Introduction to Quantum Field Theory

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Lecture 16 : Theory of Scalar Fields

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1 Lorentz transformation

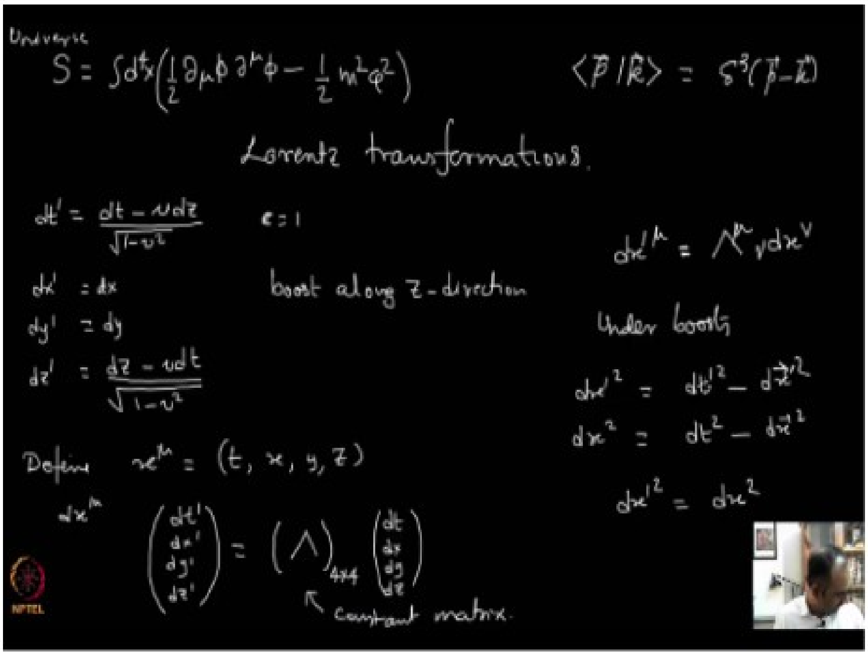


Figure 1: Refer Slide Time: 00:01:41

So, we will start a new chapter in this course and we will start discussing about symmetries now. Symmetries are at the core of the thought process which we use in doing modern physics at fundamental level today and if under some transformation, laws of physics remain unchanged then that transformation is called a symmetry.

And the entire chapter the entire discussion which we will be doing for next few lectures will be about laws under which or the transformations under which the laws of physics remain unchanged, or equivalently we will be discussing about the symmetries. Till now, in this course, we have been talking about a universe which was quite boring. So, all we had in our universe was just real scalar fields; real Klein-Gordon fields.

And there was no interaction at all because it just had particles which are free, let me go to the. So, we had this action that is what we had and we saw the states which we found, single particle states. If you look at this inner product, if you take some other state P and this state k it will be just proportional to delta cube of p – k. And similarly, if you take some other state which has more than one particles, you will just find delta corresponding delta functions.

So, a state with 2 particles goes again to state of 2 particles and the momenta of these particles remain the same. So, if you have 2 particles in momentum p 1 and p 2 and if you look at the inner product of this state with another state which has momentum k 1 and k 2, then you will see that the k 1 and k 2 have to have the momentum p 1 and p 2 and permutations of this which means your state evolves to the same state.

I mean, it does not become, a 2 particle state does not become 3 particles state or a 4 particular state. So, there was and this was happening because there was no interaction at all. We will study about interactions later in this course. So, this world which you have constructed here, our universe right now, is this, because I am telling you the action of this universe and this is having only real scalar fields which do not talk to each other because there is no interaction term.

Now, if you were to proceed to make a universe which has interactions, you will have to include more terms in here for example, I could include a phi cube term or a phi four term and then we will start getting interactions. But should I be keeping in mind certain things before I construct such a universe? To be more specific, I know that translation is a symmetry.

So, I know that space is homogeneous and if I take my entire system, imagine we will doing classical mechanics. And we will looking at a system of particles, forgot quantum field theory for a moment. You know that if you take that entire system and you put it elsewhere, the dynamics which happens in the system will remain unchanged.

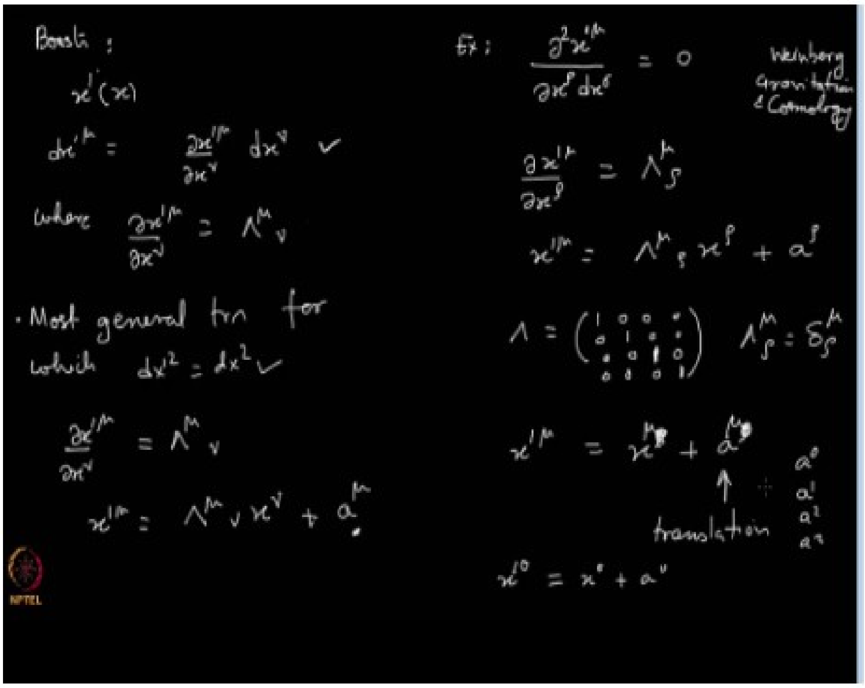
Or if you take the system, do whatever experiment you wish to do, get the results. And if you do the same experiment 1000 years later, you expect the same results. You expect so because of invariants under time translation. Now, these symmetries, symmetries because under this transformation of translation and in time or space, the laws are not going to change, everything is going to evolve in the similar fashion in the same fashion; the dynamics will be the same. These symmetries have consequences. Now, the consequence for these ones is that the energy is conserved because of invariance under time translations and momentum is conserved because of invariance under space translation. So, we have energy conservation and momentum conservation coming from space time translation invariance. Now, if I am going to make a quantum field theory, I would like to have energy momentum conservation built into my theory.

Now, how do I ensure that? It is easy to ensure because all I have to do is take my action and make sure that this action is invariant under space time translations. If I do so, then I am guaranteed that my theory will have automatically energy momentum conservation built into it. Now, the other symmetries which are known, so, we know that if I go from one frame of reference to another frame of reference that is called boost.

So, you go from one frame to another frame, the physics does not change that is what special relativity tells us. Now, I would like to ensure, this also holds true whatever model of universe I construct, meaning when I write an action, I would like to ensure in fact, I should ensure that that action is invariant under Lorentz transformations.

And so, all these new models of the universe that you would like to construct have to respect the known symmetries that already we know are there; space time symmetries like translation, rotation and boosts. These are known to be the symmetries of physical laws and we should ensure that they are encoded correctly when we write down the action.

So, let me see if I wanted to say something else, yes. So, that is one thing. Also symmetries are very useful because they will manifest themselves in the dynamics. So, when you see interactions happening, when you see this state evolving to that state, they will carry the signature of the symmetries that have been imposed on the action.



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$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{1}$$

Single particle state $|\vec{k}\rangle$

$$\langle \vec{p} | \vec{k} \rangle = \delta^3(\vec{p} - \vec{k}) \tag{2}$$

Action is lorentz invariant

So, that is the motivation for learning about symmetries. Now, we will talk more in about this later, but let us start afresh even though I have assumed to some degree that you are aware of Lorentz transformation, Lorentz symmetry. And I have used notation which assume that you know already so, you have the action which is in front of you is already written in a way, which is manifestly in variant and Lorentz transformations.

But I am going to begin it afresh, I will not assume anything, so that we record everything here at one place. And I will begin by a discussion of Lorentz transformations. So, you have already seen these things. So, if you take a time interval dt prime; in the prime frame and see how it is related to un-primed variables. So, v is the relative velocity of 2 frames of reference. And suppose the boost is only along this direction.

So, frame number and the prime frame is moving velocity v with respect to the un-primed frame and you know this is your where I put c is equal to 1 and these are prime is; let me not write like this because I am doing a boost only along x and y directions. These displacements will not change and of course, your dz prime will be dz – v dt. So, right now, I have a boost along z direction.

By boost, I just mean going from one frame to another frame. So, you can have a passive view that you go from one frame to another frame that is the coordinates are changing or you can have an active view where you take the particle and you boost this one; take the particle itself and boost or take the system and boost it which is same as leaving the system the way it is and going to another frame. These 2 are equivalent.

So, that is what I mean by boost. Now, I define x mu to be these 4 and raise t time and the spatial coordinates, then I see that the dx mu which is basically this dt prime dx prime dy prime dz prime is; you have to write this equations in matrix form. So, there is some matrix which I will call it lambda which you can determine from these equations and this is a full 4 cross for matrix of course.

And then you have your column vector. Note that the matrix lambda is a constant. So, this is a constant. The entries involve only things like 1 over root 1 – v square or v over root 1 – v square. So, this is a constant matrix and this transformation I will write as dx prime mu so, the prime differentials in the prime coordinates or the displacements in the prime coordinates are related by the solution, dx mu, this is un-primed and in for the matrix I have to put the row and column indices. I put the row index as a top index so that what you have on the left is; a free index is on the top. So, this is on the top but I define this matrix with the column index as a lower index. So, first index is up; second next index is low. And of course, there is a summation over all the mu's and there is a summation always within up and down indices. So, that is my dx prime mu.

Now, let me say that under boosts, we know the dx prime square which is dt prime square – t square remains unchanged. So, this is, let me write first this part, this is prime and we know that under Lorentz transformations, this is an invariant. This combination is an invariant combination.

Discussion on lorentz transformation

$$dt' = \frac{dt - vdz}{\sqrt{1-v^2}} \quad ; \quad c = 1 \tag{3}$$

$$dx' = dx \quad ; \quad dy' = dy \quad ; \quad dz' = \frac{dz - vdt}{\sqrt{1-v^2}} \tag{4}$$

Define,

$$x^\mu = (t, x, y, z) \tag{5}$$

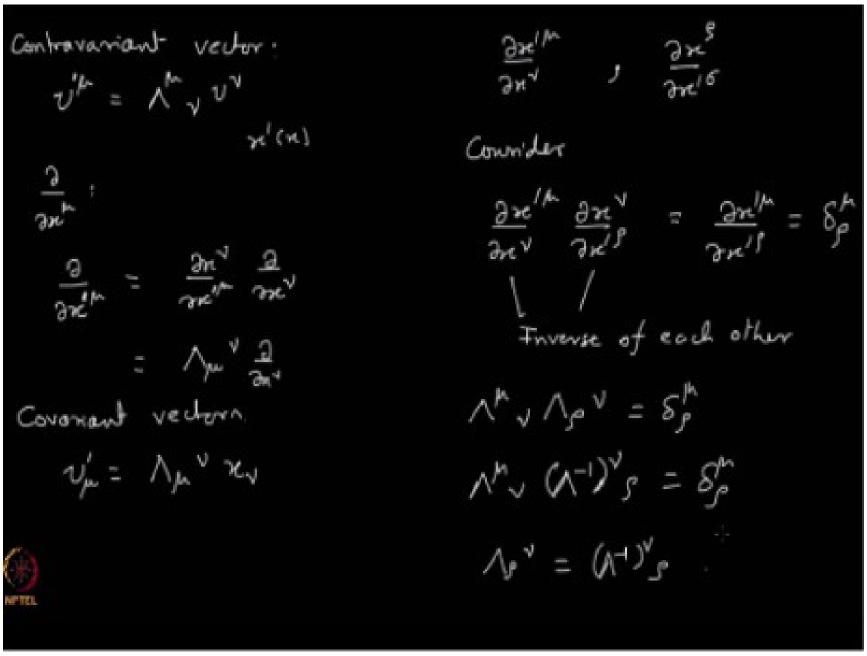
$$dx^\mu = \begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \Lambda_{4\times4} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix} \tag{6}$$

$\Lambda \to$ constant matrix, rows as top index

Under boost

$$dx'^\mu = \Lambda^\mu{}_\nu \, dx^\nu \tag{7}$$

$$\begin{aligned} dx'^2 &= dt'^2 - d\vec{x}'^2 && (8)\\ dx^2 &= dt^2 - d\vec{x}^2 && (9)\\ dx'^2 &= dx^2 && (10) \end{aligned}$$



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1.1 Boosts

Now, what are boosts? Our boosts are this; the x prime mu, let me leave it, x prime is a function of x. So, if I look at this differential dx prime mu, it is clear dx prime mu; by the chain rule I can write, it is del del over del x nu. Now, I am doing a derivative with respect to un-primed variable x and then I have to include this. Sorry, I do something silly, I have to write del x prime mu over del x nu dx nu versus the summation over mu is implied.

And this is exactly what transformation law is and as we have seen, this matrix you have here is the matrix lambda or lambda mu, where which remember is a constant matrix. For example, if you look at the element, lambda 0, up and 0 down here, both mu and nu 0, then it will be 1 over 1 – v square, there is no x, y or z involved in that. That is a constant matrix that is fine.

Now, let us ask the following question. Instead of looking at boosts, we ask what is the most general transformation of this kind, which leaves the interval invariant? So, if you look at dx prime square and dx square, they should remain unchanged. So, we are looking for those most general transformations for which this will be true. So, we are asking, what is the most general transformation for which dx prime square is dx square.

And we should not be surprised if you find transformations other than boosts, which leave this invariant. Now, what I will do is; I will take this expression. And then what we are basically asking is: what is the property of this matrix lambda? This is lambda because the transformation is going to be specified by this matrix lambda which will keep this invariant.

Now, if I assume that this matrix is a constant, so, for boost it was a constant, but now, I am looking at most general transformation. So, I have to make now, I am making an assumption that suppose this, we are interested in only those transformations for which this matrix is constant. And let me call it lambda mu nu just a notation again. Then, what I get is by doing the integral that x prime mu is lambda mu nu and then you integrate, you get plus the constants of integration.

So, you will have some constant here, 4 constants because there are 4 indices mu 0, 1, 2 and 3. So, this is what I get. Just to be clear, you do not have to assume that these are this matrix is constant. What you can do is: you can show and I leave it as an exercise; you can show that if you demand, dx prime square is dx square, you can arrive at the following condition that d 2 x mu over dx rho dx sigma, sorry this is prime. So, d 2 x prime mu over dx rho d sigma. This is zero. So, you can do this or you can look into the book by Weinberg on gravitation and cosmology, the second chapter has steps of doing this which is fairly easy. So, you can just look up there. So, if this is the case, then of course, second derivative is vanishing. So, first you integrated so, you will get del x prime mu over del x rho as some constant lambda mu rho.

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \tag{11}$$

Most general transformation for which

$$dx'^2 = dx^2 \tag{12}$$

Property of this matrix $\Lambda^{\mu}{}_{\nu}$

transformation for which $\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ is constant

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu} \tag{13}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{14}$$

Exercise:

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} = 0 \tag{15}$$

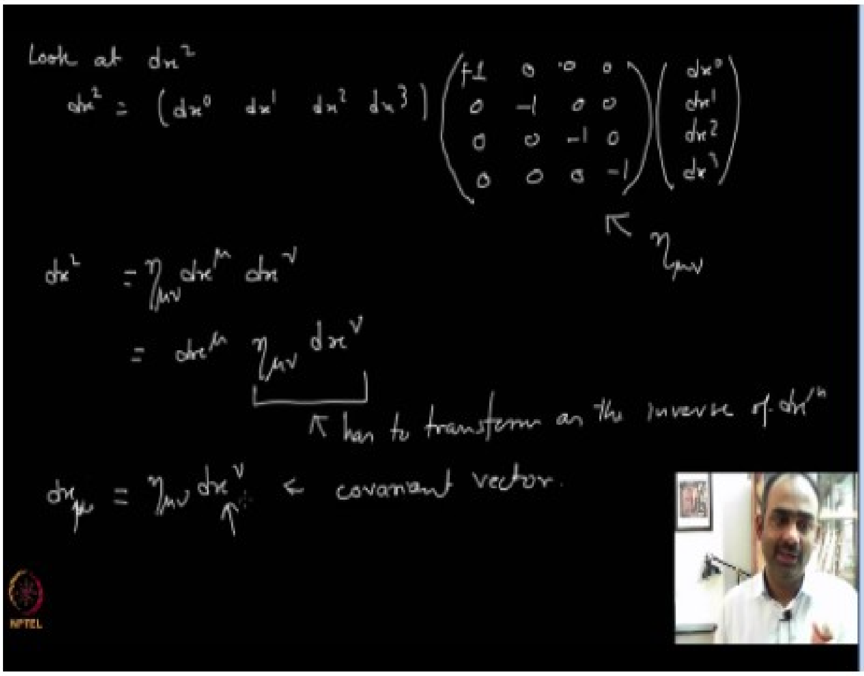


Figure 4: Refer Slide Time: 00:33:24

$$\begin{aligned} \frac{\partial x'^{\mu}}{\partial x^{\rho}} &= \Lambda^{\mu}{}_{\rho} && (16) \\ x'^{\mu} &= \Lambda^{\mu}{}_{\rho}\, x^{\rho} + a^{\rho} && (17) \end{aligned}$$

Let's take

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ; \quad \Lambda^{\mu}{}_{\rho} = \delta^{\mu}_{\rho} \tag{18}$$

$$x'^{\mu} = x^{\mu} + a^{\mu} \quad ; \quad a^{\mu} = (a^0, a^1, a^2, a^3) \tag{19}$$

When you do the first integral, you get a constant. So, that is why this is constant here, then you do a second integral and you get x prime mu as lambda mu rho x rho plus the second integration this. So, this is exactly what I wrote earlier, but there, I assumed, but if you do this exercise, then you do not have to make that assumption and you arrive at this result. So, this is the most general transformation which is going to leave this invariant. Good.

Now, let us see if lambda mu nu, if I take the lambda matrix whether this is allowed matrix and it is clear that it is because if lambda is one identity, then it is going to leave this relation true, this will hold true. So, lambda is going to identity is an element; is an allowed matrix which will satisfy the constraint that we have. So, let us take lambda to be this one and if I take that, then these lambda mu rho is just delta mu rho, well delta mu rho is the Kronecker delta.

So, if mu and rho are equal this takes the value one; otherwise it is 0. So, what we get is; x prime mu is x rho + a rho under this matrix, which means that I can easily interpret what a rho is, sorry I have, I am messing up with the indices. So, this should be mu here and this should also be mu. So, which means that this is just a shift of coordinates. So, the time coordinate is shifted by a 0 and the space coordinates are shifted by a 1, a 2 and a 3.

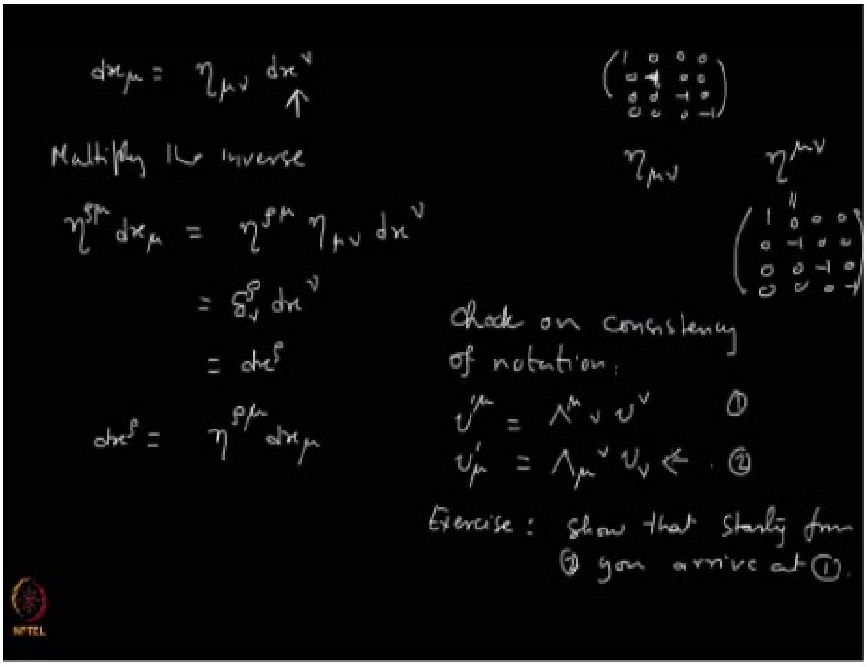
And a 0, a 1, a 2 and a 3 can take whatever values there, I mean all real numbers are allowed. So, these you can change continuously if you wish. So, this is the interpretation of this thing here. This is just introducing or inducing a translation on the coordinates so, this is translation. That is good. For example, x prime 0 which is t prime will become t + a 0, etcetera. So, that is 1 lambda I have taken but to know what are all possible transformation, I will have to find out the constraint that is imposed by this condition on lambda and then we will have to figure out what the elements could be

Contravariant vector

$$V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu} \tag{20}$$

But before I do that, I want to define our contravariant vector. So, contravariant vector is a vector; is an object which has 4 components. Let me call it v mu. But to tell that something is a vector, I have to tell how it transforms. And the transformation law is this that these 4 numbers v 0, v 1, v 2, and v 3, they transform like this. So, if you do a transformation, then the components transform like this and this is exactly what you have here.

So, I have taken this, the differential elements of the coordinates, the way those differential elements transform under Lorentz transformation. So, the set of transformations which keep this invariant, I am going to call as Lorentz transformations. So, the way these 4 numbers, x 0, x 1, x



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2 and x 3, the way they transform if any 4 numbers or 4 components, if they transform exactly in the same manner, I will call it to be a contravariant vector.

So, that is the definition of a contravariant vector for us. Now, let us look at another object which is del over del x prime sorry, del over del x mu. So, I am looking at a derivative. So, let us see how it will transform. So, what I am interested in knowing is this object, this is after you have done the transformation. Now, because your x prime is a function of x, I can again use chain rule and I can write it as del over del x nu. And then I have del x nu over del x prime mu, these are summation over mu implied. Now, I see this object does not transform in this manner. Here, you had del x prime mu over del x nu, a prime goes up and un-primed goes down. But here, it is opposite. It is not the same transformation. So, I will write this one, this will also be a constant matrix, I will write this one as lambda, but now I want to distinguish it from the previous one.

So, I will write the new index below. So, the first index, row index is low and the column index is up. So, this is a completely different object from this one. And that distinction I am making by, making the first index row and the second index up. So, here I have defined an object which has first index up, second index row and here I have defined another object which has first index low and second index up and clearly, these are 2 different objects and different matrices.

So, this is a transformation rule for the derivative. So, I defined covariant vectors. So, any 4 numbers, any 4 component objects that transforms under Lorentz transformation exactly in this manner. The way this derivative is transforming that quantity, I will call a covariant vector. So, let me define v mu whose transform transformation rule is this. So, now, I have defined 2 objects contravariant vectors and covariant vectors.

Now, there has to be a relation between this matrix and that matrix and that relation has to be given by this relation. Let me try to explain that I see, we have 2 matrices del x prime mu over del x nu and you have del x, let us call it rho over del x prime sigma. This is what is entering into contravariant and this is what is entering into covariant. Now, consider this object which will shed light on the relation between these 2.

Consider the following. Let us take del x prime mu over del x nu times del x nu over del x prime rho where this is a summation over nu. What is this subject? This is just del x prime mu over del x rho x prime rho. Now, what is this? This is going to be 1 if mu and rho are same and this is going to be 0 if mu and rho are different.

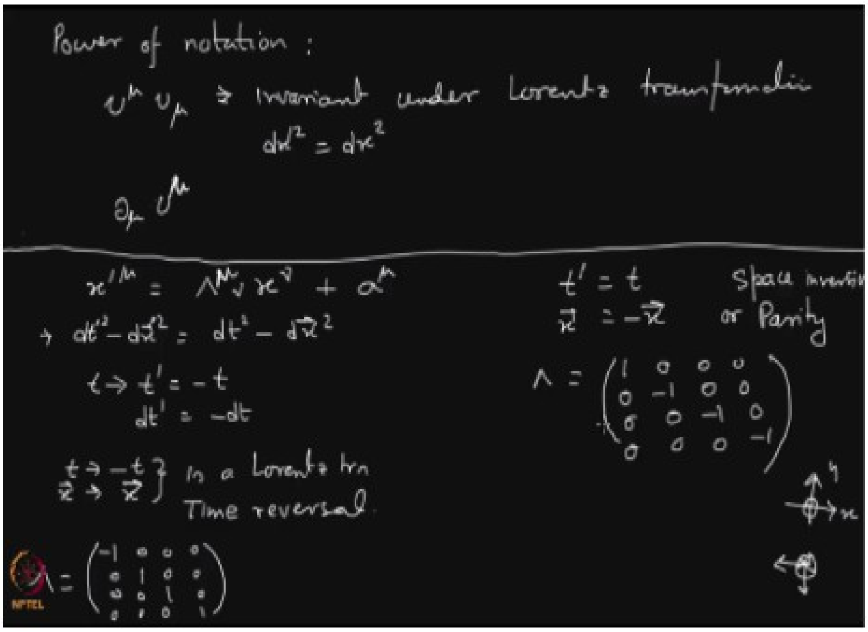

Figure 6: Refer Slide Time: 00:45:03

Transformation of derivatives $\frac{\partial}{\partial x^{\mu}}$

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} \tag{21}$$

Opposite transformation compared to x^{μ}

$$\frac{\partial}{\partial x'^{\mu}} = \Lambda^{\mu}{}_{\nu} \frac{\partial}{\partial x^{\nu}} \tag{22}$$

So, it is clearly this object which tells you that this matrix when multiplied with this matrix gives you identity which means this and this are inverses of each other; which means the way contravariant vector transforms is just the inverse of the way covariant vector transforms and that is not a surprise that is what we have actually built into it. This is happening because of this relation. I think, we will have more clarity very soon.

So, what I am saying is that this and this are inverses of each other. Why? Because the product of these 2 matrices is giving you an identity matrix. So, what is this? This is lambda mu nu. And what is this? This is lambda let us see, the prime here. So, this is this. So, new index is the second index and on the top. So, here the new index will be the second index on the top and this will be low.

So, these are 2 different kinds of matrices and that is the product which means, I can write it as lambda mu nu because this is inverse of this guy. I will write it as lambda inverse and of course, then I put because this is clearly the inverse of this matrix, I am writing here it has an inverse but now, I should put the indices exactly in this manner because it is the same kind of matrix now.

So, let me write it down again lambda sorry rho nu is lambda inverse nu rho that is good. Let us go further.

Covariant vector

$$V'_\mu = \Lambda_\mu{}^\nu V_\nu \tag{23}$$

Relation between two transformations

$$\frac{\partial x'^\mu}{\partial x^\nu} \quad , \quad \frac{\partial x^\rho}{\partial x'^\rho} \tag{24}$$

Consider

$$\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\rho} = \frac{\partial x'^\mu}{\partial x'^\rho} = \delta^\mu_\rho \tag{25}$$

Inverse transformation

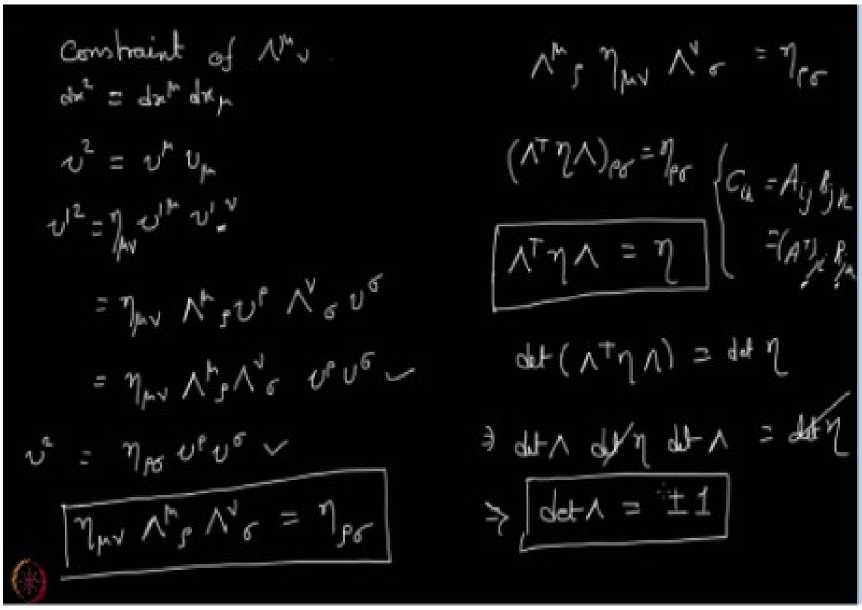

Figure 7: Refer Slide Time: 00:45:03

Now, let us look at dx square which is what is your dx square. dx square is just dx 0, dx 1, dx 2, dx 3 times a matrix, you know, when you do a square you have dx 0 square − dx vector square and of course, if I just put a column vector, it will give you dx 0 square + dx vector square. So, I have to have a matrix and that matrix I will take to be this. It has to be this and I will give this matrix name, I will call it eta but then I have to put indices on this and I put both the indices low mu nu.

And then I can write dx square as this is dx mu, this is the same object dx nu. See for a column vector, row vector, you have to just put it the same way is no distinction and then I have the matrix eta mu nu. I do not have to write in between. I can but I do not have to. This is; eta is a matrix but eta mu nu is an element of the matrix. So, it commutes with everything. So, there is no issue of non-commuting property of matrices.

So, these are just elements of the matrix. This is good. Let me write it again as we put it at its original location, same thing. Now, this left hand side is an invariant object, it does not change under Lorentz transformation. It becomes dx prime square, but dx prime square is same as dx square. But this object changes. It becomes dx prime mu. And this object has to change according to the inverse of this one, because then only the product of 2 is going to give you an not changing thing.

So, if I look at this, this has to transform as inverse of dx mu. The way this is transforming. If it is transforming with lambda, which we have been calling lambda, then this has to transform with lambda inverse that is what I am trying to say. But then you know, those objects which transform according to the inverse matrix, we call them contravariant covariant vectors.

So, this is a contravariant vector, then this has to be a contravariant vector, which means that eta mu nu dx nu is an object, which is covariant. And for covariant updates, we have been putting the index down, we had a v mu. So, I will define it to be dx nu. And that is the definition of this object. Till now, we did not encounter any dx with the index below. This is the first time, I write it. And what is the definition of it? This is the definition of it. And I have already argued that this is a covariant quantity, so, it makes sense to define this object as an object with index below. So, this is clearly a covariant vector. Good. So, we see that given a contravariant vector dx mu, I can associate with it a covariant vector dx nu, which is given by this. So, from contravariant, which has an index up, I arrive at covariant, which has an index down by contracting it with the eta mu nu.

So, this eta mu nu pulls down this index new below and makes it mu. So, this is not correctly, it should be mu. How about the opposite? How do you go from here to there? From a; if you are given a covariant vector, how do you construct a contravariant? And that is easy to see, because, what you have here is a column which is a matrix times a column; now, if you multiply inverse of this matrix on both sides, then I can free up the right hand side. And I will have the contravariant object in terms of covariant on it. So, let us do that.

$\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ and $\frac{\partial x^{\nu}}{\partial x'^{\rho}}$ are inverse of each other

$$\Lambda^{\mu}{}_{\nu}\Lambda_{\rho}{}^{\nu} = \delta^{\mu}_{\rho} \tag{26}$$

$$\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\rho} = \delta^{\mu}_{\rho} \tag{27}$$

$$\Lambda^{\nu}{}_{\rho} = (\Lambda^{-1})^{\nu}{}_{\rho} \tag{28}$$

Look at dx^2

$$dx^2 = \begin{pmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \tag{29}$$

So, right now, I have dx mu as eta mu nu dx nu and now our goal is to get is the following. That given this, how do I get down? So, now let me see what is the inverse of this matrix, which is 1 0 0 0, 0 0 sorry, −1 0 0. Clearly, if you take this matrix and multiply itself, you are going to get identity matrix because all these −1's get multiplied with −1's and they become plus and then you get an identity matrix. So, I will define inverse of eta mu nu to be again, another matrix which I call eta mu nu up, but which has the same entries. So, if I do so, these 2 are inverses of each other, then I multiply on both sides with the inverse matrix. So, let us see. So, what do you get? You have dx mu, I multiply with rho mu, so, I have a contraction over mu and there is a summation over mu and here also rho mu; I have to multiply, I already have eta mu nu dx nu.

So, what is this object? This is this times that. It is a contraction of this mu summed over, you are going to left with only rho and nu. So, this is an identity matrix of course, because it is a product of these 2 matrices and the indices will be rho up and nu down dx nu. And what is this? It says that unless mu equal to rho, this will be 0. So, you see that now I have the upper index here and this is the way I have to construct it.

So, let me write it again. So, for to get the contravariant vector starting from covariant vector, I should raise the index using the terms eta mu nu with up indices. That is good. Now, all our notation of this introducing, you know, some things up, some things with down, all these things, let us see whether it is all consistent. And one way to check all this is consistent. One thing we should definitely one thing we should check, to check the consistency of this notation is, is this, is this.

Let me write it on the new page. So, let us check the consistency of our notations. So, we had dx, or let me for a moment, not write dx, because I hve already defined what are covariant and contravariant vectors. This was lambda mu nu v nu and also a defined covariant to be lambda first index mu, this index is the same index here. So, the first index is always the index on the left, this one is same as this nu v nu.

Nice, all the notation that I have been using is consistent, then if I start from here, let us say and raise the indices using the eta mu nu. I should end up with this matrix. And that I leave as an exercise for you to check that you contract with the appropriate details on the both sides and arrive at this one. Contract with, let me just say that show that if you start from 2, you arrive at 1. That will tell you that all the notation has been consistent. Because you arrived from this kind of matrix to that kind of matrix when you are changing all the indices. So, you will have to use eta mu nu for that purpose. That is good. Now the power of the notation is this.

$$dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dx^\mu \eta_{\mu\nu} dx^\nu \tag{30}$$

Elements of matrix no issue with with commutative properties

$\eta_{\mu\nu} dx^\nu$ has to transform as the inverse of dx'^μ

$$dx_\mu = \eta_{\mu\nu} dx^\nu \quad \leftarrow \quad \text{Covariant vector} \tag{31}$$

Inverse of $\eta_{\mu\nu}$ is $\eta^{\mu\nu}$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{32}$$

Multiply with inverse

$$\eta^{\rho\mu} dx_\mu = \eta^{\rho\mu} \eta_{\mu\nu} dx^\nu \tag{33}$$

$$ta^{\rho\mu} dx_\mu = \delta^\rho_\nu dx^\nu \tag{34}$$

$$= dx^\rho \tag{35}$$

To get contravariant form, from the covariant

$$dx^\rho = \eta^{\rho\mu} dx_\mu \tag{36}$$

Checking the consistency of notation

$$\begin{aligned} V'^{\mu} &= \Lambda^{\mu}{}_{\nu} V_{\nu} & (37) \\ V'_{\mu} &= \Lambda_{\mu}{}^{\nu} V_{\nu} & (38) \end{aligned}$$

Exercise: Show that if you start with expression of V'_{μ} you arrive at V'^{μ}

Power of notation

- $V^{\mu}V_{\mu} \rightarrow$ Invariant under Lorentz transformation

$$dx'^2 = dx^2$$

- $\partial_{\mu}V^{\mu} \rightarrow$ Invariant

If I take v mu and contract with v covariant mu, then this object is invariant under Lorentz transformations. And what are Lorentz transformations? Those are the ones which keep this invariant, this update invariant which also means that if you had a del mu of some object which carries the index. So, some because this is a derivative it has to act on something which has space with depends on space time.

So, let us call it some v mu v depends on x, v mu there is no prime here; by construction this will be nu invariant because this is going to use this object is going to transform according to the inverse matrix of this one. So, they will kill each other and give you a scalar. So, whenever you have a repeated index up and down that is guaranteed to be a scalar that is the power of this notation.

We have just now encoded everything the Lorentz invariance or the way they transform into the notation itself and that is why this notation is very, very, very powerful useful. Now, let us proceed further. So, we have already written down the solution that you know x prime mu is lambda mu nu x nu plus a sorry a nu. This is the coordinate and then we are looking at the constraints coming from this relation that t prime square – dx vector square should remain as the dt square – dx vector square, should be a prime here.

There is the dx prime square is equal to dx square. And this will put some conditions or constraints on lambda mu. Now, before we systematically start looking at the constraints and or tell us about lambda mu, let us make some simple observations at least some of the matrices which will definitely satisfy this constraint. So, when you can see that if you just took t and change to –t, so, if t goes to t prime which is just minus of t, then clearly dt prime will be minus dt, but here we are looking at squares of these.

So, clearly this is going to remain unchanged because this will go to the same thing. So, dt prime will still be dt and I am not doing anything to this. So, this is clearly a possible Lorentz transformation. So, t is going to minus t and x going to x not changing anything is a Lorentz transformation. Note that I am making a distinction between Lorentz transformation and boosts.

Boosts are the one which boosts are the transformation which take you from one frame to another frame which are one inertial frame to another inertial frame which are related by velocity v. But when I say Lorentz transformation, I mean all those transformations that obey this. For example, this is of course not a boost. This is just time reversal. This is called a time reversal. You have reverse the time. So, that definitely belongs to this class.

So, you will have lambda mu nu's which will just be what so. In this case, what will be the lambda mu nu? It will be t goes to minus t, so it will be – 1 here. And all others will be +1, because I am not changing the space coordinates sorry, what am I doing? So, this will be the lambda for this transformation. And of course, you can see that if you left the time and change, but change all the space coordinates, so, if you took t prime is equal to t.

So, you do not do anything there. And if you took x vector and interchange, although all of them, so, x 1 goes to − x 1; x 2 goes to − x 2 and x 3 goes to − x 3. Now, still this will remain unchanged. So, there is another matrix. So, let me write this matrix also. This is called space inversion or parity. So, this transformation is called parity transformation or space inversion transformation. And what is the matrix?

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}{}_{\nu} x^{\mu} + a^{\mu} && (39) \\ dt'^2 - d\vec{x'}^2 &= dt^2 - d\vec{x}^2 && (40) \end{aligned}$$

This equation will put constraint

$$\begin{aligned} t \to t' &= -t && (41) \\ dt' &= -dt, \quad \text{but} \quad d\vec{x'}^2 = d\vec{x}^2 && (42) \end{aligned}$$

They says 1 0 0 0 because time has not changed 0 −1 0 0; 0 0 −1 0; 0 0 0 −1. And of course, there can be other transformations, but I wanted to point these 2 out. Let me ask you see the space in version of parity is not a rotation; you cannot arrive at this transformation by rotation. You should try that if you take x, y and z is coming out of the page. And if you do the inversion, you get x in this direction, y in this direction and z into the page.

Convince yourself that there is no way you can arrive from here to there by any rotation. So, let me this is going, z is going into, you will not be able to reach from this to this. How about having only, let us say, so, let us keep t unchanged. Let us keep z unchanged and only invert x and y. Is that also something that you cannot arrive at by doing your rotation? Or you can please think about it and check. So, that is good. Let us move ahead and try to find the constraints on the matrix lambda.

$$\begin{aligned} t &\to -t \\ \vec{x} &\to \vec{x} \quad \text{is a Lorentz transformation} \end{aligned} \qquad (43)$$

Time reversal

$\Lambda^{\mu}{}_{\nu}$ for this transformation

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (44)$$

So, how do I do that? Just because I am saying that dx square has to remain unchanged. It is and because this is just equal to dx mu dx mu, remember that this is already we have done. If I take any arbitrary for vector v mu and contract it with its covariant part, then I call it v square and this object has to be invariant. That is purely because these things transform exactly the way these transform.

So, if this combination remains invariant, then this combination also has to remain invariant. That does not have to be checked that is automatic. I mean, that is the reason why we have built things like this. So, let us go now and look at what is the constraint that is coming from here that this is same as v prime square. So, let us look at v prime square. Now, v prime square is v prime mu v prime mu.

And I will ensure that this is v prime, v prime square is v square. So, now, let me write v prime mu. This is lambda mu rho v rho. What is v prime mu? Remember it is with a lower index mu sigma v sigma. No, this is not up; this should be done. That is fine. Now, I will write it as lambda mu rho. I think that is not; I can proceed like this but it is not so nice that way. It is hardly different but still I do not know.

To do it that way, let me write it as v prime mu here and mu is the same thing. Whatever I had earlier it, I have written slightly differently. Now, this I will write as lambda mu rho v rho and this one, this object, I will write lambda nu sigma v of sigma v sigma. So, let me collect terms lambda mu nu lambda mu rho lambda nu sigma v rho v sigma. But v prime square is same as v square that I am going to use anyway but let me first write v square. v square, I will write as eta, not mu but rho sigma v rho v sigma.

Now, this and this have to be the same which means eat mu nu lambda mu rho lambda nu sigma is eta rho sigma. This is going to work because this has to be true because this relation is true for any vector v. So, for whatever vector v you choose, these 2 have to be called and that implies that unless these factors are equal, it is not going to work. So, they have to be equal and this is the constraint I get.

The constraint is if you take the matrix lambda and you contract with eta mu nu, you should get eta mu nu that is a constraint that lambda has to satisfy. So, let me put it in a box. That is an important thing to know. Now, let me write it in a matrix form. And to do that, I will do the following. I will write it as lambda mu rho. This is the space then eta mu, I can write here; this issue lambda nu sigma rho sigma that is the same equation.

I can move around eta and lambda because these are elements of the matrices. Now, you say if this lambda, this mu was the second index and was contracting that would be lambda times eta because when you multiply 2 matrices, let us say this is the matrix C ik; element is ik. The column of this one is contracting with the row of the second one. But here, it is the row of the first one, which is contracting with the row of the second one.

And you know that is the transpose because if you interchange this, I can write it as a transpose A i jk B jk, then the row index is contracting with the row index here. So, clearly, this contraction tells you that you have lambda transpose, it is getting contracted with eta. And then again, the second index is getting contracted with lambda, so it is just lambda and this is rho and sigma, rho and sigma free indices and you get eta rho sigma.

Or that is the condition the same condition which we have here is now written explicitly in terms of matrix notation. This immediately tells us that at least some of the things, we can infer about lambda. So, always whenever you have a matrix, you start to ask what its determinant because determinant is an important property of a matrix. So, let us ask what is the determinant.

So, if I take determinant on both sides, I get, this is determinant of lambda transpose is same as determined of lambda. Determinant of a transpose matrix is same as the determinant of the matrix. And I have determinant of eta and then determinant of lambda. And these 2 cancel, which implies determinant of lambda square is 1 which further implies that determinant of lambda can be either $+1$ or -1.

So, that is one interesting observation we have made. Let us just go back and see about the time reversal and parity transformations that you know, are part of the Lorentz transformations what, which of these is true for those because I have 2 possibilities. So, let us see. If I look at this one, the determinant is -1. So, time reversal has a determinant -1. This one also has a determinant -1 and one more matrix, we have seen already and that was somewhere here.

Space inversion/Parity

$$\begin{aligned} t' &= t \\ \vec{x} &= -\vec{x} \end{aligned} \tag{45}$$

$\Lambda^{\mu}{}_{\nu}$ for this transformation

Now, we said that this identity matrix is also part of it and this one is definitely determinant +1. So, we already have some examples here with us, so, we could verify that indeed at least the ones we saw all had this or that. I can derive one more constraint on one of the elements, I can immediately see by doing the following.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{46}$$

there can be more transformations.

Parity transformation is not same as rotation transformation, only changing x, y coordinates keeping t and z constant.

Constraint on matrix $\Lambda^{\mu}{}_{\nu}$

$$dx^2 = dx^{\mu} dx_{\mu} \tag{47}$$
$$V^2 = V^{\mu} V_{\mu} \tag{48}$$
$$V'^2 = \eta_{\mu\nu} V'^{\mu} V'^{\nu} \tag{49}$$
$$V'^2 = \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} V^{\rho} \Lambda^{\nu}{}_{\sigma} V^{\sigma} \tag{50}$$
$$V'^2 = \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} V^{\rho} V^{\sigma} \tag{51}$$
$$V^2 = \eta_{\rho\sigma} V^{\rho} V^{\sigma} \tag{52}$$
$$\tag{53}$$

$$\eta_{\mu\nu} \, \Lambda^{\mu}{}_{\rho} \, \Lambda_{\nu}{}^{\sigma} = \eta_{\rho\sigma} \tag{54}$$

True for any vector V

$$\Lambda^{\mu}{}_{\rho} \, \eta_{\mu\nu} \, \Lambda_{\nu}{}^{\sigma} = \eta_{\rho\sigma} \tag{55}$$

for matrices

$$C_{ij} = A_{ij} B_{jk} \tag{56}$$
$$= (A^T)_{ji} B_{jk} \tag{57}$$

$$(\Lambda^T \eta \Lambda)_{\rho\sigma} = \eta_{\rho\sigma} \tag{58}$$
$$\Lambda^T \eta \Lambda = \eta \tag{59}$$

Things to infer about Λ

$$\begin{aligned} \det(\Lambda^T \eta \Lambda) &= \det(\eta) & (60) \\ \det(\Lambda)\det(\eta)\det(\Lambda) &= \det(\eta) & (61) \\ (\det(\Lambda))^2 &= 1 & (62) \\ \det(\Lambda) &= \pm 1 & (63) \end{aligned}$$

Focus at time reversal and parity, both have $\det(-1)$. Identity transformation had $\det(1)$

Extra constraint

So, you go back to that equation in the box which I wrote let me write it again here. Eta mu nu lambda mu rho lambda nu sigma, which is eta rho sigma. Now, what are the free indices in this equation? These are the eta and rho; all others are summed over, those indices are not even there actually, there is a summation, which we do not make explicit, but let me take now, rho is equal to sigma is equal to 0 the time component, so, this one will be rho 0 0 and that will be –1 sorry, we are using metric +1.

So that will be +1. So, let me write down. eta mu nu lambda mu and I am putting rho = 0, lambda nu sigma, I am putting a 0 and that will be eta 0 0 which is 1. Now, you know that eta is a diagonal matrix. Many unless mu and nu are equal, this will vanish and it will multiply this and gives 0. So, I have this eta 0 0. So, I am summing over all possible mu and nu that is what I am doing.

But because of this property, I can save some work and just look at those terms in which mu and nu takes equal values. So, if mu is equal to 0 and nu equal to 0, you get lambda 0 0; lambda 0 0 +. When mu is equal to 1 and nu is equal to 1, you get eta 1 1, which is –1. So, let me write –1 here. Again, it is –1, so –1 is there and you have lambda, mu is equal to 1, which I will write in this way.

This is a summation over i; i from 1 to 3. So, it is lambda 1 0, lambda 1 0, plus lambda 2, 0 2 0 and so forth. In the fact that eta i r – 1, so I have put this –1 and this should be equal to 1 and this is 1. So, I have lambda 0 0 square is equal to 1 + summation of i = 1 to 3 whole square. Now, this is a positive number because it is a square of real numbers.

So, this is positive. So, clearly, we have lambda 0 0 square is greater than or equal to 1, which means that lambda 0 0 is either greater than +1, or lambda 0 0 is less than –1. Because if it is less than – 1, then also the square will be more than 1. So, that is another constraint that we have found. Another property of lambda 0 0 that we now know that its value has to be either more than 1 or less than 1.

It cannot take a value equal to 0.5 for example because that lies between –1 and +1. We will continue further on Lorentz transformations in the next video.

$$\begin{aligned} \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} &= \eta_{\rho\sigma} & (64) \\ \text{Let's take,} \quad \rho = \sigma &= 0 & (65) \\ \eta_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} &= \eta_{00} = 1 & (66) \end{aligned}$$

$$\eta_{00} \Lambda^{0}{}_{0} \Lambda^{0}{}_{0} - \sum_{i=1}^{3} \Lambda^{i}{}_{0} \Lambda^{i}{}_{0} = 1 \quad (67)$$

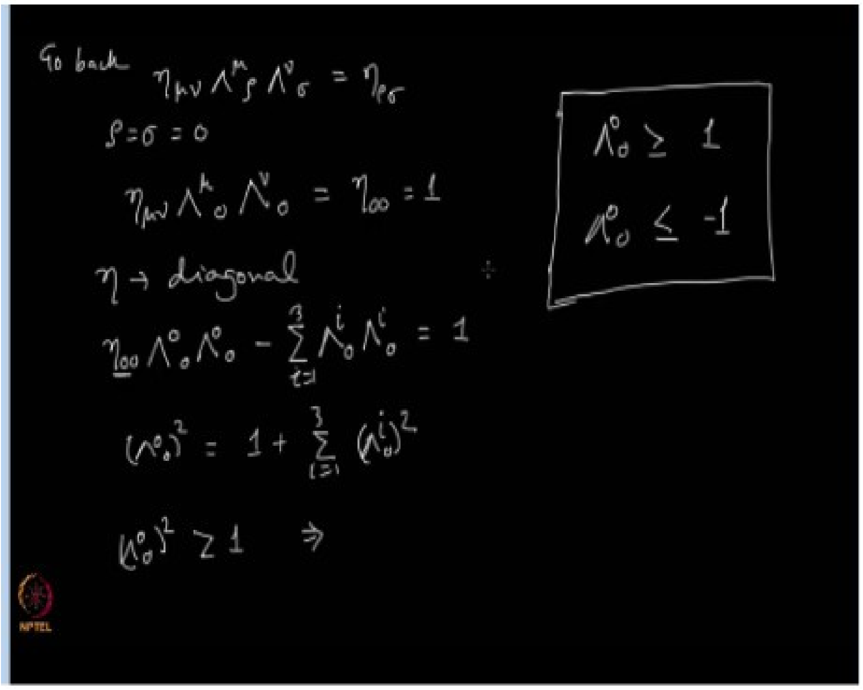

Figure 8: Refer Slide Time: 01:01:03

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda_0{}^i)^2 \tag{68}$$

$$\Lambda^0{}_0 \quad > \quad 1 \tag{69}$$
$$\Lambda^0{}_0 \quad < \quad -1 \tag{70}$$