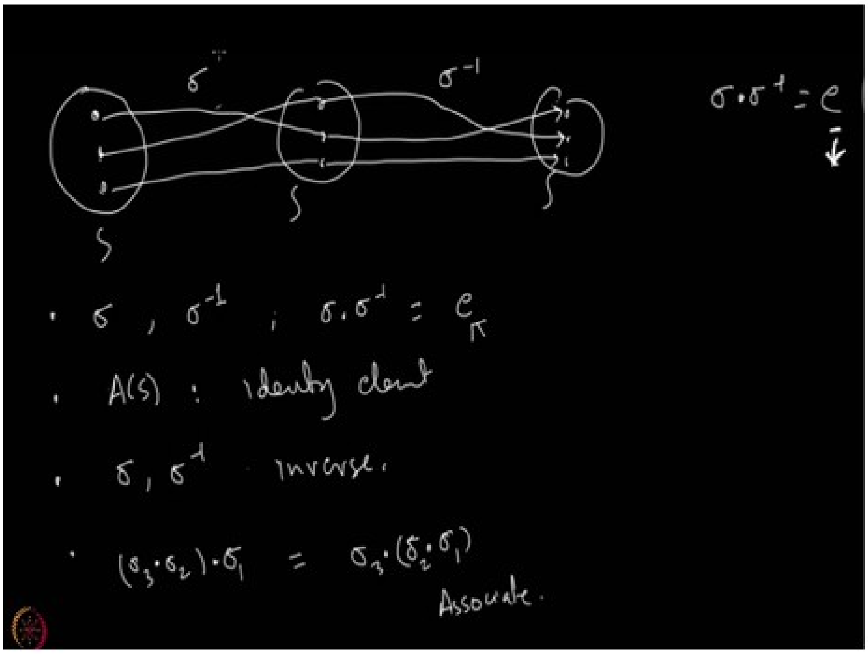
So good. So we have seen that all these maps I mean this set is closed under the multiplication if you are defined here. And also see that given a map sigma you have an inverse map to it. You can always map or find another map which will take which will give you inverse. Let me tell it more clearly. So if you have this, so this goes to that one, this goes to that one, and this goes to that one, okay. You can always find another map which will let me draw here S. So this one is going to this. What we want is go back. How do we go back? This guy should go here. If this goes here, then this one has been mapped to the same point, okay. This point x 2 has been mapped to x 1. And what I want is go back to x 2.

This one has been mapped to x 3, nothing to be done, it should go back. So see if this is sigma, then this is sigma inverse because composition of these two will give you identity which will be equivalent to doing nothing. So which will be equivalent to mapping this element back to that, this element back to this, and this element back to that, okay. By e I mean one of the maps sigma which maps each element to itself, okay.

So clearly, we have a map e okay, for which, okay I have jumped a bit. So right now, what I have said is for each map sigma, there exists another map which we can call sigma inverse, which when you multiply with sigma gives you another map e, which is the identity map, okay. An identity map is the map which maps each element to itself, okay. So they are two things.

Identity, there is an identity element in the set of all maps. And also each element in A(s) has an inverse, okay. Let me write it nicely. Or maybe not. So I have an inverse element sigma inverse for each element sigma. And sigma inverse is one of the sigma 1, 2, 3 and 24 okay. Okay, inverse. And of course, you know that these maps are this composition rule is associative.

So if you start with sigma 1 and then do a composition of this with sigma 2 dot sigma 3. This is same as sigma 1 dot sigma 2, this composition followed by sigma 3, okay. So that is associative property, okay. Now, these are all elementary observations, which we have are made about the maps, okay. And that those elementary or basic properties I am going to make as a basis of defining a new mathematical structure



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Defining a new mathematical structure by abstracting out the basic observation

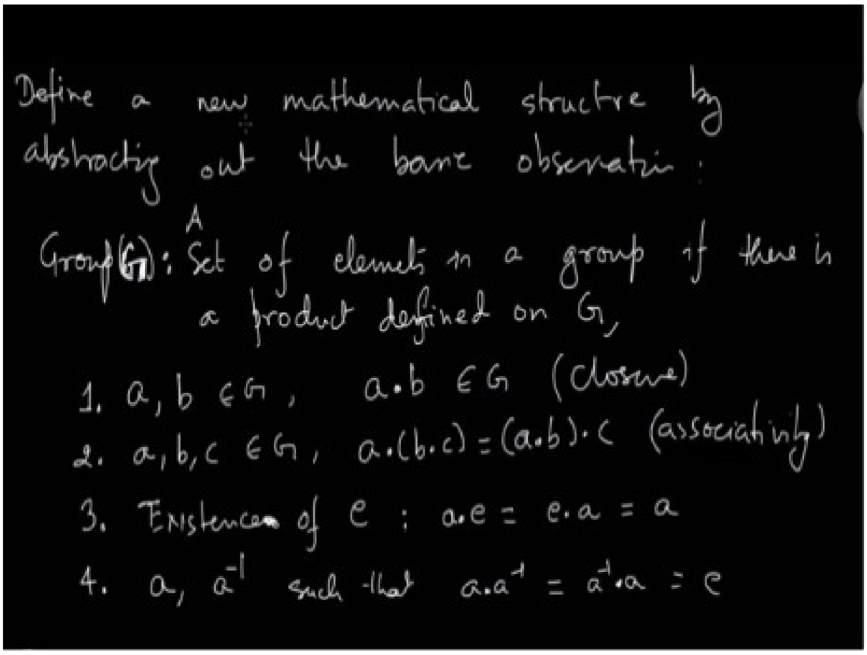
Group(G): A set of elements is a group if there is a product defined on G,

(1) $a, b \in G, \quad a \cdot b \in G$ (Closure property)

(2) $a, b, c \in G, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associativity)

(3) $e \in G, \quad a \cdot e = e \cdot a = a$ (Existence of identity)

(4) $a \in G$, $\exists\ a^{-1}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$



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So I define a new mathematical structure, okay new meaning in terms of this course what we are doing here. It is a highly developed subject. Let us see in a moment. Define a new mathematical structure by abstracting out the basic observations that were made, okay. So I defined a group. Let me see. Okay. And what do I mean by a group? I mean the following.

That a group is a set of elements okay, a group G, okay? The group will be denoted by G, capital G. Okay, and what is that group? It is a set of elements, okay. So a set of elements is a group if the following conditions are true. So first we need a multiplication to be defined on the elements. If there is a product or multiplication defined on the elements of G, okay.

So here when we are talking about the maps, the product was composition. When we were talking about Lorentz transformation, the product was matrix multiplication. But all we need is some definition of product, okay. So that is one requirement that you have to have a product and then the product should satisfy the following properties. If a and b are two elements of the group okay of the set G.

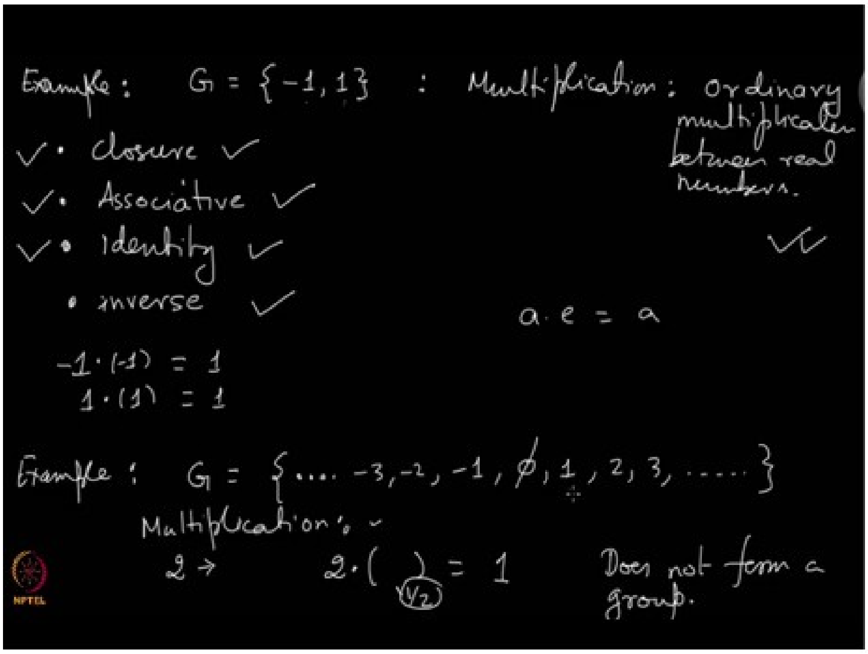
If these two belong to G, then a times b or a product b, and that is the product you have defined there, that should also belong to G. Which means that the set should be closed under multiplication. It should not happen that when you multiply these two

elements you get an element which is outside G. If that happens, then we do not have a group. So this has to be satisfied.

Second, if you have three elements a, b and c belonging to group G, then we want to have associative property to hold true, okay. We have seen that is true for maps under composition. And we want to have that property to be built in into this definition. So that is the associativity, okay. And that is again a requirement on the multiplication, okay. Third, there should exist, there should be existence of identity elements.

So we should have an element which is identity and what is the meaning of identity? I mean that I should have an element e, which is belonging to G such that when I multiply it with any other element a okay. This is, a is generic, could be any element of the set. I should always get the same element back. If that is happening then I call e as the identity element and I demand that such an element should exist in the set G.

And lastly, if you are given an element a, any element a, then there should exist an element a inverse okay, which will be one of the elements in G such that a times a inverse, okay. And remember this product is not some, I mean unless you define what you mean by this product, we do not know how to find out these objects. So that has to be defined first, okay. So anyhow, that is the structure we have for a group.



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And clearly you saw that structure was present here actually. This is the basis on which we have defined the group, okay. So what we have done is taken the set of all maps from S to S. And whatever structure was visible there on the set of all mappings, we have made that as our basis for the definition of a group, okay. So we have now defined a mathematical object called group. Let me give you an example of it.

Example: G=$\{-1, 1\}$: Multiplication; Ordinary multiplication between real numbers

1. Closure
2. Identity
3. Associativity
4. Inverse

So, G forms a group under multiplication

Example: $G = \{\cdots - 3, -2, -1, 0, 1, 2, 3, \cdots\}$, it cannot form a group under multiplication as inverse is not in the group

Okay, before going to Lorentz transformation, which is what we are really interested in. Let me give you an example which is outside, which is not based on any physics, it is just simple abstract mathematics. And sorry, I wanted to write not existence, but sorry. Okay, this is a nice example. So let us take the following. Take a set of elements G, which has only these two numbers, okay.

And I want to check whether this is a group. Now if I want to check this is a group, it cannot be checked, because it is a meaningless statement unless you define what is the multiplication, okay. What is a multiplication defined on this group? So let me say that the multiplication is the following. So the multiplication is ordinary multiplication between numbers, between real numbers, okay.

So now I have specified what is multiplication. So let us check whether this is a group. Is it closed under multiplication? Yes, let us see. What are all possible multiplications that you can make? You can multiply this element to itself, you can multiply this element to itself or you can multiply these two. And in all the cases, whatever you get is again a number in G. For example, this is again 1.

1 times 1 is 1. -1 times 1 is -1. So it is clearly closed, okay? So closure is satisfied, okay. Then what next? Associativity. Because it is ordinary multiplication and ordinary multiplication is associative. So this is also true. Do we have an identity element? Yes. 1 is identity element, because 1 times 1 is 1. -1 times 1 is -1, okay? So that is an identity element. So identity is there.

Do we have inverse? Okay, let us check. If we have inverse, then we would have all the property satisfied. So what should I multiply to -1 to get identity? See this is identity. Remember, a times e is a. This is what we have to verify. So -1 times what gives you 1? And that is -1. So -1 is inverse of itself. The only other element is 1 and you know 1 times 1 is 1.

So we do have inverse for each element. So G forms a group under multiplication. Okay, that is good. Let me give you another example. Let us take the set G of, G which consists of all the numbers, all the integers. So let us say that is -3, -4, -5 everything, all negative numbers. 0 is not here, I am just crossing it out. Okay, all other positive integers as well.

And I should tell what multiplication I am looking at. And multiplication is same as before. I am talking about ordinary multiplication between real numbers okay, the same thing as here. Now does this set form a group under multiplication between under this multiplication? And the answer is yes, you can easily check. If you multiply any number okay with any other number, you will get a number which belongs to the set, so it is closed, okay.

Associativity is guaranteed because you are talking about multiplication between real numbers and that is associative. So that is also true. So closure is satisfied. Associativity. Identity. Again of course, 1 is the identity element, okay. Do we have inverse for each element? Yes, we do have inverse for. So I think yeah. So do we have inverse of each element 2? Let us look at for example 2.

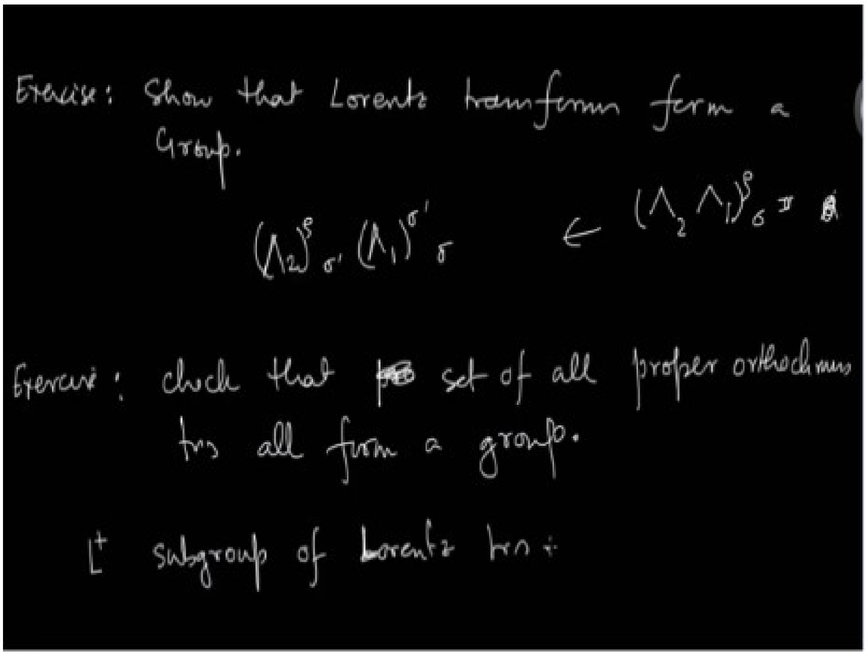
And what we are asking is two times what gives you the identity and identity is 1, okay. And you see that that number is half and half is not there. So you see this G cannot form a group. Okay, because half is not here. So does not form a group. Okay so G, this set of elements G is not forming a group, okay. Now I will give you an exercise, how that Lorentz transformations form a group under multiplication, okay. Form a group, let me add like this. And you have to ask about multiplication and multiplication you already know. You have, if you have a matrix lambda 1 and another matrix lambda 2, how are these multiplied? These just multiplied by this. So this is, let us call it lambda 2, this is lambda.

Okay, maybe I should write here. So this is this multiplication is defined in this way. Lambda 2 rho, okay I am sorry again. Lambda 2 rho. Let us call it sigma prime, lambda 1, sigma prime. And this is sigma. So that is your matrix multiplication and that is the multiplication you have defined on the set. So check that this set of all Lorentz transformations form a group, okay.

Exercise: Show that Lorentz transformation forms a group under multiplication

$$(\Lambda_2)^{\rho}{}_{\sigma'}(\Lambda_1)^{\sigma'}{}_{\sigma} \rightarrow \text{multiplication} \tag{3}$$

Exercise: Check that set of all proper orthochronous Lorentz transformation all forms a group, L^+ a subgroup of Lorentz transformation



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Then another exercise, check that all proper orthochronous transformations, the set of all, set of all proper orthochronous transformations also form a group okay under the same multiplication law.

And that will be easy because once you have shown that all Lorentz transformations form a group, showing this will be easy because the only additional thing you have to ensure is that when you multiply two proper orthochronous transformations you stay within that proper, within that set meaning you again get determinant 1 and lambda 00 element positive, okay.

And again also you have to check about the inverses that they also belong to the same set. So once you have done that, you would have shown that this group okay is a subset of this bigger group, okay? So you would have shown that, I think I will call it L+ for proper orthochronous transformations that this group is a subgroup. This is the definition of a subgroup, subgroup of Lorentz transformations, okay?

Meaning L+ will form a group by itself under the same multiplication rule, which you have for Lorentz transformation. And that is why it is called a subgroup, okay. And remember that the identity element always has to be there in the group. So L+ anyway has an identity element. So it is going to work out okay that it turns out to be a subgroup. Okay, so we will stop this video here. And we will continue our discussion further in the next video.