Introduction to Quantum Field Theory

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Lecture 14 : Theory of Scalar Fields

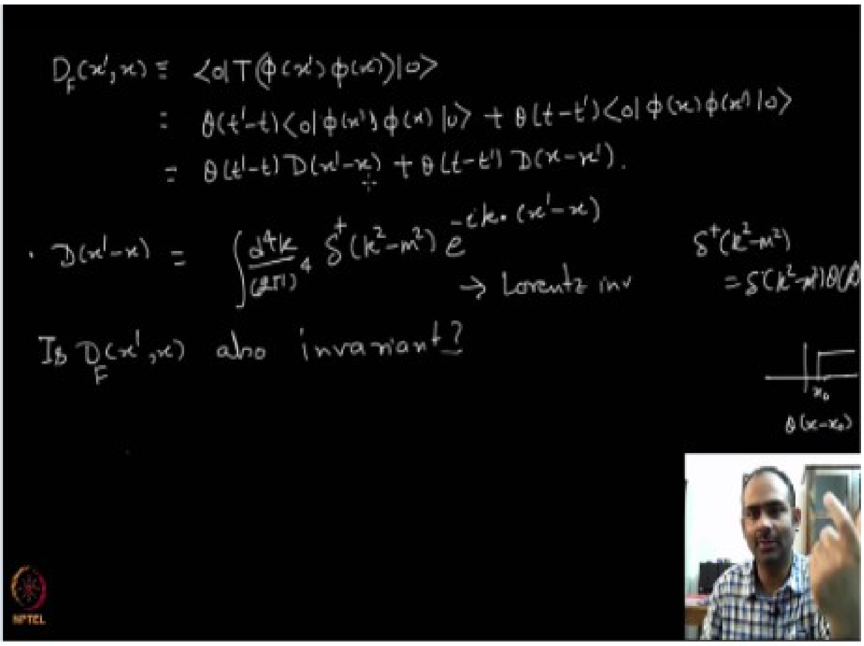

Figure 1: Refer Slide Time: 00:13

Last time, we had defined this propagator, which is basically the vacuum expectation value of the time order product of 2 fields phi x prime and phi x. The fields are at 2 different space time points and this, we can write as we also saw last time as theta of t prime – t phi of x prime phi of x + theta of t – t prime phi of x phi of x prime and this quantity, we called D of x prime x.

And we found that it was translationally; it was invariant under translations and that is why we had put x prime – x + theta of t – t prime D of x – x prime. Also, we saw that D of x prime – x is D 4 k over 2 phi to the 4 and then we had a delta function with the delta + k square – m square and then we had e to the – i k dot x prime – x. Well, let me remind you that delta + is basically k square – m square is just a shorthand notation for writing theta of k 0, which will this quantity is non vanishing only when you have the sign of k 0 was positive.

Theta is the unit step function, this one. This, you have if theta of x – x 0 and this is the point x 0. In this case, this is theta of sorry, what I meant to draw is this, this goes to infinity, extends to infinity and this is the point x 0 and this is theta of x – x 0, some point x 0. That is the step function. And we also saw that this object D x prime – x is Lorentz's invariant and it is easy to see.

So, if you do a transformation of x prime – x to some lambda of x prime – x meaning you do a Lorentz transformation and go to another frame, so, here this will go to lambda of x prime – x, but then you can do a change of variables and go from k to lambda k. If you do so, this will be k dot x prime – x. This will go to this way, this object x prime – x has anyway gone to lambda times x prime – x and k will go to lambda k.

And this you know, it is a Lorentz's invariant object. So, this will come out to be x prime k times x prime – x and then we saw that these pieces are anyway invariant. So, k square remains unchanged when you go from k to lambda k and D 4 k also remains unchanged. So, you can re-label the variables and again get back the same equation. So, this we saw that this is Lorentz's invariant. Let me remove this.

Now, what do you want to ask is whether D F x prime x, this is also a Lorentz invariant object that is the question is. First, let me write down Lorentz invariant. And our question is, is D x the propagator, D x prime, x also invariant? Again, that is what we are trying to do in this video. What happened? Oh, that is strange. Now, back to normal. So, that is the goal.

And what I want to do first is look at D x prime – x that is what I want to look at and see what it is explicitly when the interval between x prime and x is space like.

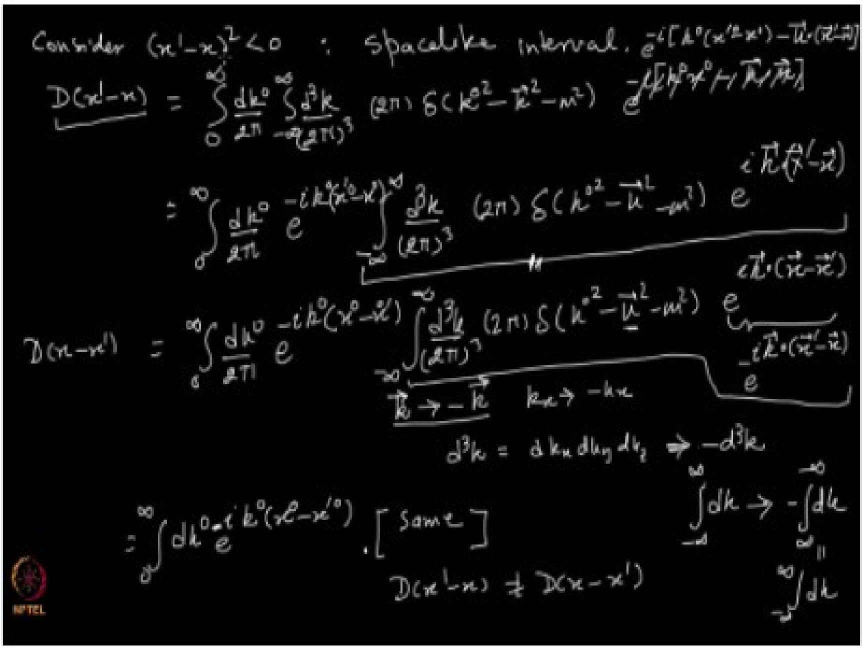

Figure 2: Refer Slide Time: 06:10

The Feynman propagator

$$
\begin{aligned}
D_F(x',x) &= \langle 0|\, T(\phi(x')\phi(x))\,|0\rangle \qquad (1)\\
D_F(x',x) &= \theta(t'-t)\,\langle 0|\,\phi(x')\phi(x)\,|0\rangle \;+\; \theta(t-t')\,\langle 0|\,\phi(x)\phi(x')\,|0\rangle \qquad (2)\\
D_F(x',x) &= \qquad (3)
\end{aligned}
$$

$$D(x'-x) = (2\pi)\int \frac{d^4k}{(2\pi)^4}\delta^+(k^2-m^2)e^{-ik\cdot(x'-x)} \qquad (4)$$

$$\delta^+(k^2-m^2) = \delta(k^2-m^2)\theta(k^0) \qquad (5)$$

Where $\theta(x - x_0)$ is the unit step function defined as

$$\theta(x - x_0) = 1 \quad , \quad x > x_0 \tag{6}$$
$$\theta(x - x_0) = 0 \quad , \quad x < x_0 \tag{7}$$

Is $D_F(x' - x)$ also lorentz invariant?

Consider $(x' - x)^2 < 0$: spacelike interval

$$D(x' - x) = \int_0^\infty \frac{dk^0}{2\pi} \int_{-\infty}^\infty \frac{d^3k}{(2\pi)^3} (2\pi)\delta^+(k^{0^2} - \vec{k}^2 - m^2) e^{-i(k^0\cdot(x'^0 - x^0) - \vec{k}\cdot(\vec{x'} - \vec{x}))} \tag{8}$$
$$= \int_0^\infty \frac{dk^0}{2\pi} e^{-ik^0\cdot(x'^0 - x^0)} \int_{-\infty}^\infty \frac{d^3k}{(2\pi)^3} (2\pi)\delta^+(k^{0^2} - \vec{k}^2 - m^2) e^{-i\vec{k}\cdot(\vec{x'} - \vec{x})} \tag{9}$$

So, consider x prime – x square to be less than 0 that is we are looking at a space like separation. And then let us see what happens to this subject. I had missed 2 phi to the 4 in the previous expression let me put it here. So, here we had a 2 phi factor also. Let me write it down. So, we have here this thing D 4. Let me write d k 0, d cube k that is our d 4 k.

And I am splitting the factors of 2 phi like this and the integration limits are from minus infinity to plus infinity and here also from minus infinity to plus infinity in both the cases. Then you have 2 phi delta of k 0 square – 3 vectors square – m square and then you have theta of k 0. So, that I can use here and put the limit from 0 to infinity because now, you have a support only from k 0 = 0 to infinity.

So, if your k 0 is less than 0, all the integral integrand vanishes. So, that is why I have put the limit to be 0 here. And then you have e to the power – i k dot x, which is k 0, x 0 minus the dot product of the 3 vectors. That is fine. Let me write one more step, e to the – i k 0 x 0 d cube k. Not necessary that I do this, it is looks a little bit neater. So, that is what we have. Now, I want to look at D of x – x prime.

So, I am changing the order. Now, if I change the order, then I will have just like what I had here integral 0 to infinity, D k 0 over 2 phi e to the – i k 0. Something is wrong. So, here I made a mistake. So, this is e to the – i k 0 x prime – x 0 – k dot x prime – x. That is what we have not this and here also, this will be x prime 0 – x 0 and this will be x prime – x. So, let us look at this one now.

x 0 – x prime integral these limits are minus infinity to plus infinity 2 phi cube 2 phi, same thing again and then we get e to the minus; e to the i k dot x vector – x prime vector, just the order is interchange; exactly the same thing, but this piece is; if I interchange the order, I can write it as e to the – i k dot x prime – x. So, the argument here the x prime – x is same as here.

The only difference between these 2 is minus i. That is fine now. What I can do is: I can do a change of variable and go from k to – k, which means that k x goes to – k x; k y goes to – k y; k z goes to – k z. And if you do that, then your d cube k which is basically dk x dk y dk z will go to – d cube k and the integration limits will go from instead of minus infinity to plus infinity, they will get reversed.

They will go from infinity to minus infinity but then you have overall minus sign here and each integral limit you can interchange again because using this. So, let me that explicitly. So, dk if you take k to going into – k, it will become – dk and infinity minus infinity, you can take this infinity and interchange the order. So, which is same as minus infinity to plus infinity dk.

So, you get back the same thing. So, under k going to – k replacement, this remains the same, this remains the same because this is squared. This gets changed; you get a + sign here which is what you had earlier for d x prime – x. So, it becomes the same factor like this. So, from here to there, this place, you get exactly the same if you change the variable from k to –k. So, this piece will be identical because this will change.

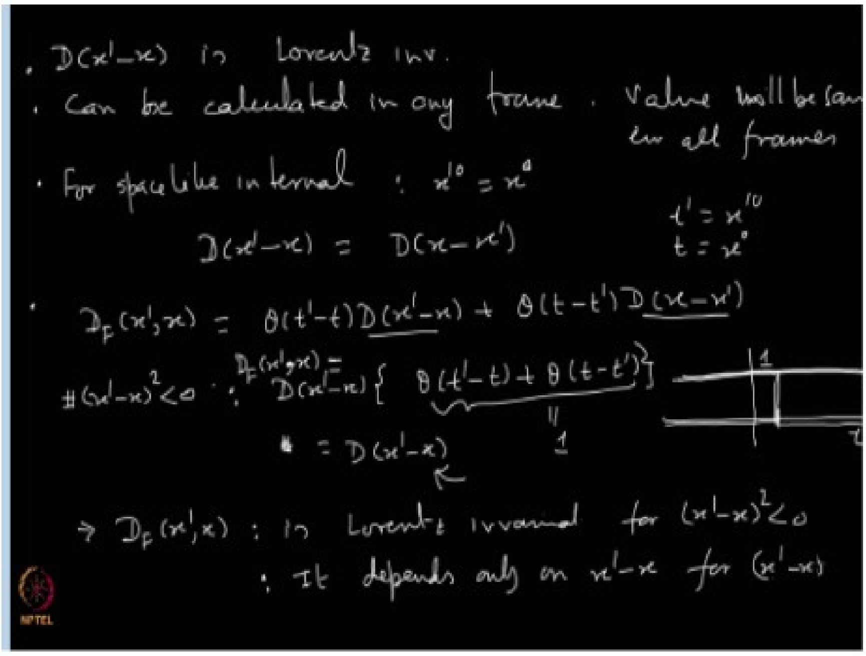

Figure 3: Refer Slide Time: 16:18

So, let me write it down. So, you get 0 to infinity dk 0 e to the i k 0 x 0 – x prime 0 into a piece which is same in the same as above; same as this one. I say in general D x prime – x and D x – x prime are not same. They are not same because of the difference here. Here, I cannot take k 0 to – k 0 and get the same thing as here that is not possible. That is not going to happen. there is a minus here. So, you see, you have x prime 0 – x 0; here you have x 0 – x prime 0. So, there is a difference in sign here. But now the limits are not from minus infinity to plus infinity. They are from 0 to infinity. So, that makes a difference, which means that D x prime – x is not equal to D x – x prime in general, that is the general statement.

$$D(x-x') = \int_0^\infty \frac{dk^0}{2\pi} e^{-ik^0\cdot(x^0-x'^0)} \int_{-\infty}^\infty \frac{d^3k}{(2\pi)^3}(2\pi)\delta^+(k^{0^2}-\vec{k}^2-m^2)e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \tag{10}$$

If we change sign of $\vec{k}$

$$\begin{aligned} \vec{k}\to-\vec{k}, \quad & k_x \to -k_x \\ d^3k \;=\; & dk_x dk_y dk_z \to -d^3k \\ \int_{-\infty}^{\infty} dk \;\to\; & -\int_{\infty}^{-\infty} dk = \int_{-\infty}^{\infty} dk \end{aligned} \tag{11}$$

But now, we are going to use the fact that we have already shown this object to be a Lorentz invariant object, meaning I can calculate this in any frame, whichever I wish and the value would be the same in all other frames. So, that is what I am going to utilise to make a statement about what happens when x prime – x is space like. Now, if the separation between 2 space time points is space like; it means that allows you to go to a frame where they happen simultaneously.

There is no notion of which event happened first and which event happened later, if the events are separated by a space like interval. So, I can go to a frame where t prime is same as t. This is what I call t prime and this is what I call t. I should have used that that would have been a bit neater, but anyway. So, I go to a frame where t prime is t or x prime 0 is equal to x 0. So, let us do that. So, go to D x prime – x is Lorentz invariant can be calculated in any frame and of course, the value will be same in all the frames and because this is space like; I am choosing space like an interval,, I am looking right now only at space like intervals. So, not because, let us say for space like, I can always set x prime 0 is equal to x 0. So, I will go in such a frame. And that is useful because if I go into such a frame, then this thing is 0, x 0 – x prime 0 is equal to 0.

And the same thing here. So, the piece which was giving a difference just drops out so it becomes C to the one which is sorry, e to the 0 which is one. So, we get then integral 0 to infinity dk 0 over 2 phi, this is one and then you have this piece which is same in both expressions. So, you see that for space like interval, D x prime – x is same as D x – x prime, but this is of course, not true for time like separation between x prime and x, because then you cannot set this to 0.

If they are not at the same time in some frame, then they are not going to be at the same time in any frame if they are separated by time like interval. But for space like interval, that is good, we can show this equality. And now this will be useful to make some statements about the D F of x prime x. Now, let us write it back again. This was theta of t prime – t. So, let me write down t prime is x prime 0 and t is x 0.

And we had D of x prime – x + theta of t – t prime D of x – x prime. So, let us take first the case where x prime – x is space like; the interval is space like, so for such a case, this and this, they are equal. And if they are equal, I can just take them as common. And then what I have here is theta of t prime – t + theta of t – t prime. And what is that? Some that is unity that is one because you say here, so, let us choose this to be t and choose t prime to be something.

So, you see, one function gives you unit when you are below this, the other theta gives you; this one gives you this; one is giving you this, the other theta is giving you this. So, the sum of the 2 is 1 all over. So, this is identically equal to 1 which means D F so, this is equal to; this is; so, for experiment is x square less than 0, D F x prime x is equal to D of x prime – x.

Now, because this object is Lorentz invariant, it means that D F x prime x is also Lorentz invariant if the interval is space like and because this depends only on the difference of x prime and x. D F the propagator also depends only on the difference between x prime x and this we have shown for space like separation. So, let me write down. So, we conclude here the D F x prime x is Lorentz invariant because this object is for x prime – x square less than 0.

And also it depends only on the separation x prime – x because on the right hand side, the dependence is only on x prime – x and not x prime and x individually. So, this has to be true for the propagator also, this; what we have shown .

Now

$$D(x-x') \;=\; \int_0^\infty \frac{dk^0}{2\pi} e^{-ik^0\cdot(x^0-x'^0)} \int_{-\infty}^\infty \frac{d^3k}{(2\pi)^3}(2\pi)\delta^+(k^{0^2}-\vec{k}^2-m^2)e^{-i\vec{k}\cdot(\vec{x'}-\vec{x})} \tag{12}$$

- $D(x-x')$ lorentz invariant
- Can be calculated in any frame, value will be samein all frame

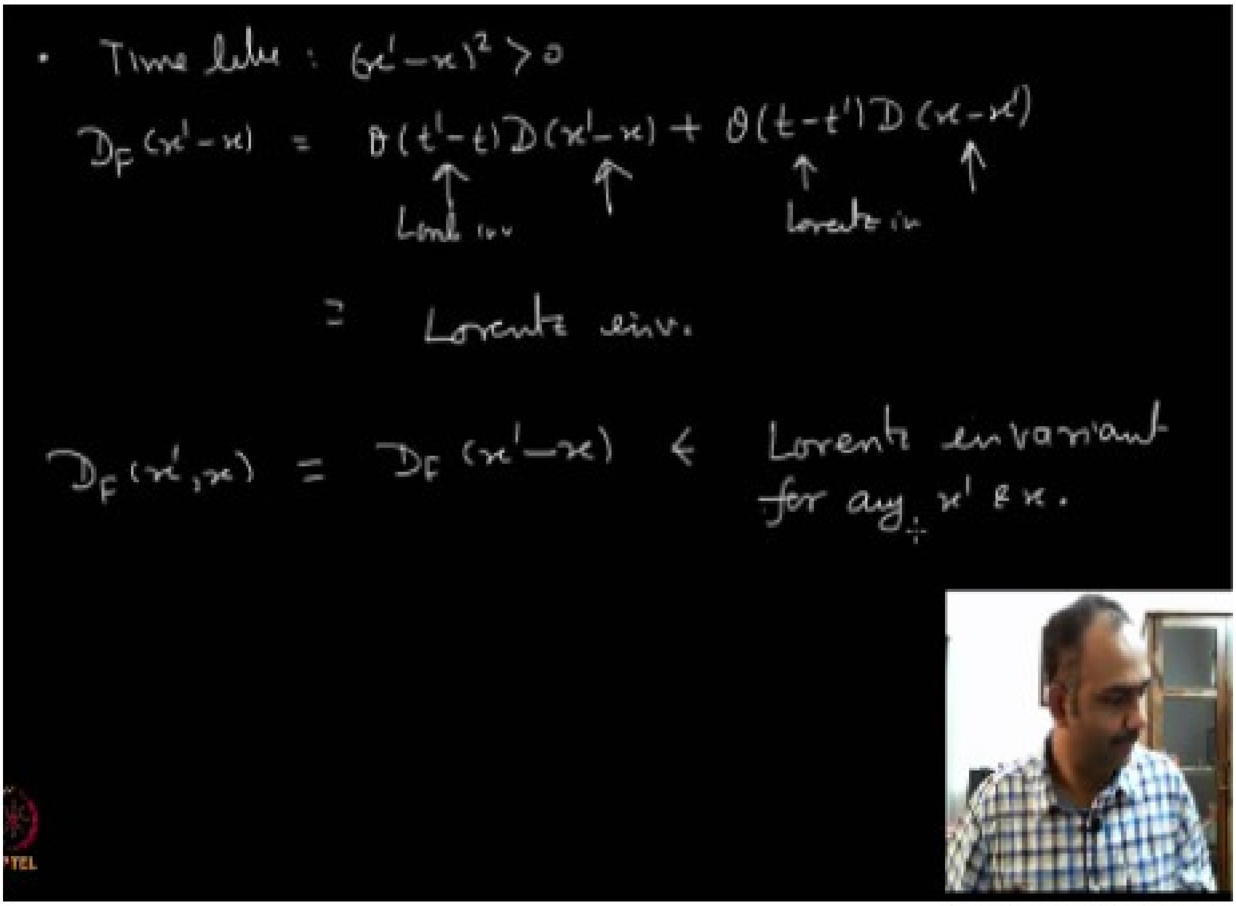

Figure 4: Refer Slide Time: 22:38

- For space-like interval

$$D(x - x') = D(x' - x)$$

Now, let us ask what happens when the separation between x prime x is time like, greater than 0. Now, let us see you have D F x prime – x is equal to theta of t prime – t D of x prime – x + theta of t – t prime D of x – x prime. Now, this is Lorentz invariant. This is Lorentz invariant. This; we shown already irrespective of whether the separation is time line like space like. This is always Lorentz invariant.

This object, this one and this one, these 2 are also Lorentz invariant. If the separation is timeline, remember the sin of t prime – t is invariant if they are separated by time like separation. So, this is also Lorentz invariant. This is also Lorentz invariant which means that the right hand side is Lorentz invariant. So, what we have shown is the D F and also here you see that it depends only on the separation between x prime and x and not individually.

$$D_F(x', x) = \theta(t' - t)D(x' - x) + \theta(t - t')D(x - x') \tag{13}$$

$$\text{if } (x' - x)^2 < 0 \quad ; \quad D_F(x', x) = D(x' - x)\{\theta(t' - t) + \theta(t - t')\} \tag{14}$$

$$D_F(x', x) = D(x' - x) \tag{15}$$

$D_F(x', x)$ is lorentz invariant for $(x' - x)^2 < 0$, it depends only on $(x' - x)$ for $(x' - x)^2 < 0$

Consider time-like interval : $(x' - x)^2 < 0$

$$D_F(x', x) = \theta(t' - t)D(x' - x) + \theta(t - t')D(x - x') \tag{16}$$

The RHS in above equation is Lorentz invariant

$$D_F(x', x) = D_F(x' - x) \tag{17}$$

Lorentz invariant for any x and x'