Introduction to Quantum Field Theory

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Lecture 18 : Lorentz Transformations Continued

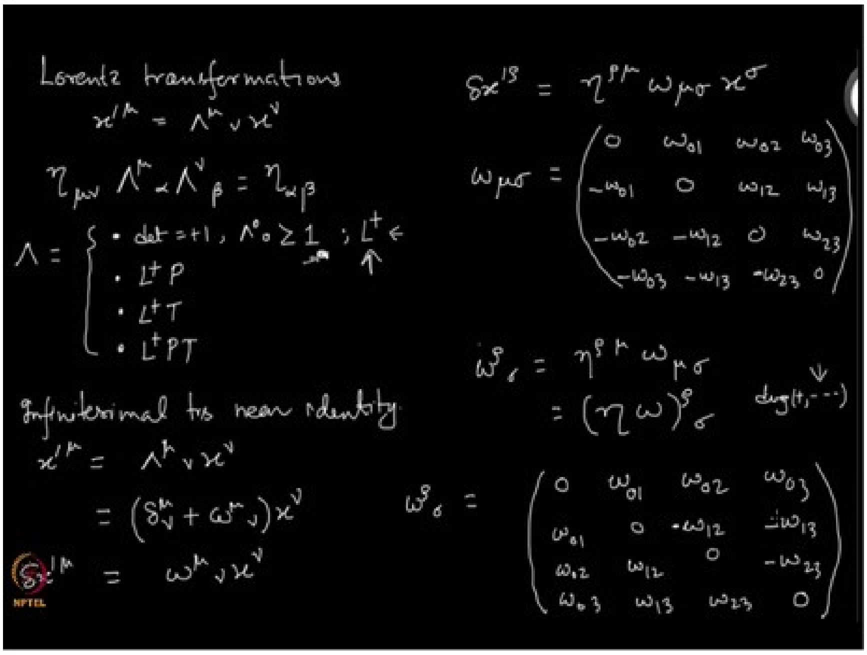

Figure 1: Refer Slide Time: 00:17

Okay, so we were looking at Lorentz transformations last time. Lorentz transformations. And we had already found that apart from an overall shift of the coordinates which I was writing with plus a mu okay, this is the most general transformation and okay that plus a mu is not counted as part of Lorentz transformations, okay. So this is what how coordinates change in the Lorentz transformations.

And we saw that the constraint on lambda okay, which was that if you have eta mu nu then mu alpha lambda nu beta, this should give you back again eta alpha beta, okay. So that is the constraint. And then we further saw that Lorentz transformation matrices lambda fall under four categories, okay. One is proper orthochronous transformation, so determinant is +1 and lambda 00 is greater than 1, okay That is proper orthochronous. Let us denoted by L+. So all the transformations which fall under this category, I will say that they belong to this part of the set, okay.

And then we also saw or I think I asked you to do as an exercise one of these that any Lorentz transformation lambda can be written as a product of an orthochronous, proper orthochronous Lorentz transformations time a parity operator or time this times time reversal operator or a combination of both that you have both time reversal and parity, okay.

So these are four categories in which you can classify the matrices lambda. And the one here, this is the one which is continuously connected to the identity matrix, okay. The no transformation at all. Identity corresponds to no transformation at all, okay. So now what I want to do is I want to look at infinitesimal transformations near identity. So clearly I am talking about this part because identity element is only in here.

These do not include identity elements because parity and time reversal are discrete transformations. So we are in this set in in this set, okay. And any transformation near identity I can write as follows. So of course, in general I will write lambda mu nu x nu and lambda mu nu if there was no transformation at all then you are sitting at the identity element and that is just delta mu nu.

But we are saying we are infinitesimally away from identity. So let us say omega mu nu, these are the parameters that parameterize how much further away you are from identity. So right now these are infinitesimal, okay. What they are, we will see in more detail how to interpret them. But right now we are just saying these are infinitesimal parameters which take us away from identity, okay.

Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{1}$$

The constraint

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta} \tag{2}$$

Four parts

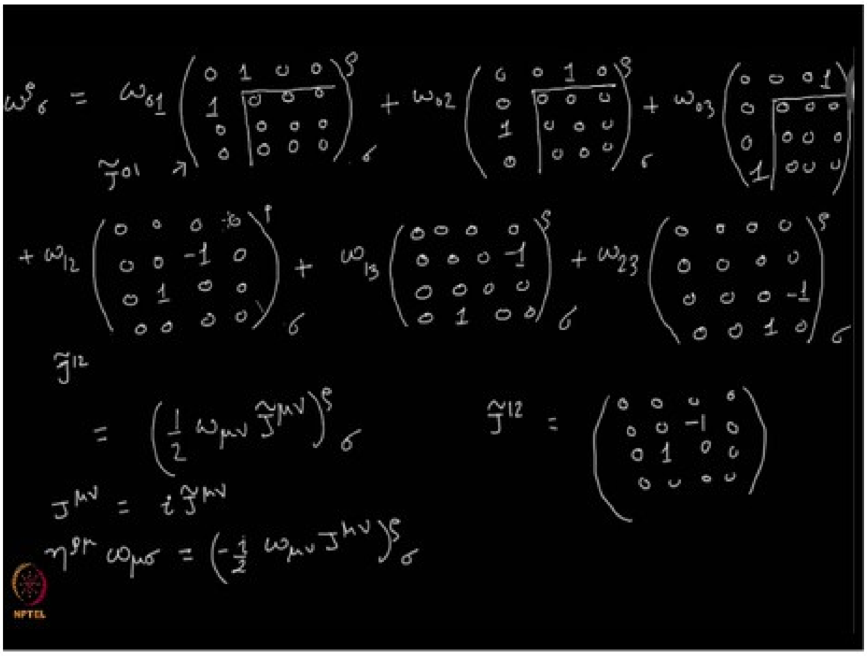

Figure 2: Refer Slide Time: 12:37

$$\Lambda = \begin{cases} \det = +1,\ \Lambda^0{}_0 \geq 0,\ L^+ \to \text{(includes identity)} \\ L^+P \\ L^+T \\ L^+PT \end{cases} \tag{3}$$

So this is this. Now I can write the same thing. So here the first piece delta times x nu gives you again x mu. So the change in x is the x prime minus x which is just this part, okay. So your delta x prime mu is omega mu nu x nu, okay. Now what I want to do is I want to bring the first index down so that both the indices on omega are lower indices and rewrite this. So I can bring it down using an eta what do you call it?

Yeah, metric tensor here. So let me do that. Instead of putting mu I will just write rho. So del x prime rho is eta rho mu. omega mu nu x nu. Let us see. Nu is contracted, mu is contracted. Only rho is free and that is fine because only rho should be free as you have on the left hand side, okay. We have already argued this in the last lecture that the omega with both the indices below okay is anti-symmetric.

It is a 4 cross 4 anti-symmetric matrix. So that is what I am going to utilize now. So I write omega okay. Yeah, whatever I wrote was correct, but I just want to use sigma instead of nu. Does not matter you can write nu. So if I look at omega mu sigma, if you look at this 4 cross 4 matrix, okay, these are which is anti-symmetric, it looks like this. So because it is anti-symmetric, all the diagonal entries are 0.

And this one I call, this is omega 01. This is omega 02, and this one is omega 03. And due to anti-symmetry it will be minus omega 01 just negative of this, this one should be 0, okay. And then here, you should have minus omega 02, okay. This one, let us call omega not let us call but it is omega 13. This entry would be 0. This one will be minus omega 12, okay.

This one is omega 23, omega 03 this one, this is fourth entry. So this one is minus omega 03, minus omega 13 and minus omega 23 and 0. So this is negative of this. This is negative of that, okay. So that is fine. So that is what it looks like. But you see, we have to know what is omega mu nu with mu up and nu down. Not the omega with both the indices down. But that is easy, we can find that out.

So what we need is eta this one this eta rho, let me write it like this where is it omega this is omega rho sigma. Rho sigma which is same as eta rho mu, omega mu sigma, this is what we are trying to find, okay. One index up one index down. And that is easy to find, because this is just we take the eta matrix, take the omega matrix which is here with both down okay. Eta you know anyway. And then multiply them and look at what is the rho sigma element, okay. And that is what will give you this one. So that you can do easily. You can take the eta to be, you know what eta is. So diagonal plus, minus, minus, minus, okay. Take that matrix multiply with this one and read the omega sigma elements. That you can do and you can another way to do is this. You realize that you are raising the first index on omega.

That is what you are doing right by taking the eta. And when you raise the index with eta it will not change the sine of the 0 when rho is when mu is 0, okay. But when mu is either 1, 2 or 3 then because when you are raising it will change the sign because it is this matrix, okay. So it will do nothing to the 0 index of mu but it will change the sign of, change the sign of the matrix elements if mu is either 1, 2 or 3, okay.

So that way it is again easy. So let me then write down what will be omega rho sigma. So omega rho sigma will be this, okay. Let us first write and then put the bracket. So this is fine. First index is 0, so I am looking at 01 element, nothing changes. Nothing changes, Here this will change the sign, okay. This is omega 10, which is same as minus omega 01 but the element is omega 10, right?

This is the entry name is omega 10. So first index is 1. So it will change the sign. So I will put minus omega. It will change the sign so it will become omega 10, omega 20. Sorry, what am I doing? It will become omega 01, omega 02, omega 03. The signs have changed, okay. And then of course this. The first index is 1, so nothing will happen, sign will not change. Sorry sign will change, 3 okay.

And this will become omega 12, 0 minus omega 23, omega 13, omega 23 and 0, okay. So this is what the matrix would be. Now remember that all these omegas which we have written, so there are 6 of them which are independent. The lower half is anyway completely determined by the upper half. These are independent of each other.

Look at infinitesimal transformation near identity

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{4}$$

$$x'^{\mu} = (\delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}) x^{\nu} \tag{5}$$

$$\delta x'^{\rho} = \eta^{\rho\mu}\,\omega_{\mu\sigma}\,x^{\sigma} \tag{6}$$

Using the anti-symmetry property of $\omega_{\mu\nu}$

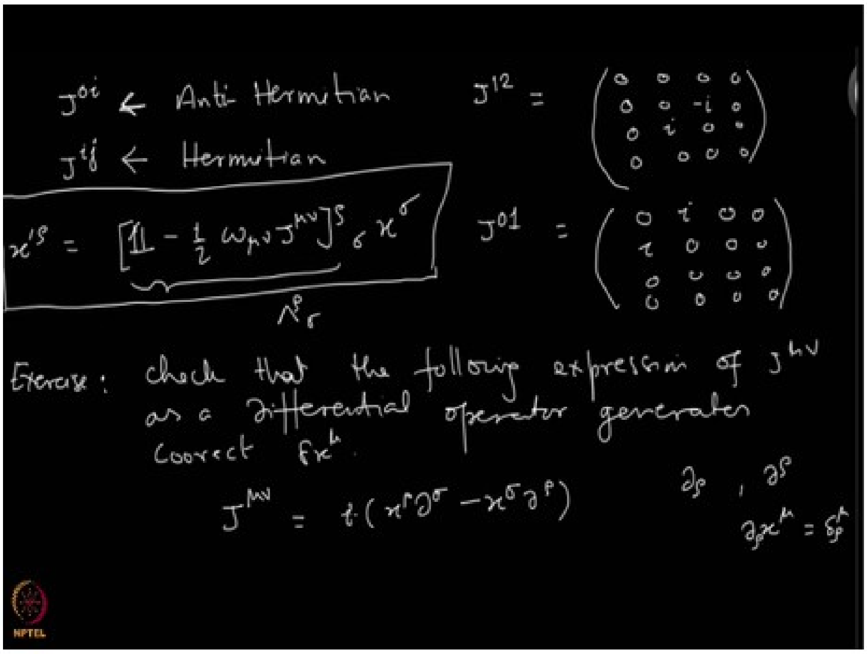

Figure 3: Refer Slide Time: 22:17

$$\omega_{\mu\sigma} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{12} & -\omega_{23} & 0 \end{pmatrix} \tag{7}$$

So I can write the matrix omega rho, is it rho sigma? That is correct. So this can be written as the following. This is fairly trivial but let me nevertheless write it explicitly and bore you guys a little. So I will write it as omega 01. What I am doing now is putting all the, I mean just I am writing it as sum of 6 matrices. That is all I want to do, okay. So omega 01 omega 01 that you can factor out.

It will leave behind a 1 here and 1 here and others. Anyway I am going to split, so I will put all others to be 0. And then add to it with omega 02 omega 02 element, put everything else to be 0. So that is what I am doing. So you can see that it will be the following 0 1 0 0, 1 0 0 0 and remaining are all zeros, okay. So what you have here is, okay, you get this block. It has all zeros here, okay.

Then, what do I have? Omega 02 and here you will have 0 0 1 0, 0 0 0 0 and then 1, okay. While I am writing this, you can also write on your own without looking what I am doing. And then we can just match at the end. All these are 0 again. See again this part is completely 0. All the things all the entries in here are 0, okay? Plus then you have omega 12. 0 0 -1 0. 0, this should be +1, okay?

I hope you are also doing it with me so that we never have to spend time on this again. Okay. And then you have all these zeros, this one is -1. Third row is fully 0, and then this is 0. Something looks not good. Is that correct? Let us check omega 13, let us go back. Omega 13, so 1 here and 1 here, that is fine. Okay, that is correct. And the last one is omega 23. So you got 0 0 0 0 and you have only entries in this one, okay.

So if you have to take the rho sigma element, if you have to find the omega rho sigma, then I should be taking rho sigma element of these matrices, okay? That is what we should be doing, okay. So rho will be the first index, which will be the rho index and the sigma will be second index which will be the column index, okay. So now I can write it in short as half okay, there are 6 generators here.

So these are, these I will call generators, okay. And the reason I call them generators is because you see, these matrices are, see the change in x is this much and this change is being generated by these omega mu nu's and omega mu nu's apart from these parameters are controlled by these matrices. So these matrices are the ones which are generating the change in x. So we call them generators, okay.

So I will write this as, so omega 01 omega 02 omega 03 these are parameters, so I write them as omega mu nu. And these matrices six of them I write as J mu nu, okay. So this are mu runs from 0 to 3, nu runs from 0 to 3. So it is 4 times 4, 16. But then I remember that because this is anti-symmetric, this will also be anti-symmetric. Okay I want to first put tilde on this.

So I do not want to call them as J's but J tilde, okay. Now as I was saying that, because omega is anti-symmetric in mu nu okay, J tilde has to be anti-symmetric in mu nu, okay. It will kill the symmetric part. If there was a symmetric part to J tilde, it will kill that part. So let us take the anti-symmetric part, I mean let us take J tilde to be anti-symmetric, okay.

And anti-symmetric under mu nu interchange, okay. It is anti-symmetry under mu nu. But note that mu and nu, they do not label the entries of these matrices. See mu nu labels, which matrix you are talking about? Whether you are talking about this matrix or that matrix or that matrix. The entries are labeled by the indices row and sigma, okay. So that has to be kept in mind. So let us see.

$$\omega^{\rho}{}_{\sigma} = \eta^{\rho\mu}\omega_{\mu\sigma} \tag{8}$$

$$\omega^{\rho}{}_{\sigma} = (\eta\,\omega)^{\rho}{}_{\sigma} \tag{9}$$

Raising the first index,

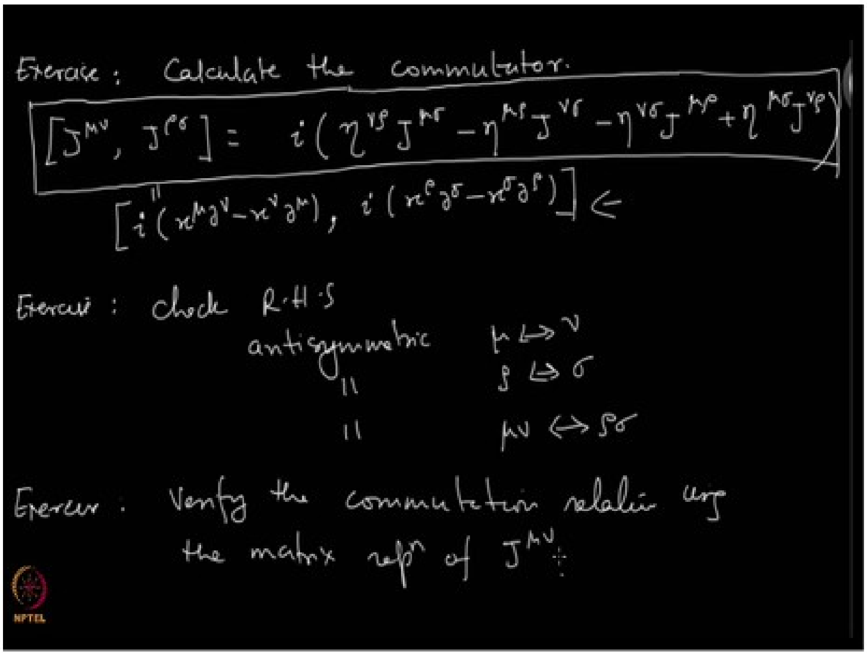

Figure 4: Refer Slide Time: 28:11

$$\omega^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & -\omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & \omega_{12} & \omega_{23} & 0 \end{pmatrix} \tag{10}$$

$$\begin{aligned} \omega^{\rho}{}_{\sigma} = \omega_{01} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \omega_{02} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \omega_{03} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \\ \omega_{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \omega_{13} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \omega_{23} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{11}$$

We call these matrices as generators

So we have for example, J tilde 12 is 0 0 -1 0 and 0 1 0 0, 0 0 0 0. Let us see where it is omega 12, this one, okay. This is the one I am talking about. Okay, now I want to define J. So j mu nu I will define it to be i times J tilde mu nu. And if I do so then I get eta rho mu, omega mu sigma. That is the quantity we have been looking for considering since some time.

So this becomes minus i over 2 omega mu nu J mu nu and rho sigma, okay. And the reason I have put a i here is that I want to make certain generators as Hermitian. So let us see which ones. Okay that is, oh I have already written something here. Let us see. Okay, let us skip this one. Let me put a cross. So let us go back. So what will happen with because of this vector of i is, let us see where is J 01.

This is J 01 or J tilde 01. J tilde 01, okay. So you have 1 here, 1 here. If you put a factor of i, which is here, you will have an i here and i here, okay. That is one thing. The other one let us look at J 12. Where is 12 here? So J tilde 12. If I multiply a factor of i to get J 12, it will be -i here and +i here. Now this is, that will be Hermitian right?

Because if you take a transpose and complex conjugation that will give you again the same thing because of the factors of i, okay. So what I have achieved by multiplying a factor of i in the definition of J is that J 0i. Let me write about J 0i after I have written about J ij. These generators J ij okay, these are Hermitian matrices. So these generators become Hermitian. And this one still remain anti-Hermitian. Okay, my interest was in making something Hermitian. If this is not becoming Hermitian J 0i, that is fine, we cannot do anything.

But the interest was in making this part Hermitian. But you do not have to do that, okay. But it is advantageous because we are going to do quantum mechanics eventually. Okay, so let us just I think it will be good idea to write them here for later convenience, later reference. So J 12 becomes 0 0 0 0 what becomes 1 2 1 2 this. So this will become -i this will become +i. –i i 0. Okay, that is our generator J 12.

And so clearly this is Hermitian. If you take a transpose and complex conjugation, you get back the same thing. And let me also write down J 01. So 0, this will give you what? J 01. So J tilde 01 is both, okay. So we will get i 0 0, i 0 0 and everything else is 0. Okay so good. And now we have shown that x prime rho is just del rho sigma -i over 2 omega mu nu J mu nu, okay.

If I do not put a rho sigma here, that will be better. Let me write it again. Instead of writing delta let me write identity -i over 2 omega mu nu J mu nu, okay. And then you have to take the rho sigma element of this and contract with x sigma, okay. And this is what your lambda mu nu is, lambda rho sigma. That is the infinitesimal Lorentz transformation, okay. That is good, let us see what I want to say here.

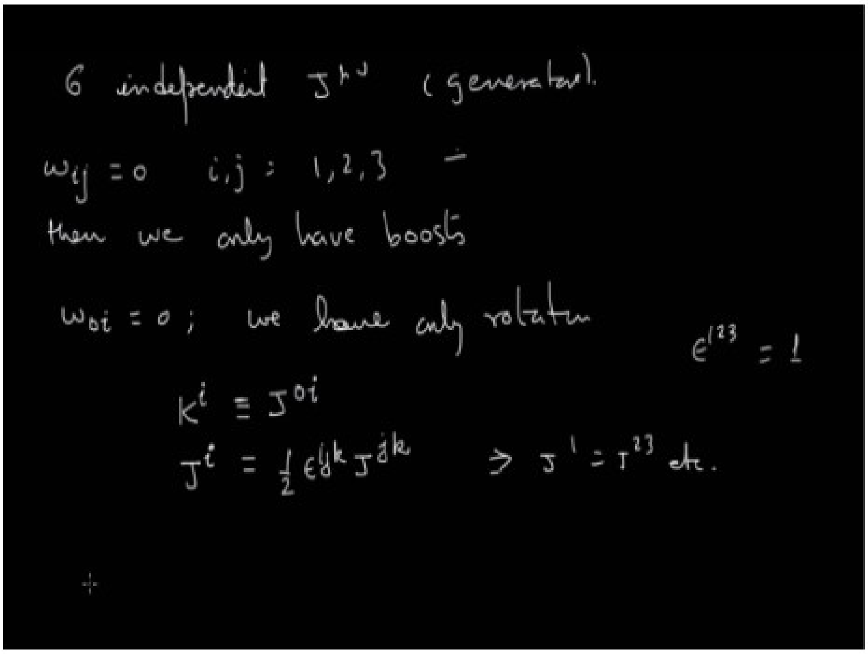

Figure 5: Refer Slide Time: 32:47

$$\omega^{\rho}{}_{\sigma} = \left(\frac{1}{2}\omega_{\mu\nu}\tilde{J}^{\mu\nu}\right)^{\rho}{}_{\sigma} \tag{12}$$

Okay, good. Now note that we have already, I think this is the place where I want to, okay maybe later. So anyway I was about to make a remark, but I will save it for little later. So this is what we have shown and this is a result which we want to put in a box okay. Now I have an exercise for you. Check that the following expression of J mu nu as a differential operator generates correct delta x okay, delta x mu or whatever you want to write.

So check that if you take J mu nu the generator, not in this matrix form but in this form, i x rho del sigma minus x sigma del rho, okay. You take this one okay and plug in here. Then check that this is producing the correct transformation which you expect. So that you should be able to verify. All you will have to remember in this entire derivation is that del rho and del rho are related by, you have to raise the index by using eta mu nu, okay.

That is what you have to do. And another thing is that del x mu over del x rho with rho up, that is what gives you, okay this is what you should use. And because here you have one index up you have to contract with eta mu nu to raise the index. Okay, if you use this you will be able to show that this indeed reproduces correct change in the coordinate x.

ω is anti-symmteric. Take $\tilde{J}^{\mu\nu}$ to be anti-symmetric. For example

$$\tilde{J}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

We define

$$J^{\mu\nu} = \iota \tilde{J}^{\mu\nu} \tag{14}$$

$$\eta^{\rho\mu}\omega_{\mu\sigma} = \Big(-\frac{\iota}{2}\omega_{\mu\nu}J^{\mu\nu}\Big)^{\rho}_{\ \sigma} \tag{15}$$

The factor ι because we want some of them to be hermitian

$$J^{0i} \to \text{anti-hermitian} \quad ; \quad J^{ij} \to \text{hermitian} \tag{16}$$

J^{12} will become,

$$J^{12} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\iota & 0 \\ 0 & \iota & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad J^{01} = \begin{pmatrix} 0 & \iota & 0 & 0 \\ \iota & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

$$x'^{\rho} = \Big(\mathbb{1} - \frac{\iota}{2}\omega_{\mu\nu}J^{\mu\nu}\Big)^{\rho}_{\ \sigma} x^{\sigma} \tag{18}$$

Where

$$\Lambda^{\rho}{}_{\sigma} = \Big(\mathbb{1} - \frac{\iota}{2}\omega_{\mu\nu}J^{\mu\nu}\Big)^{\rho}_{\ \sigma} \tag{19}$$

Exercise (1): Chechk that the following expression of $J^{\mu\nu}$ as a differential operator generates correct δx^{μ}.

$$J^{\mu\nu} = \iota(x^{\rho}\partial^{\sigma} - x^{\sigma}\partial^{\rho}) \tag{20}$$

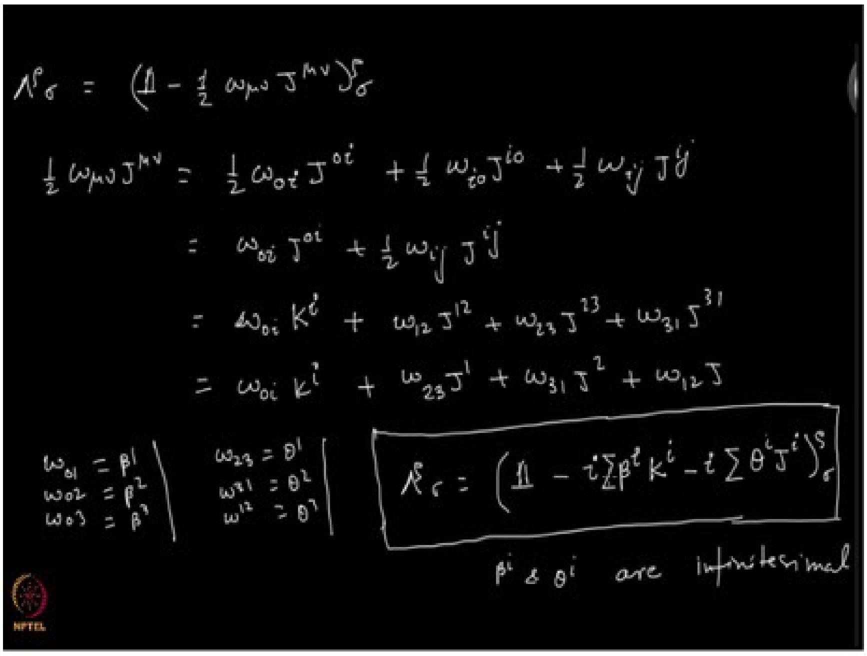

Figure 6: Refer Slide Time: 37:20

Then you do another exercise, again not difficult. Calculate the following commutator. Calculate the commutator and the commutator is this. Take J mu nu and J rho sigma and find their commutator. And you will be able to show that this is equal to let me first write before I give the expression. Okay, maybe I should give the expression i eta nu rho J mu sigma eta mu rho J nu sigma nu sigma J mu rho.

And then eta mu sigma J nu rho, okay. This is what you have to show and you start by doing the following. You write the expressions for J mu nu and J rho sigma. So you will be basically looking at this object J mu del nu minus x nu del mu. And you put and you commute with find the commutator with this one x rho delta sigma x sigma del rho, okay.

It is a fairly easy exercise to show that this results is true, this result is true. Now note that whatever symmetry the left hand side has, has to be obeyed on the right hand side. So left hand side here is anti-symmetric under mu nu interchange. So right hand side should also be anti-symmetric under mu nu interchange. Right hand side is anti-symmetric under rho sigma interchange.

That should also be true on the right hand side. And also if you take this J mu nu and put here and take the J mu nu here and put there you get a minus sign because you have a commutator, which means you also have anti-symmetry under interchanging mu nu with rho sigma, okay. So you should make these checks and show and check clearly that this is indeed the case on the right hand side, okay.

Maybe I should write it down. Check that right hand side is anti-symmetric under mu nu interchange, anti-symmetric under rho sigma interchange and also anti-symmetric under mu nu getting interchanged with rho sigma, okay. Okay. Now why I am interested in this commutator and I will talk about this in a later video. But because this is such an important thing, I will put it in a box, okay.

That is a, that is going to be a very important object for us and why, I will tell you later. So please do this exercise. You should do it both ways. You can use this to show it, which is going to be very easy. And okay and you can also show using the explicit expressions of these matrices, okay. Both ways it will work, okay. And that is also another exercise. Maybe I should write that also down.

Verify the commutation relation using the matrix representation of J mu nu. Okay, please do these exercises okay. Now we see that there are 6 independent generators, okay. And we are talking about proper orthochronous transformations. You see we are connected to identity, we are slightly away from identity. So obviously, we are talking about proper orthochronous transformations. And if you put omega ij equal to 0 where i and j run from 1 to 3, then we are only talking about boosts, okay. I think that is clear. Let us go back.

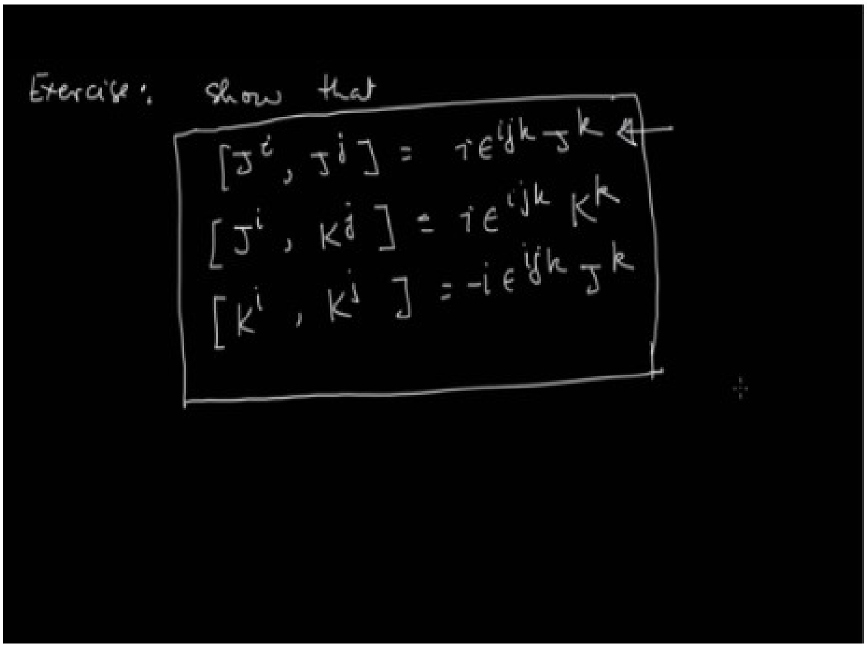

Figure 7: Refer Slide Time: 42:57

Exercise (2): Calculate the commutator

$$\Big[J^{\mu\nu}, J^{\rho\sigma}\Big] = \iota\Big(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\nu}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\nu\sigma}J^{\nu\rho}\Big) \tag{21}$$

So what I am saying is if I put omega 12, omega 13 and omega 23 to be 0, then you are left with only these three parameters. If we keep them to be nonzero, then you see this mixing is between the time component and the x component. This is in the time component and the y component. And this one is time component the z component, okay. So this is clearly boosts, these are about boosts.

And similarly, if you put omega 0i to be 0, then we have only rotations, okay. So suppose you were looking at a Lorentz transformation such that these parameters omega 01, omega 02 and omega 03 were 0, then what you have here is only rotation, okay. It is just mixing between, okay my battery is going low. Just a second. Okay. So okay, I think that is the reason why it was a bit dim. So good.

T

Exercise (3); Check RHS is

- Anti-symmetric with $\mu \leftrightarrow \nu$
- Anti-symmetric with $\rho \leftrightarrow \sigma$

- Anti-symmetric with $\mu\nu \leftrightarrow \rho\sigma$

Exercise (4); Verify the commutator using the matrix representation of $J^{\mu\nu}$.

Generators

hen now we understand which generators are generating rotations. So these are J 12, J 13 and J 23. And which ones are generating boosts? They are J 01, J 02 and J 03, okay. So what I will do is I will give names. So as I said J 0i are generating the boosts, so I call them, I give them a name K i okay, that is the definition. And also J jk, these ones are generating rotations.

So these ones I will define to be J i okay, where epsilon ijk takes value 1 when ijk are 1, 2 and 3. And if you, let me write down. Epsilon 123 is 1 and every time you interchange any two indices, you pick up a minus sign, okay. So this is J i. So these are three generators for boost, three generators for rotations, and you have already seen that you have three for boosts and three for rotations, okay.

So for example here, this implies that J 1 is J 23 etc., okay. Check. Check how J 1 and J with one index and J with two indices are related. So this is one of them here. Now yeah, so let us go back and write down the following. Show that, shall I show this ersion? Okay, maybe I will do it even though it is fairly trivial and should be left as an exercise. But I will nevertheless do it. Let me do it on the next page. So you had lambda rho sigma as identity minus i over 2 omega mu nu J mu nu rho sigma okay that is what we had. So let us look at this piece. Half omega mu nu J mu. Let us look at this. This is what? Let us put half. So mu there is a summation over mu and also summation over nu. So let us start with 0, okay. And then let us take see omega 00 will be 0, because these are anti-symmetric, so there is no such element.

But if this is i, where i is 1 to 3, this is non zeros. So this makes sense. Then you have half omega i0 J i0; that is also not going to vanish plus half omega ij J ij, okay where ij take values 1 to 3. Now this one is, these two together you see, because of anti-symmetry, this is just double of this. So half I mean not double of this, same as this. So I can combine the two and write omega 0i J 0i, okay.

This is we have already called define J 0i to be K i. So this becomes K i. Omega 0i K i plus this one will become let me write down explicitly. Omega 12 J 12, this you can easily verify, okay. And I can just write down this as omega 12 and our J 12, J 23 and J 31 can be written in terms of J 1, J 2 and J 3. So this becomes J 23 J 1 plus omega 31 J 2 plus omega 12 J 3, okay

Six independent (generators)

$$\star\ \omega_{ij} = 0 \quad , \quad i, j \to 1, 2, 3 \tag{22}$$

Then we only have boost

$$\star\ \omega_{0i} = 0 \quad , \quad i \to 1, 2, 3 \tag{23}$$

Then we only have rotations.

We define boosts K^i and rotations J^i as,

$$K^i = J^{0i} \quad , \quad J^i = \frac{1}{2}\epsilon^{ijk} J^{jk} \tag{24}$$

Where $\epsilon_{123} = 1$.

$$\Lambda^\rho{}_\sigma = \left(\mathbb{1} - \frac{\iota}{2}\omega_{\mu\nu}J^{\mu\nu}\right)^\rho_\sigma \tag{25}$$

$$\begin{aligned}
\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} &= \frac{1}{2}\omega_{0i}J^{0i} + \frac{1}{2}\omega_{i0}J^{i0} + \frac{1}{2}\omega_{ij}J^{ij} && (26)\\
&= \omega_{0i}J^{0i} + \frac{1}{2}\omega_{ij}J^{ij} && (27)\\
&= \omega_{0i}K^i + \omega_{23}J^1 + \omega_{31}J^2 + \omega_{12}J^3 && (28)
\end{aligned}$$

$$\begin{aligned}
\omega_{01} &= \beta^1\\
\omega_{02} &= \beta^2\\
\omega_{03} &= \beta^3
\end{aligned} \tag{29}$$

$$\begin{aligned}
\omega_{23} &= \theta^1\\
\omega_{31} &= \theta^2\\
\omega_{12} &= \theta^3
\end{aligned} \tag{30}$$

$$\Lambda^\rho{}_\sigma = \left(\mathbb{1} - \iota\sum\beta^iK^i - \iota\sum\theta^iJ^i\right)^\rho_\sigma \tag{31}$$

So lambda matrix if I drop, do not show the indices then it is just or maybe let me show it. Identity minus i, okay before I do that let me do something more. So let me define omega 01, omega 02, omega 03 as beta 1, beta 2 and beta 3. Okay, these are just new names for the parameters I am giving. And also define omega 23 as theta 1. Omega 31 as theta 2. Omega 12 as theta 3. Okay, just renaming things.

Then I get with these new names the lambda matrix rho sigma to be identity minus i, okay half I have already taken care of in this. Beta i K i. This is a summation over i from 1 to 3 minus theta i J i, okay. Again this is summation from 1 to 3, rho sigma, okay. So this is the most general form of any Lorentz transformation, which belongs to the proper orthochronous part of all Lorentz transformations, okay.

And any other Lorentz transformation can be built by multiplying this with a parity operator or time reversal operator or the product of those, okay. And these are right now all betas and thetas are infinitesimal parameters, okay. So you will be able to identify these theta i's with the angle of rotation. You should make sure that you understand. And beta i's with the boosts. So this will be related to the velocities. So find out the expressions of beta i's in terms of velocities, okay. Good, now another exercise.

Show that if you take a commutator of J i and J j, you get the following. And if you take a commutator of J i and K j, okay for example J 1 and K 2 etc., you get i epsilon ijk K k is summation over k here in both these expressions, so i and j are only the free indices. And then K i with K j is –i. So that minus sign is very crucial. That is going to be very useful at some point of time for us.

Not sure whether I am going to do that in this course. But yeah, these are again the commutation relations. The same commutation relations, which we have already written down before but now in terms of these generators. And I am sure you are already familiar with this one. This one you have seen before in your quantum mechanics course, okay. And these two ones are new for you.

Where β^i and θ^i are infinitesimal

Execrise: Show that

$$\begin{aligned} [J^i, J^j] &= \iota\epsilon^{ijk} J^k \\ [J^i, K^j] &= \iota\epsilon^{ijk} K^k \\ [K^i, k^j] &= -\iota\epsilon^{ijk} J^k \end{aligned} \tag{32}$$

And we will talk more about these commutation relations in another video. But of course, these are going to be again very important commutation relations, okay. So we will stop this video here. Please do all the exercises. And I hope this was not difficult. I have done it in great detail. So should be easy. Okay. So see you in the next video.