Introduction to Quantum Field Theory

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Lecture 15 : Theory of Scalar Fields

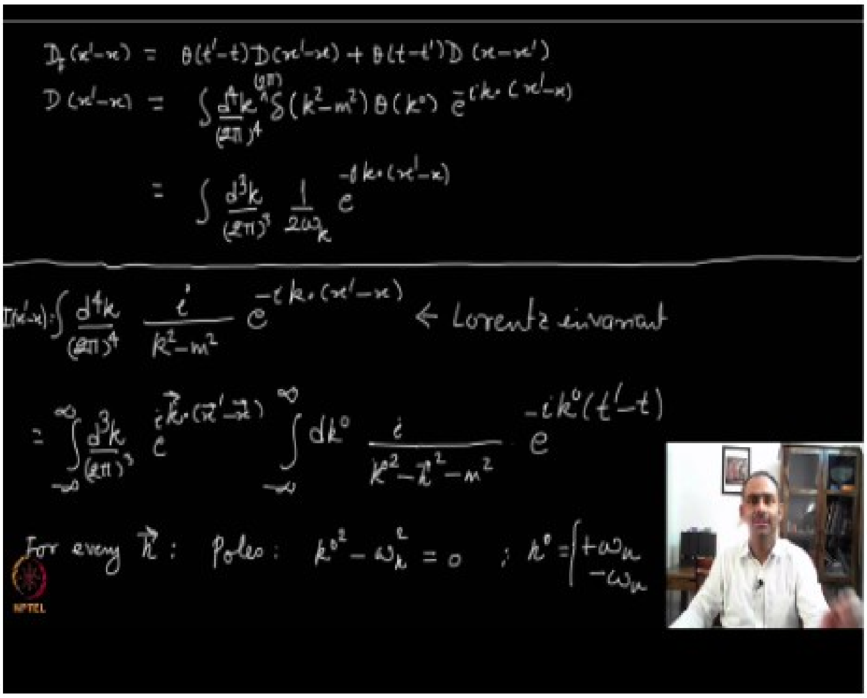

Figure 1: Refer Slide Time: 00:14

Last time, we were looking at the propagator which was given by this expression. I think, I did not write the expression yet. But we wrote this as theta of t prime – t D of x prime – x + theta of t – t prime D of x – x prime. There is the expression for D F. And we argued that D F is a Lorentz invariant quantity that is what we showed. And also, let us remind ourselves that the expression for D of x prime – x is the following.

We wrote it as D 4 k over 2 phi 4 and then k square – m square. Then we had a theta of k 0 and e to the – i k dot x prime – x. And we argued that this is Lorentz invariant quantity and please make sure that you are able to show that this is a Lorentz invariant. And also, you recall that this originally, we wrote the expression for D x prime – x as follows. So, it was d cube k over 2 phi cube. I realised that I missed a factor of 2 phi here.

It was d cube k over 2 phi cube 1 over 2 omega k times e to the – i k dot x prime – x. And of course, this is Lorentz invariant. Because this is an; I think, I also showed that openly give us an exercise to show that this is Lorentz invariant. So, this is where we stopped last time. Now, I want to look at some integral, just for fun. And integral, which I want to look at is the following.

So, I look at the integral d 4 k over 2 phi to the 4 Iota over i over k square – M square e to the – i k dot x prime – x. Let me give it a name. I have written x prime – x because it is only a function of x prime – x. It does not depend separately on x prime and x. It always comes as this combination. Now, if I look at this integral and if I repeat the arguments that we were giving last time, when we were analysing this integral, you see that this is a Lorentz invariant object because the denominator is invariant.

The measure is invariant. And this will be invariant if you appropriately change k to lambda k, if you are going from x prime – x to lambda of x prime – x, the same reasoning which we gave last time will work and this is Lorentz invariant. Now, let us look at this a little more closely. I will write it as d cube k over 2 phi cube and the limits are from minus infinity to plus infinity and then we have the e to the i k dot x prime – x.

And then we have the integral from minus infinity to plus infinity, d k 0. And then I should write this one, this is iota over k 0; it is a matter, but let me write it like this – m square times e to the – i k 0 t prime – t. So, all I have done is just rearrange the terms. So, this is same as the above. Now, the integration is over the real line for k 0 and also the k. But I can always look at it as an integration in the complex plane and the contour of integration is over the real axis.

So, that is what I will do. So, if I look at the k 0 integral, the integrand has a pole or has poles at the place where the denominator goes to 0. So, for any k, for every k, we have the following poles of k 0 in k 0 plane. And how do you get the poles? You get the poles by solving the denominator and equating it to 0. So, you have k 0 square minus this object, you remember this is just omega k, so you have minus omega k square.

$$
\begin{aligned}
D_F(x', x) &= \theta(t'-t)D(x'-x) + \theta(t-t')D(x-x') &(1)\\
D(x'-x) &= \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2-m^2)\theta(k^0)e^{-ik\cdot(x'-x)} &(2)\\
&= \int \frac{d^3k}{(2\pi)^3} \times \frac{1}{2\omega_k} e^{-ik\cdot(x'-x)} &(3)
\end{aligned}
$$

Integral I want to look at,

$$
\begin{aligned}
I(x'-x) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik\cdot(x'-x)} &(4)\\
&= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(x'-x)} \int_{-\infty}^{\infty} dk^0 \frac{i}{k^{0^2}-\vec{k}^2-m^2}e^{-ik^0\cdot(t'-t)} &(5)
\end{aligned}
$$

For every $\vec{k}$ poles

$$k^{0^2} - \omega_k^2 = 0$$

Having a pole at contour of integration is problem,

- Integral is not defined → poles on the contour of integration

$$
\begin{aligned}
I^- &= I_P + I_\gamma &(6)\\
I^+ &= I_P + I_{\gamma'} &(7)
\end{aligned}
$$

Change the integration contour, take the semi-circle to zero

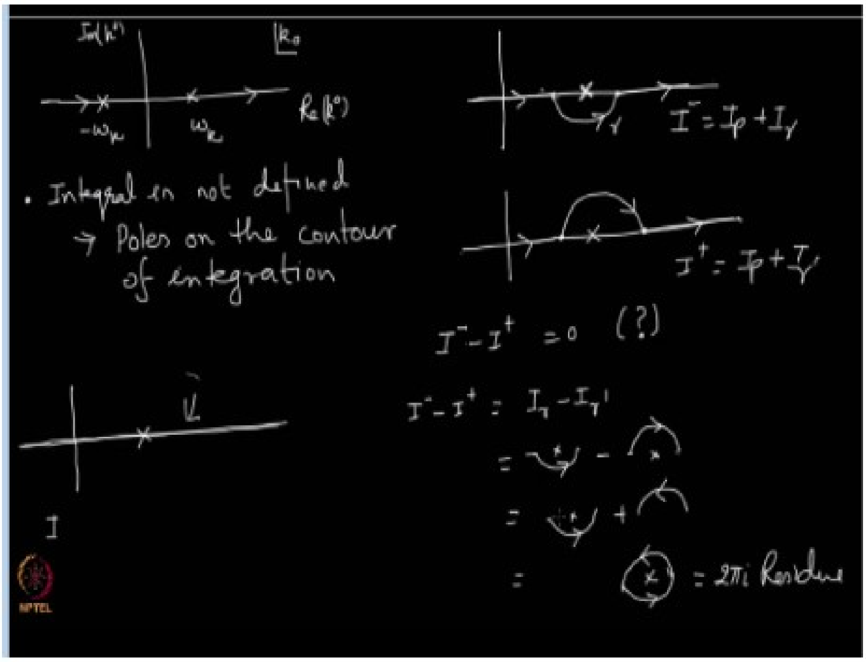

Figure 2: Refer Slide Time: 07:24

Two alternatives
So, this is how you will find the poles and clearly the poles are located at k 0 is equal to, let me write it here, k 0 is equal to + omega k that is one place and – omega k. These are the location of the poles. So, let us see. So, here if I see, k is 0. So, this is the way of denoting the complex plane in general. So, that is the standard way of writing. So, when I write k 0 like this, I mean the complex plane of k 0. So, here on this axis, you have real k 0 and here you have imaginary k 0. So, the integral is from minus infinity to plus infinity, which means the integral is over this real axis. I have to integrate over this line and I have seen that there are 2 poles and those 2 poles, I will denote by crosses.

So, one is at a value minus omega k and one is at omega k. So, these are the 2 locations of the poles; having a pole is not a problem, but having a pole on the contour of integration is a problem because your integral is not defined them. So, the problem is with this integral which I am right now looking at is that it is not defined because the pole lies on the integration contour so, which is a situation here.

So, the problem is that the integral is not defined and it is not defined because there are poles on the integration contour. Let us see why this integral is not defined if you have holes on the contour of integration. Let me give you a simple way to understand this. So, let us forget that there are 2 poles. Let us take one pole on a contour of integration. So, suppose I am looking at some integral over this contour, is a real axis. And you have a pole sitting on the contour. Now, let me call it as integral I. Now, what I will do is I will take the pole of the contour. So, instead of looking at this object which I do not know how to calculate or what it means, I will look at the following object. I take this one. So, I take pole here. This is again my as before the integration contour and what I will say is that I will let this pole approach here.

And when it reaches the contour that is the integral I am anyway interested in. So, this is what the one I am looking at or equivalently you can do the following instead of taking the pole off and bringing it to the axis, we can indent it. So, what we can do is: we let the pole here; we let it be there and we change the integration contour. So, we come up to here and then we make a detour and we continue like this.

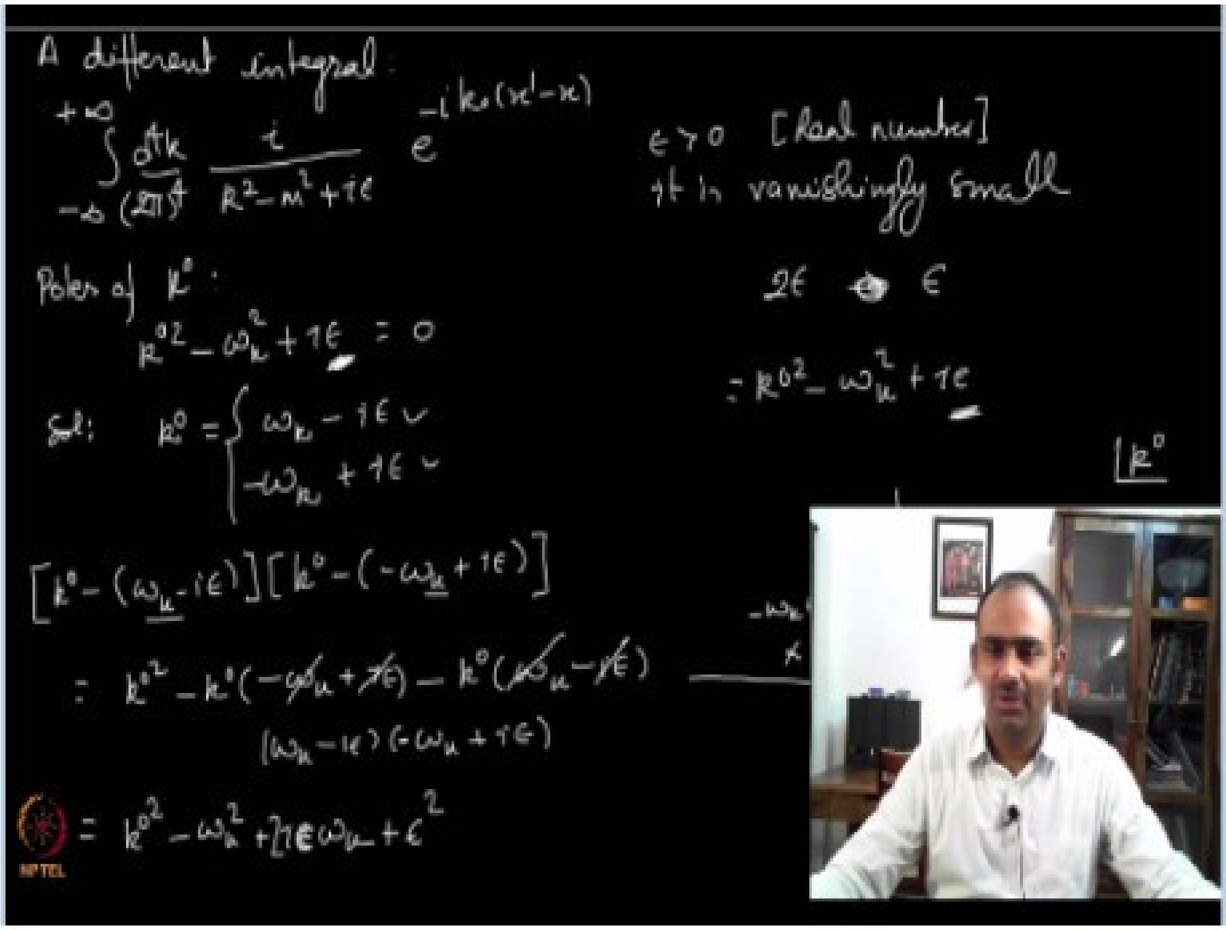

Figure 3: Refer Slide Time: 16:06

And then you take the semicircle to go to 0 and then you get a result and you would try to interpret that it is providing a continuation to this integral, but then you also have a possibility of doing this, same thing, you leave it here. The pole is there and you integrate along this path. So, you come up to here from minus infinity and here you make a small detour and continue like this. And then again take the semicircle to go to 0.

In that case, you will be really in this situation. Now, let me write the integral in this case, this integral. So, this is not this one; this is different. So, this integral like we call as I minus is equal to I p. So, I p is from minus infinity to here and then here to infinity + I of gamma and this semicircle, I am calling gamma and similarly, this one I will call I plus, the whole integral and then the integral from here to here and then from here to here, I will call as I p is the same as above plus this part and I will call it I gamma prime.

Now, let us see, if I prime in I give the same answer. If they do give the same answer, then I should get it I minus – I plus is equal to 0 and this is what I should expect if I minus – I plus are same and I could say that this is what I minus and I plus are giving the result for this integral. This is how I could define it. Now, let us see what we get. When I calculate this, I get I minus – I plus is equal to the I p the principal part is same in both, so, that drops out and I get I gamma – I gamma prime.

And what is I gamma? I gamma is this integral. So, it is basically this integral and direction is this; minus what is this one; this is this one, which is same as this plus; the minus sign, I can absorb and change the direction of the sense of the contour. So, it will become this and it was this here which is same as having, this is; the first term is going to give you this one and the second term when you add, it gives you this and the pole is there.

And this is when you do the integral, this one, this is 2 phi i times the residue of the pole which is lying here. So, you see that I prime I – and I +, they differ by 2 phi i times the residue of the pole, so, you do not get 0, so, you do not get the same thing. So, there is an ambiguity. You cannot just say that I will take the limit of i – and take the limit with the semicircle goes to 0 and I will say that this is what defines this integral, because you see that you also have another possibility and which is giving a different results.

And the result is the residue of the pole there times 2 phi. So, we see that if you have a pole on the integration contour that is not defined as I have shown you here. So, which means the integral which I wanted to play for a while is not a defined integral.

$$
\begin{aligned}
& I^- - I^+ = 0 \ (?) && (8)\\
= \ & I_\gamma - I_{\gamma'} && (9)\\
= \ & 2\pi i \times (\text{residue}) && (10)
\end{aligned}
$$

So, now I will look at a different integral. So, let me look at this one, d 4 k over 2 phi to the 4 i over k square – m square this and e to the – i k dot x prime – x. This was not defined. So, what I will do is; I will put a + i epsilon here, where epsilon is positive. It is a real number. I need a real quantity. And it is vanishingly small. So, it is very small; it is,and let us see whether this integral is defined or not whether we still have poles on the integration contour.

And of course, we do not have because of this i epsilon. So again our integration continues from minus infinity to plus infinity for all the 4 variables. And let us look at where the poles are located in the k 0 plane. So, what we want to look at the poles of k 0, so what I want to do is; k square – m square is basically k 0 square – omega k square, as you saw before and you have a + i epsilon and let us find out where this becomes 0.

So, I will give you the solution, it is easy to find out. The solution is: k 0 is omega k – i epsilon. That is solution number 1. And the solution is – omega k + i epsilon. These are the 2 solutions. I think, this is correct. Now, let us see whether this is correct. All I will do is take k 0 – first solution and then the second one k 0 – the second solution. So, this is k 0 square. And then you get when you multiply this piece with the k 0, you get – k 0 – omega k + i epsilon.

Then you multiply this and this, you get – k 0 omega k – i epsilon. And then when you multiply, so this is; these are this and then you finally multiply the 2 terms here. You get minus into minus is plus and omega k – i epsilon into omega k + i epsilon. Let us see, k 0 omega k with a plus sign; k 0 omega k with the minus sign. So, these 2 cancel. This is – k 0 i epsilon; this is + k 0 i epsilon. So, this term also goes away.

And we are left with k 0 square; somewhere it has, this one, I have forgot a sign and this one gives this time; this is – omega k square. And then you have – i epsilon times – omega k gives you, i epsilon omega k sorry, I should use the same symbol. And when this one multiplies; it also gives you i epsilon omega k. So, you have 2 i epsilon omega k. And then this one gives you, i times i is – 1, minus, minus plus, + epsilon square.

A different integral

$$
I(x'-x) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x'-x)} \tag{11}
$$

$\epsilon > 0$ and it is vanishingly small

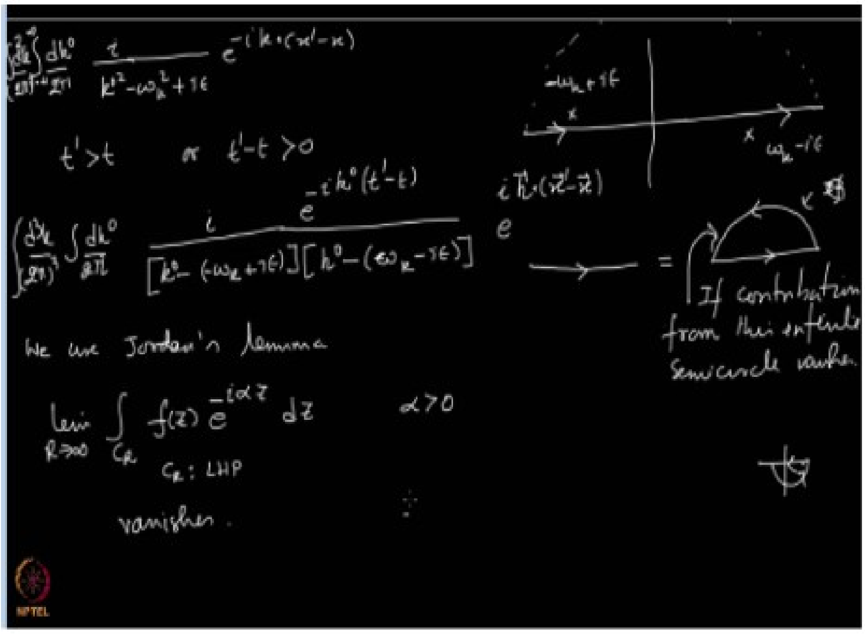

Figure 4: Refer Slide Time: 23:29

Poles of k^0

$$k^{0^2} - \omega_k^2 + i\epsilon \;=\; 0 \tag{12}$$

Gives

$$k^0 = \begin{cases} \omega_k - i\epsilon \\ -\omega_k + i\epsilon \end{cases} \tag{13}$$

But we are taking epsilon to be vanishingly small. Now, if epsilon is vanishingly small, 2 times epsilon is also vanishingly small. So, I can just instead of writing 2 epsilon, I can just write the same thing as absolute, just because epsilon square. I am not interested in the value of epsilon, I just want it to be something very small. So, this object then becomes k 0 square – omega k square, 2 i epsilon omega k.

I can write it the same thing as i epsilon omega k is also absorbed there. If I can observe too, I can actually absorb omega k also. And I dropped epsilon square term because it is 1 order of higher in smallest. So, you see that this is exactly what we started with. So, clearly, these 2 are the solutions. So, these 2 are the solutions that is what I have shown. So, let us see where the poles are in the k 0 plane. The poles are here. So, one is that omega k – i epsilon.

So, minus i epsilon means it has a negative imaginary part. So, it is here. And you have – omega k + i epsilon. So, there is a positive imaginary part. So, here, so one is – omega k + i epsilon and this is my omega k – i epsilon. So, at least the integral appears to be defined, well defined because there is no pole lying on the integration contour. So, this is good.

So, unlike the previous one, this one is nice. And now that we have come so far, let us do the integral over k 0. So, I will do an integral. Let us do that. So, let us go to the next page.

$$\begin{aligned} & [k^0 - (\omega_k - i\epsilon)][k^0 - (-\omega_k + i\epsilon)] && (14) \\ = \;& k^{0^2} - k^0(-\omega_k + i\epsilon) - k^0(\omega_k - i\epsilon) + (-\omega_k + i\epsilon)(\omega_k - i\epsilon) && (15) \\ = \;& k^{0^2} - \omega_k^2 + 2i\epsilon\omega_k + \epsilon^2 && (16) \end{aligned}$$

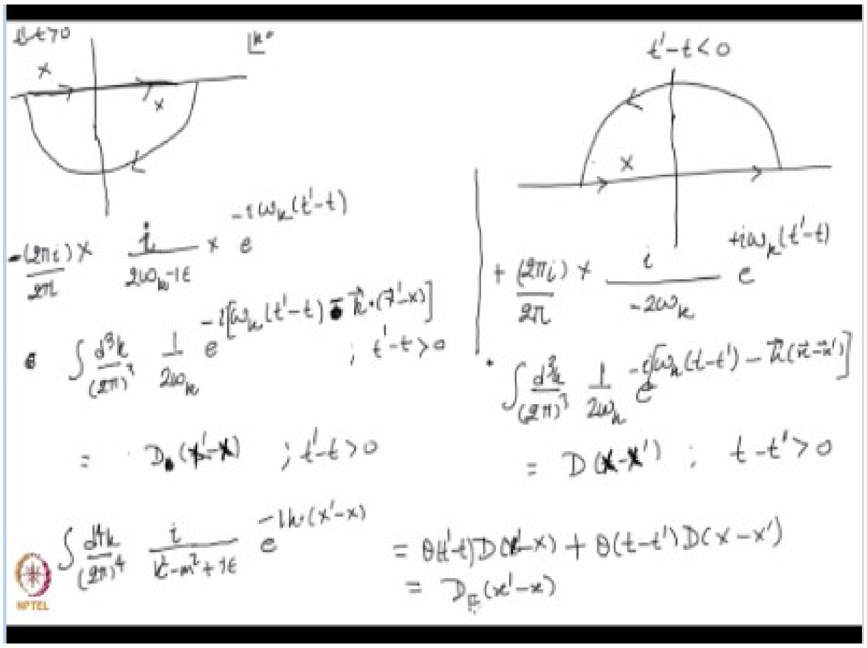

Figure 5: Refer Slide Time: 33:56

As ϵ is vanishingly small we can write 2ϵ as ϵ and $2\omega_k$ is absorbed

$$= k^{0^2} - \omega_k^2 + i\epsilon \tag{17}$$

So, I have d k 0 i over and of course, d cube k is also there. And all the factors of 2 phi, k 0 square – omega k square + epsilon, and e to the – i k dot – x. And the situation is this; have a pole here omega k – epsilon and then you have a pole there, omega k + i epsilon. And what is integration contour for k 0? It is the real axis that is this is where you have to do the integral the, this is from minus infinity to plus infinity.

But now, I can do the integral. Now, it would be nice if I could close the contour because if I could close the contour, let us say like this and then take the limit where the radius of this semicircle goes to infinity; in that infinite limit I will have a; the contour will be from minus infinity to plus infinity and a infinite semicircle closing it but I can do so, without the changing the value of the integral only if the contribution that I get from the semicircle is 0.

Where is the contribution from the semicircle is 0, I can add it to my integral and the advantage would be that then the result you can easily get using Cauchy's theorem and all you have to do is just find out the value of the residue at this pole and multiply it by 2 phi times 2 phi i. But now, whether you should close it like this or you should close it below in the lower half plane, how do you decide that? So, there are 2 issues.

How do I decide which way to close? And second whether I can really close it because I have to ensure that the value of the integral from this semi circles, this infinite semicircle is 0. So, let me make it. Let me repeat what I am trying to say. I am saying the integral over this, this real axis I have drawn here; the integral over this is same as integral over this plus integral over the semicircle provided the contribution you get from the semicircle is 0.

If it is not 0, then you cannot write this 2 as same. If contribution from this infinite semicircle, infinite many infinite radius vanishes. So, let us see. What I will do is: first I wish I had not run this but anyway. So, first let us consider a situation where t prime is greater than t or t prime – t is greater than 0, same thing. Let us look at this situation. Then the k 0 integral is the following.

You have integral d k 0 over 2 phi i over k's what were the;, these are the poles. So, you have k 0 – omega k + i epsilon times k 0 –; sorry here I should have put a minus. There is a minus and then you have a – omega k – i epsilon. And then on the top of it, you have e to the – i k 0 t prime – t and then you have e to the i k dot x prime – x and then you have d cube k over 2 phi cube.

So, I have written down everything now. Looks correct. Now, recall. Now, because we are taking t prime – t to be greater than 0. Let me just seeing the notes. So, now, what I want to do is: I want to use Jordan's lemma. So, you will recall that if you have an integral over a semicircle and it is f f of z e to the – i alpha z dz where alpha is a real number and positive, then and where the semicircle is in the lower half plane.

So, you are closing the contour in the lower half plane like this. Then if your function f of z vanishes fast enough is like 1 over mod of z, then this integral is gives you 0. So, this integral vanishes if alpha is greater than 0 and you close the contour in the lower half plane that is important. If this integral will given to you, you cannot close the contour into upper half plane that will not work. So, that you that will not vanish.

And this is what is very useful for us because look at the situation here. Instead of z, what we have is k 0, we are in the complex plane of k 0 and t prime – t is what is alpha here which is positive. So, if alpha is positive, you have e to the – k 0 times whatever here and the function of z here, you can see that this goes fasters 1 over mod of k 0; it actually goes as 1 over k 0 square.

Then this integral over k 0 is going to vanish over the semicircle in the lower half plane if you take that semicircle to be of infinite radius which I can put as limit are going to infinity where C R is the semi circle of radius R. So, I will use this and already it is clear what I should, I mean, already we have answered our both the questions where I should close whether upper half plane or lower half plane and whether the integration over the semicircle is 0.

So, I should close in the lower half plane for t prime – t to be; when t prime – t is positive and yes, the integral vanishes as you can see from the Jordan's lemma. So, what I have here is that the integral which was; let me go to the next page, this is right.

No poles at contour of integration

Closing the contour, using the big circle, to use Cauchy's theorem if contribution from semicircle to be zero

Let's do the integration

$$\int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} e^{-ik\cdot(x'-x)} \tag{18}$$

For $t' - t > 0$

$$\int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{ie^{-ik^0(t'-t)}e^{-i\vec{k}\cdot(x'-x)}}{[k^0 - (-\omega_k + i\epsilon)][k^0 - (\omega_k - i\epsilon)]} \tag{19}$$

We use Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{-i\alpha z} dz,\ \alpha > 0 \tag{20}$$

Integral vanishes

So, let me give just a second. You see, what we want to do; wanted to look at was an integral over k 0 and it was on the real axis. Now, I have shown that if t prime is greater than t, then I showed, I can close the contour in the lower half plane because the contribution vanishes and this integral is same as this integral. And when I do so, remember there were 2 poles; 1 pole was there; 1 pole is here and pick up this one.

And the result will be what 2 phi times the residue of the pole here. So, let us find out or the residue, so 2 phi i times the residue. Now, note that the contour is in the clockwise sense, clockwise direction, which is negative sense. So, I will get a minus sign. So, let me put that minus sign already here. Now, we want to look at the residue. And residue is what; I have moved to this anyway.

So, what is the residue? I should put. So, this one is the piece which is giving the pole. So, I should put in the k 0 here, omega k – i epsilon. So, if I do that, I will get omega, I will get, let me write say omega k – i epsilon and then you have minus, minus plus another omega k, so 2 omega k's, and again, a minus i epsilon. So, you get again, a minus i epsilon, so you get 2 omega k – i epsilon.

So, you get 1 over 2 omega k. But you still have the exponential factor. This anyway does not get. So, here, you will get e to the – i and the other, we were looking at omega k. So, you get e to the – i omega k t prime – t. So, what do I have then? I have not this, if I write the full integral, I had d cube k over 2 phi cube. Remember that and then you have, let me see if I am making any sign mistakes. I am making not that, of a some mistake, but I am missing the factor of i and this factor of 2 phi.

So, i is here and we also have a 1 over 2 phi. So, I have d cube k over 2 phi cube and then i into i is – 1; minus miuns is plus and 2 phi cancels. So, I have 1 over 2 omega k and I can leave out the – i epsilon e to the – i omega k t prime – t and remember that we have to take t prime – t to be positive. Does that sound familiar? We have seen that before. This was exactly, again I am missing factor plus you still have the k dot x prime – x; this factor, I should bring from here and of course, there is a mistake here. This is the correct one.

This is the familiar result and this is exactly what you had for D F t prime – t and because this is only the result when t prime – t is greater than 0. So, that is good. Now, let us look at the situation where t prime is less than t which is t prime – t is less than 0. Then the Jordan's lemma tells you that you have to close in the upper half plane because the sign of alpha which was t prime – t is negative now, so, you should be closing here.

The sense of the contour is positive. So, I will this minus sign will not appear and I will have the following. Again, 2 phi i times the residue and residue is i over; let us go back here; now, look at this, we are going to pick up this pole, this one. So, pole is here. So, let us see what is the residue? So, this one is the one which is giving pole and this one is the one which will give you a residue. So, k 0 will, you put – omega k and – omega k from here gives you – 2 omega k and this one, let us come in.

So, – 2 omega k; the sense is positive and let us look at exponential factor. It was – omega k so, get + i omega k t prime – t. And then of course, you have the factor of 2 phi coming from here these 2 and the factor of i which I already have taken. So, my result of the integral is integral d cube k over 2 phi cube and then I substitute all this, i square gives you a – 1.

And then there is a minus sign here so, which cancels and you get 1 over 2 omega k and this is e to the – i omega k t – t prime minus and I should bring that piece and x prime x – x prime

$$-\frac{(2\pi i)}{2\pi} \times \frac{1}{2\omega_k - i\epsilon} \times e^{-i\omega_k(t'-t)} \tag{21}$$

$$\times \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i[\omega_k(t'-t)-\vec{k}\cdot(x'-x)]} \quad ; t'-t>0$$

$$= D_F(x'-x) \quad ; t'-t>0 \tag{22}$$

Case when $t' - t < 0$

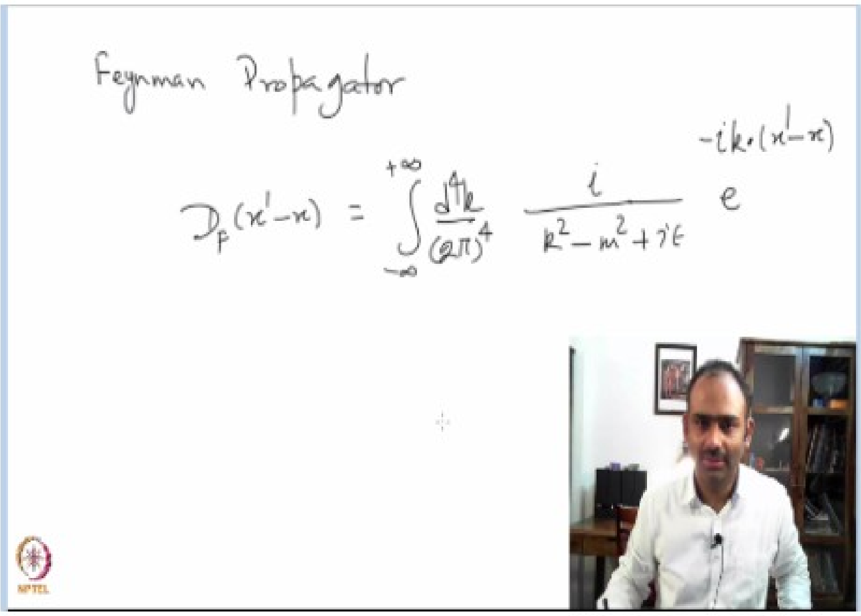

Figure 6: Refer Slide Time: 44:20

$$\begin{aligned} & \frac{(2\pi i)}{2\pi} \times \frac{i}{-2\omega_k} \times e^{-i\omega_k(t-t')} && (23) \\ & \times \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i[\omega_k(t-t') - \vec{k}\cdot(x-x')]} \quad ; t' - t > 0 \\ & = D_F(x - x') \quad ; t - t' > 0 && (24) \end{aligned}$$

So, I brought that piece also here and this you remember that this is the D; not D F. This is D. D of t – t prime and t is greater than t prime which means that the integral, I am looking at; this integral which I was interested in calculating. This is following.

Did I call it something? I should have given it a name but it is okay. Let me write it down. So, integral d 4 k over 2 phi 4 i over k square – m square + i epsilon e to the – i k dot x prime – x is equal to that is equal to D of not t prime – t, I should write x prime – x and this one should be x – x prime. This is D of x prime – x but only when t prime is greater than t which means I put a theta of t prime – t and plus if t prime is less than t, then this one should be the result.

So, I can ensure that by putting theta of t – t prime D of x – x prime and this is a familiar result to us which means, you recall that this is just the propagator. So, what we have shown is that this integral which for curiosity, we started looking at it exactly turns out to be the Feynman propagator.

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x'-x)} &= \theta(t'-t)D(x'-x) + \theta(t-t')D(x-x') && (25) \\ &= D_F(x'-x) && (26) \end{aligned}$$

So, what I have shown is that D F x prime – x or maybe we can do it a bit nicely. I have shown that Feynman propagator is the following. D F x prime – x is integral d 4 k over 2 phi to the 4 i over k square – m square + i epsilon times e to the – i k dot x prime – x. And what the i epsilon is doing is just telling you exactly how you have to close the contour. And as I have explained here. So, we will stop at this point and we will continue the discussion further in the next video.