Introduction to Quantum Field Theory

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Lecture 17 : Lorentz Transformations

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1 Lorentz transformation

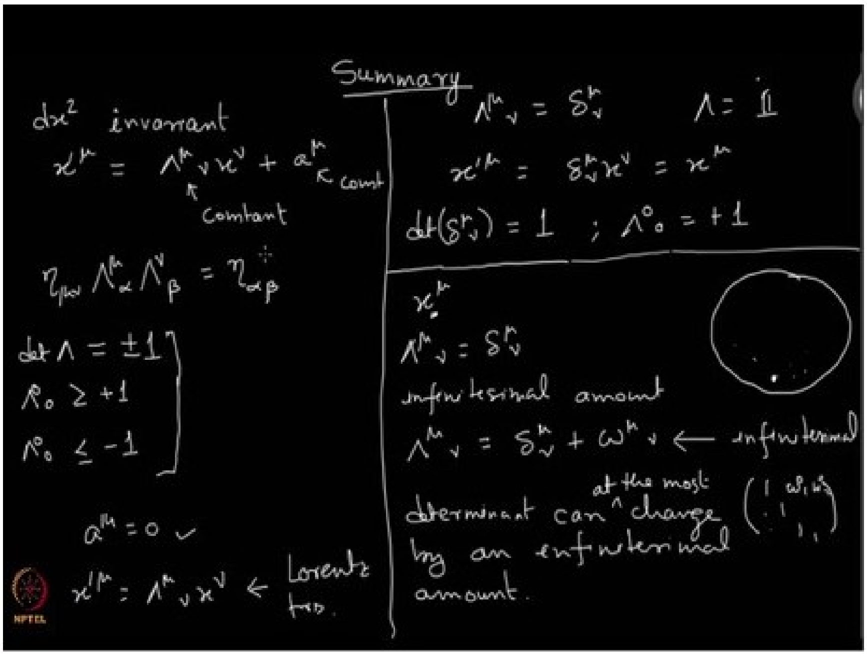

Figure 1: Refer Slide Time: 00:22

Okay, let us start by summarizing what we were doing last time. We started looking at all the transformations which keep the dx square invariant, okay. And then we found that the most general solution is that if x mu transforms in the following manner, if x mu goes to lambda mu nu x nu plus a mu where a mu where a mu's are constants, okay? This forms the most general, this is the most general transformation which keeps dx square invariant, okay.

And let me write down, these are constants. And we also at least I told you that these are constant matrices, you can derive it. So these are also constant. By constant I mean that this does not depend on x. You see you could have a transformation where this matrix also depends on x. But what we have told is that the most general transformation is that keep this invariant as lambda mu nu a constant.

Okay, and then we also saw that the constraint on this matrix was the following that if you take eta mu nu lambda mu alpha lambda nu beta and it should be again eta alpha beta okay.

So that is the constraint that you have on these matrices. And from these constraints, we could derive some simple observations that determinant of this matrix lambda can either take value +1 or -1, okay.

These are the only two possibilities. And we also found that lambda 00 is either greater than or equal to +1 or it is less than or equal to -1. These were some observations that we had made. Okay. Now I am not sure whether I told this last time. Yeah, I did not. So let me say this time. Now that is good, we are going to make use of these observations, okay. Okay, so now we are talking about new things today.

If you look at the matrix lambda mu nu equal to delta mu nu. Let us look at this matrix, which is just saying that the matrix lambda has, is a identity matrix, okay, all the entries are 1. So mu is equal to nu. This is 1 else it is 0. Is that a transformation which falls under this category? And the answer is yes. By the way, I should also remind you that we are putting a mu equal to 0, okay.

So all the transformations here with a mu equal to 0 are called Lorentz transformations, okay. So we are looking at these kinds of transformations, okay? a mu has been set to 0. We will come to a mu non-zero later. Okay, so these are Lorentz transformations. When a mu is not equal to 0, then we do not say that these are Lorentz transformations, okay? It is only said for a mu equal to 0.

Okay, so let us come back here. Let us take this matrix. And clearly this if you do a transformation by this matrix, then your x prime mu is just delta mu nu x nu which is x mu. Of course, you have multiplied an identity matrix. So you expect no change, which means that after transformation the vector remains the same, the four vector. And of course, then the x square or dx square whatever you wish to see, it remains invariant, okay.

Summary dx^2 is invariant under Lorentz transformation,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{1}$$

Where a^{μ} is a constant

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta} \tag{2}$$

Deriving few constraints

$$\begin{aligned} \det \Lambda &= \pm 1 \\ \Lambda^{0}{}_{0} &\geq +1 \\ \Lambda^{0}{}_{0} &\leq -1 \end{aligned} \tag{3}$$

Okay, so that is indeed an element in the set of transformations that we are considering. And what is the determinant of this matrix lambda, which we are looking here that we put it this way. Okay, this is just a way of writing. So clearly it is 1 because all the entries are 1 on the diagonal. So determinant is +1. And how about lambda 00? That is also +1 for this matrix, right? Because 00 element is 1. Okay.

So identity matrix is a Lorentz transformation, okay? And that transformation means no transformation at all. So you are doing nothing. Okay. Now imagine that you are given some x mu, okay? And you think of no transformation. So you are sitting in all, let us say, all the points in here denote Lorentz transformations, okay. So I am looking at the set of all transformations, which are Lorentz transformation.

So each point is one such element. So I am saying that I am starting with a vector x mu. And I do no transformation. So I am sitting at this point which corresponds to identity matrix, okay.

So I am looking at lambda mu nu is equal to delta mu nu, okay. No transformation at all, okay. Now I change the matrix lambda mu nu, by small amount okay, by let us say an infinitesimal amount.

Meaning the entries of the matrix lambda mu nu, which is different from this one, so change by some small amount. So instead of being sitting at the identity, now you change by some amount omega mu nu. And by omega mu nu, I mean some infinitesimal numbers, okay. So these are, so you can imagine that you have 1, 1, 1, 1 and these all were 0 when it was identity.

You are putting some small number here, okay? So you are putting omega 01 here, omega 02 here, which are very small numbers, okay. You are just putting some number so that you are doing some infinitesimal transformation away from that entity. Okay, that is fine. Now when you do so what will happen to your determinant?

If you look at the determinant now and you calculate it, the determinant will was to begin with 1 because you had only diagonal entries, everything else was 0. But now when you have these other entries also which are infinitesimal, when you calculate the determinant, it will at most change by an infinitesimal amount.

Because you are changing all the elements by infinitesimal amounts, your determinant can change at the most by some infinitesimal amount, okay. So let me write that, at most, at the most change by an infinitesimal amount, okay. So but then you realize that you have already found the condition that lambda mu nu can be can have determinant either +1 or -1, okay.

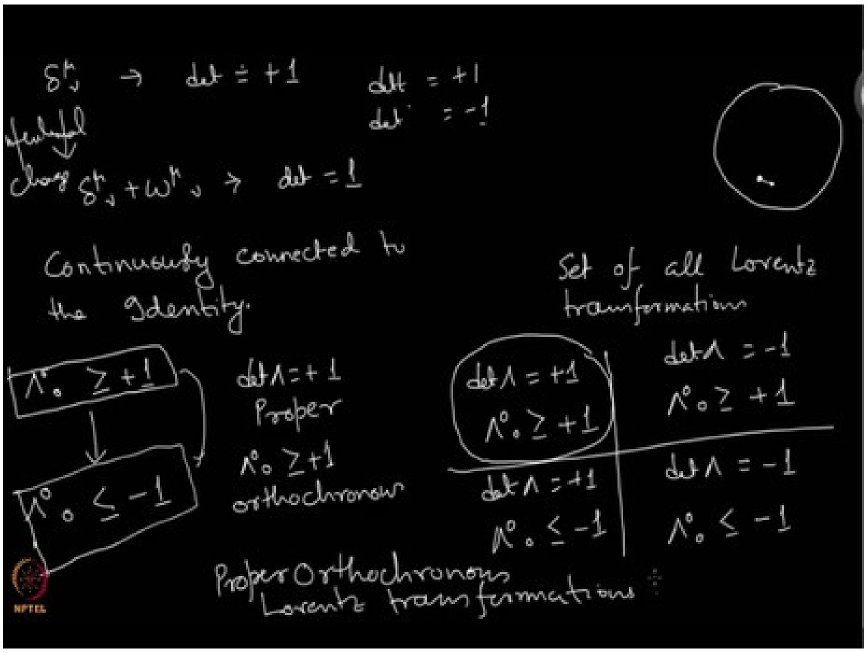

Figure 2: Refer Slide Time: 09:34

Let's look at

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} \quad ; \Lambda = 1 \tag{4}$$

For Lorentz transformation, we take $a^{\mu} = 0$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{5}$$
$$x'^{\mu} = \delta^{\mu}_{\nu} x^{\nu} = x^{\mu} \tag{6}$$

$$\det(\delta^\mu_\nu) \;=\; 1 \quad ; \; \Lambda^0{}_0 \;=\; +1 \tag{7}$$

Thus identity matrix is a Lorentz transformation.

Change of infinitesimal amount

$$\Lambda^\mu{}_\nu \;=\; \delta^\mu_\nu + \omega^\mu{}_\nu \quad ; \quad \omega^\mu{}_\nu \to \text{infinetessimal} \tag{8}$$

Now if you start with the identity element, if you start with the identity Lorentz transformation, its determinant is +1. If you do an infinitesimal change, and you reach at this, your determinant can change at most by infinitesimal amount, but the constraint says that it is has to have determinant +1, okay. Then it means the determinant will not change at all, right?

Because it, there are only two possibilities, right? It can go to either +1 or -1 okay. Now under an infinitesimal change it cannot jump to -1 because -1 is not an infinitesimal change from +1. It is a finite jump. So that is not allowed. What is allowed is you stay put a determinant +1, okay. So under these transformations your determinant will still remain the same, okay.

So as you make another, so if you go to another element in this set so you start with identity, you go to another element close by, by a close by I mean that these parameters you change by small amounts. You see that the determinant stays 1. If you make another change by a small amount you reach here. Determinant still has to stay at 1 for the same reason.

So as you keep moving continuously away from the identity you always have your determinant fixed at +1. You cannot make a jump to determinant equal to -1 because that will be a finite jump, okay. So we say that these matrices lambda mu nu which have, I mean the ones which we are talking about right now, they are continuously connected to the identity, okay.

So that is good and similarly, we can say things about elements of a Lorentz transformations that have lambda 00 equal to +1, not +1. Again, let us go here. Let us again start with the identity element and identity element has the 00 component as +1 and if you do a infinitesimal transformation okay, your lambda 00 component can at most change by an infinitesimal amount, okay.

So you these entries become slightly different from 1. So as you keep doing, so your entries keep becoming slightly different from 1, okay. So all these are continuously connected when you make the Lorentz transformations. But see that you will by doing so you are never going to make a jump to a Lorentz transformation which has lambda 00 less than equal to -1 okay, understand.

So as I keep doing Lorentz transformations which have, let me say again, as I start from identity, I make a small change in the parameters. At the most I can have lambda 00 change by a small amount. Which means that if I start from here which is the case for identity, I am not going to make a jump to such elements. It is never going to happen that you go from here to there.

And this one has lambda 00 less than -1. Because that is again a finite jump. So from +1 to -1 it is a finite jump. So all the elements which you reach by changing the parameters continuously will always have this condition satisfied if you started with lambda 00 greater than 1 value or if you had started with a lambda 00 less than -1.

Away from identity

$$\omega^\mu{}_\nu = \begin{pmatrix} 1 & \omega_1{}^0 & \omega_2{}^0 & \omega_3{}^0 \\ \omega_0{}^1 & 1 & \omega_2{}^1 & \omega_3{}^1 \\ \omega_0{}^2 & \omega_1{}^2 & 1 & \omega_3{}^2 \\ \omega_0{}^3 & \omega_1{}^3 & \omega_2{}^3 & 1 \end{pmatrix} \tag{9}$$

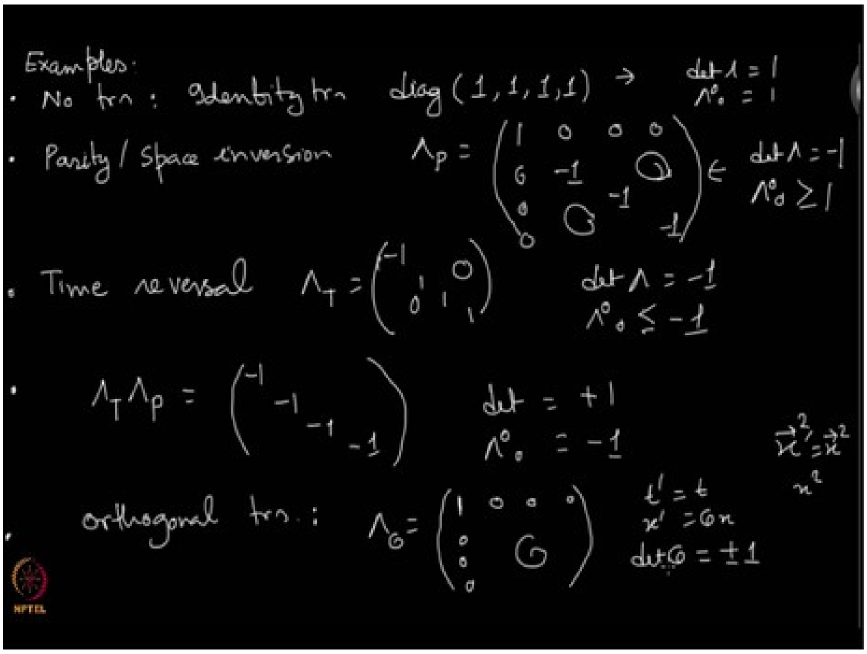

Figure 3: Refer Slide Time: 18:34

Determinant can change at the most by an infinitesimal amount, for $\delta^{\mu}_{\nu} \rightarrow \det = +1$, thus the determinant will remain same. Continuously connected to the identity

And if you again do continuous transformations, okay you change the parameters continuously, you will stay put in this part. You will always have lambda 00 to be negative, okay. Which means that I cannot by doing continuous change of parameters of Lorentz transformation, I cannot jump from determinant +1 to determinant -1 nor I can jump from this part to that part, okay.

So our entire set of transformation, so if I look at set of all Lorentz transformations okay, we see that they naturally fall apart into four kinds. One for which determinant lambda is +1 and lambda 00 is greater than or equal to +1. Second which has determinant lambda is equal to -1 and lambda 00 element is greater than equal to +1. I cannot jump from, I cannot go continuously from this to this region that cannot happen as I argued before, okay.

Again I have determinant lambda is equal to +1 lambda 00 less than equal to -1. I cannot come from here to there also, that is not possible because even though determinant lambda is +1 in both the cases, so as far as determinant lambda is concerned, that puts no restriction in coming from there to here. But the fact that lambda 00 is negative here and positive there okay, puts the restriction that you cannot, these two parts are not continuously connected.

And the final one is determinant lambda equal to -1 and lambda 00 is less than equal to -1. So the same thing here, even though determinant lambda is -1, I cannot come, these parts are not continuously connected, okay. And the ones, let me give some names. When determinant lambda is +1, we call proper transformations, okay. When lambda 00 is positive we call orthochronous transformations, okay.

So this is proper these ones are proper orthochronous transformations. So this first block, proper orthochronous Lorentz transformations, okay. That is good and you cannot go from one part to another in a continuous manner. Let me give you some examples of Lorentz transformations that are here, okay. So let us do some examples

$$\det \Lambda = +1 \to \text{Proper transformation} \tag{10}$$
$$\Lambda^0{}_0 = +1 \to \text{Orthochronous transformation} \tag{11}$$

1.1 Examples of Lorentz transformations

One of course is no transformation, which is what we call identity transformation. And that one is just let me put it this way, I just say that the entries are 111. So I am saying diagonal entries and they are all +1, okay. That is a identity transformation. Then you can clearly see that parity, I think I already talked about this or space inversion. If you look at this the lambda matrix, let me put a subscript p here, that is 1.

So I am not doing anything to the time component. And I am just changing all the space components to minus of that. So these are all 0 here. These are all 0 here. So let me put 0 here and 0 here, okay. And for this one of course, as I talked, determinant lambda is +1 and lambda 00 is +1. For this one determinant lamda is +1 sorry -1 and lambda 00 is, okay? So this one belongs to this block, okay.

Then you can look at time reversal which is let me call lambda T, which is you just change the time component, so all other things are 0, okay. So this will change T to -T and all other coordinates will remain the same. And of course, this one also has determinant lambda equal to -1 just like this one. But lambda 00 is negative or less than -1. Now if you look at a parity followed by time reversal.

So if you let us say first do a transformation, which is a parity transformation followed by a time reversal that is just multiplying these two matrices and which gives you all the diagonal entries as -1 and the determinant is now +1, but lambda 00 is -1 okay. So apart from this identity all the other examples that I have given to you they belong to either this or this or that, okay.

None of them belongs to this part. So if you have a parity or time reversal, you have either a improper transformation or a non orthochronous transformation. You do not have a proper orthochronous transformations, okay. That is good. Another example I will give you. Think of an orthogonal transformation okay, in the three dimensional part of the space without touching the time.

It is the second, let me close the window. It is raining heavily. Okay, so let us look at this. We should also ask whether that is allowed Lorentz transformation. Let us check. So I am saying look at this one. Then you put a O here. So I do not change the time, I keep it same. So that does not get transformed. And here is an orthogonal matrix, you can put an orthogonal matrix here, okay.

So basically your t prime is same as t. And your x prime is Ox, okay. Now this will of course, satisfy your constraints. So that is something you should check that this constraint is satisfied or dx square is satisfied, dx square is invariant under this, okay? That is kind of obvious because your x square will be x prime square.

I am just looking at the three vector under orthogonal transformations and that is why your x square will also remain invariant. But here you have two varieties. Determinant O is allowed to be either +1 or -1, okay. So you see again, when determinant O is +1, then these matrices also fall into the category of proper orthochronous transformations. But if determinant O is -1, then it belongs to improper one. And you see why you get a -1. And you get because you have minus minus minus here. You have a parity involved in those orthogonal transformations, okay. So these are some of the examples to see what is happening in these transformations. Now let me give you a set of nice exercises which will help you to appreciate better what kind of transformation matrices are in this entire set of Lorentz transformations. So exercises. So first you show that any Lorentz transformation. Okay, let me first write down and then show that any Lorentz transformation is either a proper orthochronous transformation, meaning either it is one of the element here or can be written as product of proper orthochronous transformation and parity transformation that we call p, okay.

Or again can be written as a product of, let we write down, can be written as a product of proper orthochronous and time reversal. Or can be written as again proper orthochronous and productive of time reversal and parity, okay. So what I am saying is if you look at this entire set, the matrices either belong to this.

Or if not, then you can write them that lambda whatever that lambda is as a product of an element in this set times a parity operator or lambda times a time reversal operator or lambda times parity times time reversal. And that will exhaust all the possibilities which are listed here. All the matrices which you can find here, they all are covered by what I have listed down here. Okay, that is a nice exercise to do and that tells a lot of things.

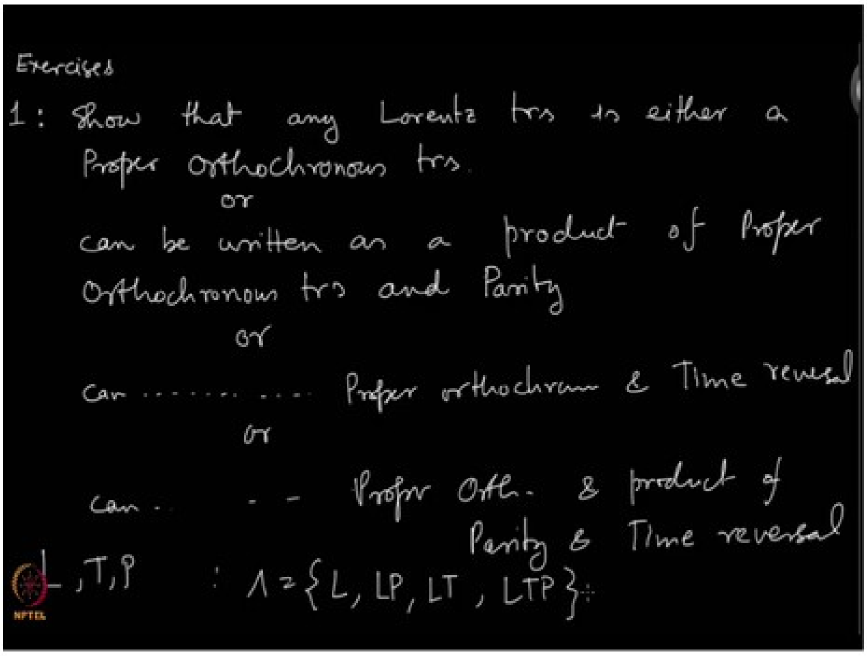

Figure 4: Refer Slide Time: 25:17

- Identity transformation i.e. no transformation

$$\det\Lambda = 1 \quad ; \quad \Lambda^0{}_0 = +1 \quad ; \quad \Lambda = \mathrm{diag}(1,1,1,1) \tag{12}$$

- Parity transformation / Space inversion (Λ_i)

$$\Lambda_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13}$$

$$\det\Lambda = -1 \quad ; \quad \Lambda^0{}_0 \geq +1 \tag{14}$$

• Time reversal(Λ_T)

$$\Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

$$\det \Lambda = -1 \quad ; \quad \Lambda^0{}_0 \leq -1 \tag{16}$$

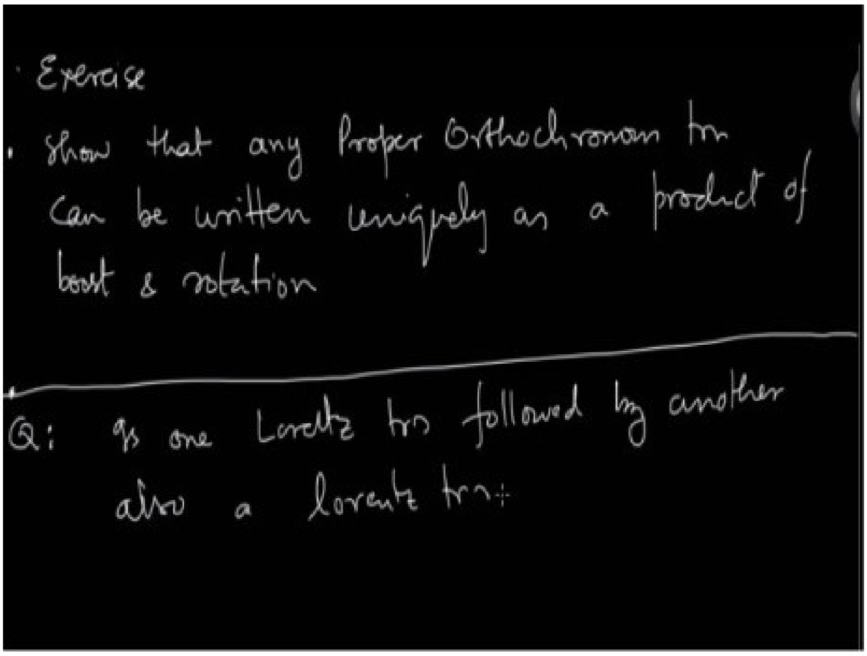

Figure 5: Refer Slide Time: 29:14

• Parity transformation followed by time reversal

$$\Lambda_T\Lambda_i = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{17}$$

$$\det \Lambda = +1 \quad ; \quad \Lambda^0{}_0 = -1 \tag{18}$$

• Orthogonal transformation

$$\Lambda_o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{19}$$

So we just write in notations. So if by L I indicate proper orthochronous transformations and if I by T and P, time reversal and parity, then what I am saying is any lambda is either L or T or P or lambda times T or there is no need to write this separately because of the identity element. So L or L times P or L times T or L times TP. This exhausts all the possibilities, okay. That will be something nice to show. Now next important exercise. Show that any proper orthochronous transformation okay, can be written uniquely as a product of boost and rotation, can be written uniquely as product of boost, boost you remember the transformations which take you from one inertial frame to another frame, boost and rotation, okay. Now this is a nice exercise because once you have done that this part which remains the non-trivial part, right?

Because everything here is just a, this times parity and time reversal. So this part has become easy if we know this part. And we are saying about this part that every element in here is either a boost or a rotation or product of these, okay. And boost you know and rotations you know. So if you know those two, you know everything, all elements in here because that can be uniquely expressed as a product of these, okay.

So product a boost and rotation and of course, pure boost is just a product of boost times rotation, which is identity matrix. So everything is included here. Okay, that is another important exercise to be done. Okay, that is good. So just a second. Yeah. So you see it really boils down to knowing boosts, rotations, parity and time reversal, okay. Beyond this, there is nothing else in this set of transformations, okay.

That is the main lesson you learn from doing these exercises. Now you can ask if you do one Lorentz transformation followed by another, is that also a Lorentz transformation? We should show that indeed it is a Lorentz transformation and because that is so easy, I will do it. So what we are asking is, let me write the question here.

Is one Lorentz transformation followed by another also a Lorentz transformation, okay. So let me show that to you that indeed the answer is yes.

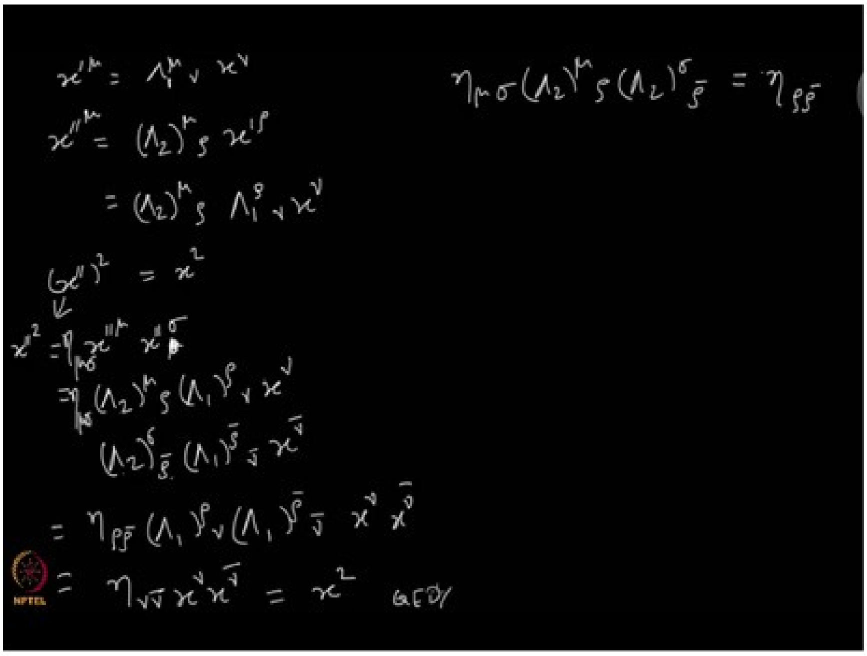

Figure 6: Refer Slide Time: 32:34

$$t' = t \quad ; \qquad x' = \hat{o}x \tag{20}$$

$$dx^2 \text{ is invariant} \quad ; \qquad \det \hat{o} = \pm 1 \tag{21}$$

Exercise2:

Show that any Lorentz transformation is either a proper orthochronous transformation(L). OR Can be written as a product of proper orthochronous transformation(L) and parity(P).

OR Can be written as a product of proper orthochronous transformation(L) and time reversal(T).
OR Can be written as a product of proper orthochronous transformation(L), parity(P) and time reversal(T).

$$\Lambda = \{L, LP, LT, LTP\} \tag{22}$$

Exercise:

Show that any proper orthochronous transformation can be written uniquely as a product of boost and rotation.

Question: Is one Lorentz transformation followed by another Lorentz transformation is also a Lorentz transformation.

So take a Lorentz transformation which is just defined by x nu. Okay, let me put 1 here. So the matrix, lambda 1 is the matrix and mu nu are the indices and then I do another Lorentz transformation. Means what? I take x prime mu, and transform it to, transform it by lambda 2, some new Lorentz transformation lambda 2. So which means your x double prime will be let us put an index mu here.

It will be lambda two okay, mu rho x prime. No need to put the bracket x prime rho. Okay, the same like here, it is just lambda 1 lambda 2. And that you know is lambda 2 mu rho. What is this quantity? That is lambda 1 I am putting from here, rho so I should put rho, and maybe nu, nu is fine, x nu, okay. This is fine. Now what I want to do is I want to check whether x double prime square is same as x square, okay.

I know that x square is same as x prime square because it is allowing some omission. But now if I have done two transformations one after the other, is this still the same as x prime square and which is same as x square? Is this true? If this is true, then indeed, this is a Lorentz transformation and let us check.

So what we are asking is let us write down x double prime square which is this thing x double prime mu x double prime mu, okay. So this is this part, which is lambda 2 mu rho lambda 1 rho nu. And then you have x nu. Let me write yeah x nu. Okay I want to do it slightly differently. So this is fine, up to here it is fine. Instead of putting this index here let me write, what should I write?

Yeah rho I have used, nu I have used. Let me put a sigma, mu eta, mu sigma. Okay, so instead of writing x up with a mu up and mu down I have put a sigma when contracted with eta mu nu. So here I should put eta mu sigma, okay? So this is this part I have written. Let me write down this one which is same as what I have written above with a sigma and let me call rho bar lambda 1 rho bar nu bar x nu bar, okay.

Now this is easy. If you look at this eta mu sigma lambda 2 mu rho lambda 2 sigma rho bar. Okay, let me write down here. Eta mu sigma lambda 2 mu rho lambda 2 sigma rho bar. You remember what is this? This is exactly what the constraint you have for lambda 2 to be a Lorentz transformation that this should be equal to eta, rho, rho bar. Remember that constraint equation which we have written several times.

Here, you see the mu and the nu are contracted with the upper indices and alpha beta is the lower which is same as the index on the right side and that is what we have here. So I am saying that this has to be this much for lambda 2 to be Lorentz transformation. So these three terms I can replace by eta, rho, rho bar and then we are left with these pieces. Let me write down lambda 1 rho nu, lambda 1 rho bar nu bar, x nu, x nu bar, okay.

But then lambda 1 is a Lorentz transformation. And again I can use the same condition here. You see the rho and rho bars are contracted with the upper indices. Nu and nu bars are the ones which are below. So which means that I should write this as this. And clearly this is just your x square. So I have shown that one Lorentz transformation followed by another Lorentz transformation is also a Lorentz transformation. Okay, now I will give you another very nice exercise which is this. Check that if L 1 and L 2 are proper orthochronous Lorentz transformations, okay. Then L 1 times L 2 or L 2 times L 1 they are also orthochronous proper orthochronous Lorentz transformation, proper orthochronous Lorentz transformation. Sorry for being lazy to write this down again and again. Okay, now you see this is non-trivial.

Here what I have shown is if you take one Lorentz transformation followed by another Lorentz transformation, it is a Lorentz transformation. But now this exercise is something even more stronger. It says if you do a Lorentz transformation L 1, which belongs to this followed by another L 2, which also belongs to this, the product will remain within this. It is not going to jump out to another one, okay.

And that you should do. That should be easy to check because all you have to do is, you see you have shown already that one Lorentz transformation followed by another Lorentz transformation is a Lorentz transformation. Which means, if you do a L 1 and L 2, multiply here, you will be either here or here or here or here, okay. Because that is a full set.

Now what you have to show is that when you do so you still get, you still remain in here meaning the determinant of L 1 times L 2 is still 1 and if you look at the product L 1, L 2 and look at the 00 component it is still greater than 1. If you show that then you have proved, you have done the exercise, okay. Okay. So that is good. Now let me make some remarks.

So we know that rotations and boosts they are you know the transformations which keep physical laws unchanged okay, that the dynamics is unchanged. That is known. That is known through enormous number of experiments and the basic understanding is that the space is isotropic. So you have rotational invariance, okay. And your boosts because different inertial frames are equivalent, okay.

Which is equivalent to saying that proper orthochronous transformations because that is all there is in this part of the Lorentz transformation. That is proper orthochronous Lorentz transformations, okay, leave the laws of physics unchanged. That is about this part. But then we started looking at a bigger class of transformations which leaves dx square is invariant. You see, boosts and rotations leave these invariant.

That is good. But then we went ahead and asked for all the transformations that leave this invariant. And we found that there are many more and this is the entire set. And basically, it is two more parity and time reversal, because you can just multiply and get everything else. But there is no reason for us to believe that parity and time reversal will also be the symmetries of nature that all the physical laws are invariant under time reversal.

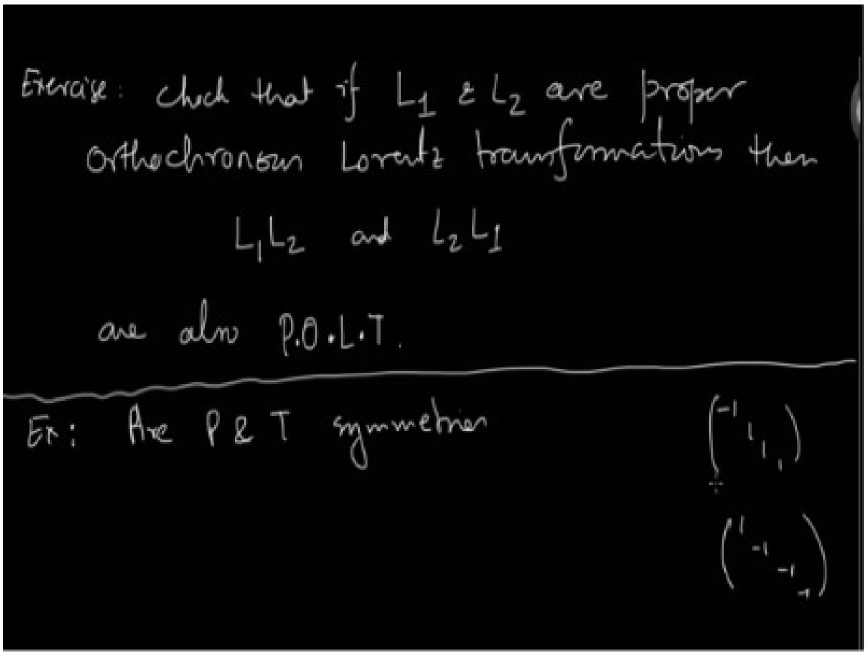

Figure 7: Refer Slide Time: 38:26

$$
\begin{aligned}
x'^{\mu} &= \Lambda_1^{\mu}{}_{\nu}\, x^{\nu} && (23)\\
x''^{\mu} &= (\Lambda_2)^{\mu}{}_{\rho}\, x'^{\rho} && (24)\\
x''^{\mu} &= (\Lambda_2)^{\mu}{}_{\rho}\Lambda_1^{\mu}{}_{\nu}\, x^{\nu} && (25)
\end{aligned}
$$

For Lorentz transformation,

$$(x'')^2 = x^2 \tag{26}$$

That is not automatic, just because we have asked for all the transformations which leave x square invariant or dx square invariant, okay. So one has to find from experiments, whether time reversal and parity are indeed the symmetry transformations under which physical laws remain unchanged, okay. So I leave it to you to search through literature and figure out whether parity and time reversal are symmetries of nature, okay.

So that is another exercise. Are P and T symmetries of nature, okay? I hope the point I wanted to make was that it is not guaranteed that they have to be, okay. If they are, good; if they are not, that is also good. And I encourage you to figure that out. Okay, let me see what else. Okay. Okay, so let me proceed.

So now that we have found that it is really rotations and boosts that we need to study because parity and time reversal are anyway simple right. It is, you know what they are, they look like this. That is easy. There is not much to be done here. Okay, so these are easy. We want to focus now on only the proper orthochronous Lorentz transformations, okay. If we know them well we know the entire transformations.

$$
\begin{aligned}
x''^2 &= \eta_{\mu\sigma} x''^{\mu} x''^{\sigma} && (27)\\
x''^2 &= \eta_{\mu\sigma}(\Lambda_2)^{\mu}{}_{\rho}\,(\Lambda_1)^{\rho}{}_{\nu}\, x^{\nu}\,(\Lambda_2)^{\sigma}{}_{\bar{\rho}}(\Lambda_1)^{\bar{\rho}}{}_{\bar{\nu}}\, x^{\bar{\nu}} && (28)
\end{aligned}
$$

$$\eta_{\mu\sigma}(\Lambda_2)^{\mu}{}_{\rho}(\Lambda_2)^{\sigma}{}_{\bar{\rho}} = \eta_{\rho\bar{\rho}} \tag{29}$$

which will become

$$\begin{aligned} x''^2 &= \eta_{\rho\bar{\rho}}\,(\Lambda_1)^{\rho}{}_{\nu}(\Lambda_1)^{\bar{\rho}}{}_{\bar{\nu}}\,x^{\nu}x^{\bar{\nu}} && (30)\\ x''^2 &= \eta_{\nu\bar{\nu}}\,x^{\nu}x^{\bar{\nu}} && (31)\\ x''^2 &= x^2 && (32) \end{aligned}$$

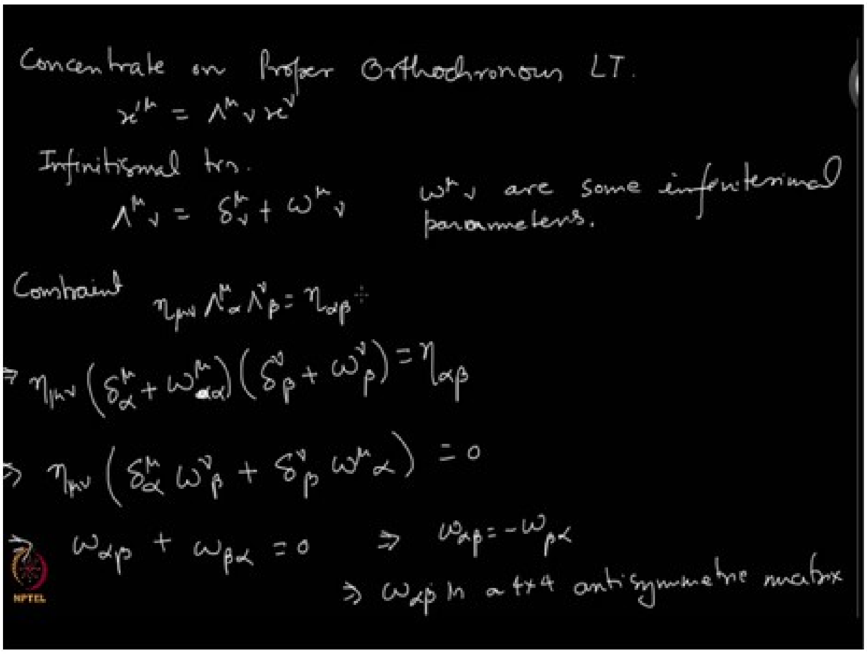

Figure 8: Refer Slide Time: 45:07

Exercise: Check that if L_1 and L_2 are proper orthochronous Lorentz transformation then L_1L_2 and L_2L_1 are also proper orthochronous Lorentz transformation.
Question: Are P and T symmetries?

Okay, so let us concentrate on that proper Lorentz transformations, okay. So let us write down this equation again that that is the transformation law, okay. I have put m equal to 0 and because that does not belong to Lorentz transformations. You do not call those as Lorentz transformations. This is here. Now think of an infinitesimal transformations, okay.

Now I know that lambda mu nu, when there is no transformation, then you are sitting at the identity matrix. If you do a little bit then you change the parameters a little bit by infinitesimal amounts and let me write it down as omega mu nu, okay, the infinitesimal changes. So omega mu nu are some infinitesimal parameters, okay.

But these cannot be chosen arbitrarily, you cannot put whatever numbers you like in this place, because you remember, there is a constraint right? And the constraint was here. This constraint needs to be satisfied. So let us see what this constraint implies for our these elements, okay. So let us write the constraint we have. Constraint was that if you take eta mu nu, do I have enough space?

Yeah, eta mu nu, lambda mu alpha, lambda nu beta, then this is again alpha beta, okay. So that is the constraint I am going to impose on here. So let us write this down eta mu nu and then you have this delta mu. I would encourage you to parallelly do the computition while I am doing it, and then match the results, okay. That way it is better. So lambda mu alpha.

Okay, lambda mu alpha plus omega mu alpha, then the second lambda is delta nu beta plus omega, nu beta. And that should turn out to be eta alpha beta. That is what we want. Okay, let us check. So they are in total four terms. Two times two is four. One term comes from multiplying the deltas, okay. Then two terms come from multiplying one omega with one delta.

So one either this omega with delta and the other is this omega with the delta. And the fourth term comes by multiplying these two omega squares two omegas. But that is a term which is of omega square. See omega is already small. So I will drop the omega square terms and we will keep only the three terms. Now here you have eta mu nu, delta mu alpha, delta nu beta.

$$x^{'\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{33}$$

For infinitesimal transformation,

$$\Lambda^{\mu}{}_{\nu} \;=\; \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \quad ; \quad \omega^{\mu}{}_{\nu} \rightarrow \text{infinetesimal parameters} \tag{34}$$

- Constraints

So this will force mu to become alpha and this will force nu to become beta. And you will get eta alpha beta, okay. And that eta alpha beta is on the right hand side. So I can cancel, so that is gone. Now let us look at the terms which are linear in omega, quadratic in omega we are leaving out. So linear omega will be this. This eta will force the mu. Sorry I should have been more careful.

This should be the second index and this should be the second index, okay. So let me write it more carefully now. So it will be omega nu alpha plus omega nu is forced to become mu here. Okay, I think, what did I do? Let me go slowly. Got confused. So here that part alpha beta is gone. So let us write eta mu nu and then we have this piece multiplying omega here.

Delta mu alpha, omega nu beta plus this time stays which is delta nuw beta, omega mu alpha is equal to 0, because I have cancelled that already. So this mu is contracted, so you have a new alpha. Eta nu alpha, but nu is also contracted. So it gets pulled down and becomes omega alpha beta plus here mu is pulled down to nu okay.

And then you get omega beta alpha, okay, which means that the omega, that the matrix omega, you see omega is a matrix right because these are all matrices, is a anti-symmetric matrix. So omega alpha beta is equal to minus omega beta alpha. So this implies that omega alpha beta is a 4 cross 4 anti-symmetric matrix. So that is the constraint that you have found by imposing this constraint which Lorentz transformations have to satisfy.

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} \;=\; \eta_{\alpha\beta} \tag{35}$$
$$\eta_{\mu\nu}(\delta^{\mu}_{\alpha} + \omega^{\mu}{}_{\alpha})(\delta^{\nu}_{\beta} + \omega^{\nu}{}_{\beta}) \;=\; \eta_{\alpha\beta} \tag{36}$$
$$\eta_{\mu\nu}(\delta^{\mu}_{\alpha}\,\omega^{\nu}{}_{\beta})(\delta^{\nu}_{\beta}\,\omega^{\mu}{}_{\alpha}) \;=\; 0 \tag{37}$$

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} \;=\; 0 \tag{38}$$
$$\omega_{\alpha\beta} \;=\; -\omega_{\beta\alpha} \tag{39}$$

A 4x4 anti-symmetric matrix, thus the constraint on $\omega^{\mu}{}_{\nu}$ is

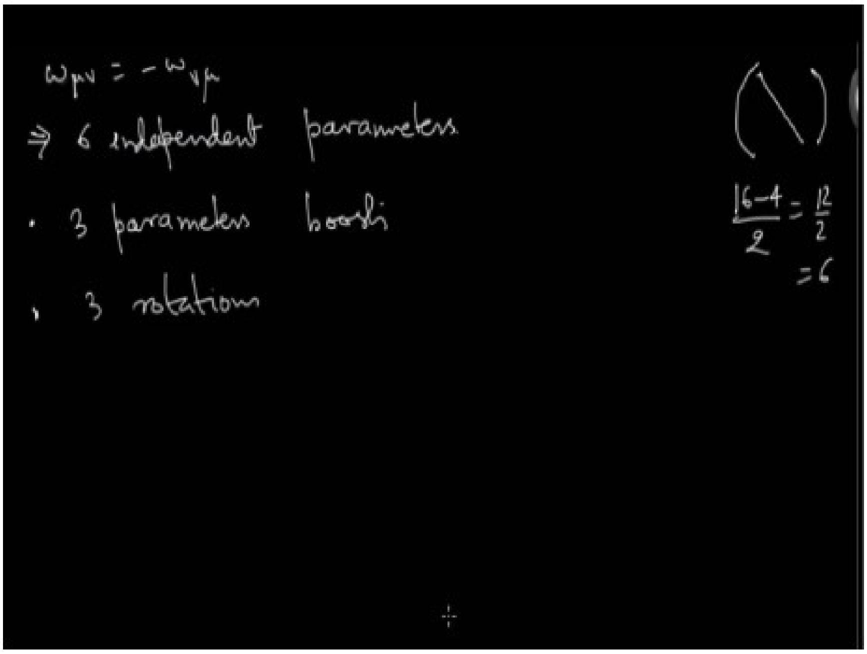

Figure 9: Refer Slide Time: 51:58

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{40}$$

6 independent parameters.

- 3 parameters for boost
- 3 parameters for rotations

So we have seen that okay so that these parameters have to satisfy this relation. Now omega is a 4 cross 4 matrix. So you have a total of now 16 entries. But because of anti-symmetry these four diagonal entries will be 0. So 16 - 4 and then you divide by 2 so that you get the upper half, I mean this corner and the other one is related by symmetry so you divide by 2. So 12 over 2, which is 6.

So you get 6 independent parameters which means that the proper orthochronous Lorentz transformations are parameterized by 6 independent parameters. And is that something expected? I guess it is because, you know I think one of the exercises that I gave to you is that, let me check. Show that any proper orthochronous transformation can be written uniquely as a product of boost and rotation.

So how many boosts do you have? You have 3 boosts okay and correspondingly three parameters coming from boost. Okay, you can boost by some velocity in x direction, some velocity in y, some velocity in z. So your three parameters for boosts and three for rotations. So you see, this adds up to 6 and that is why that is the reason why you got 6 independent parameters.

Okay, let us go further. Or maybe I will do in the next video. Okay. Yeah, I think that is a good place to stop here. Okay, we will stop here