Introduction to Quantum Field Theory

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Lecture 13 : Quantization of Klein-Gordon Theory continued (6)

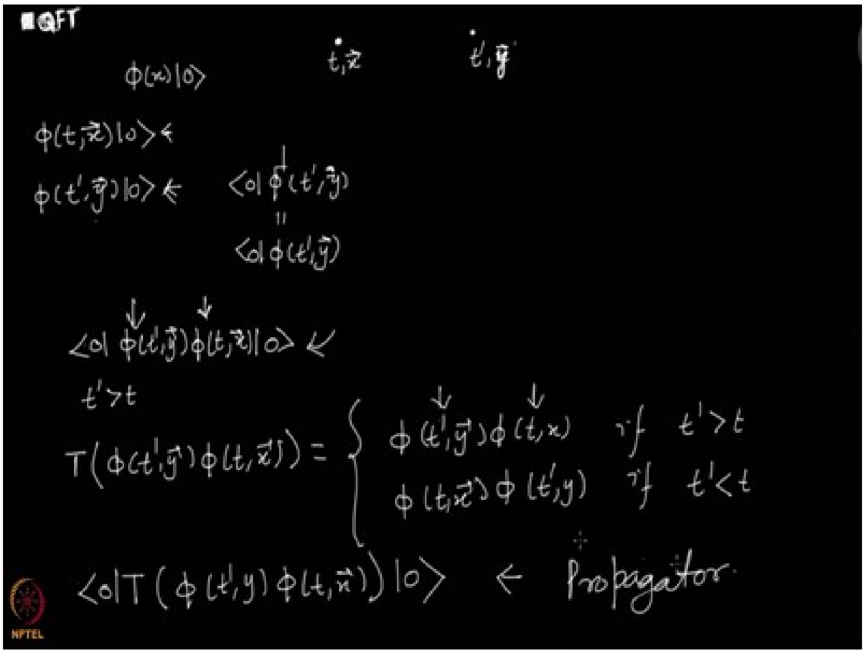

Figure 1: Refer slide time 00:25

In last video we learn how to create a particle at point x by operating with the field phi of x on the vacuum and the free theory. So if I write it more specifically this has created a particle at time t which is located at x. Suppose you want to have a particle which is located at t at position x and this particle, we want to travel to another location and reach at t prime position x prime or y then we ask what is the probability amplitude that a particle located here at time t will appear at time t prime at this location.

So that is the object we want to calculate. So let us create a particle then at location y at time t prime and that you can obtain by this. I want to know the probability amplitude of this state appearing as this state there I have to calculate the inner product of these 2 states so first I turn this ket into a bra, which is done by writing this but now remember that we are still dealing with real scalar field, which means the phi dagger is same as phi.

So this subject is again same as this. The probability amplitude for this state of this particle traveling to that location at time t is given by the following. So, you have 0 corresponding to the vacuum then phi dagger t prime which is same as phi t prime y, actually you should read like this. So you start from the vacuum you create the particle at t at x and then you are looking at the inner product with this state.

So, this is object that you want to calculate and O and here I am saying that t prime is greater than t. So that your particle is propagating from here to there another way to see the same object is this that you start from the vacuum state. This phi creates the particle and this phi when it acts on this state. It kills that particle back the vacuum. That is how you can also interpret the subject. Now I want to calculate such objects but to ensure that the field appears here or to the left has the time which is always greater than the field.

The time component the time label of this period to the; right so to do that, I will introduce a notation which is denoted t. And when I take t and act on t prime phi y and phi of t prime x then what it gives you is phi of t prime y phi of tx if t prime is greater than t. At the time argument here is bigger. Then this time argument then you write it this way writet it this in this fashion. So if t prime is greater than t and if it so happens that the prime is less than t, this is smaller time than the symbol t does is it reorders everything. So, it write phi of tx phi of t prime y, t prime is less than t.

So I just change the order of the field and remember this change matters because phi is an operator. It has it is not just a number it is quantum operator. It has a unique dagger. So the order matters. So here are the definition of t is. So if I am looking at a propagation amplitude or a propagator this is propagated then by propagator I mean this subject t prime is greater than t. So, generally if I do not want to worry about whether this is correctly time ordered then I will define output. So, I put t here phi of t prime y phi of tx. And this symbol t will always put the time the field with higher time to the left and it will always represent propagation from a lower time to a later time and this is what is called as a propagator.

$$\phi(x)\,|0\rangle \qquad\qquad t,\vec{x} \qquad\qquad t',\vec{y} \tag{1}$$

$$\phi(t,\vec{x})\,|0\rangle \leftarrow \tag{2}$$

$$\phi(t',\vec{y})\,|0\rangle \leftarrow \tag{3}$$

To get the inner product,

$$\langle 0|\,\phi^{\dagger}(t',\vec{y}) = \phi(t',\vec{y}) \tag{4}$$

$$\langle 0|\phi(t',\vec{y})\phi(t,\vec{x})|0\rangle \qquad\qquad \text{t' > t} \tag{5}$$

I will now suppress writing explicitly the time and space components I will just write it like this. And I will denote this as y, x this argument here is the first argument and second argument is there I am reading it this way. So, that is the final propagator. Now let us calculate the subject and it is not difficult. But before that I let me introduce some more notations. So this subject I can write as 0 phi of y and phi of x 0 times theta, theta is the unit step function.

And it is saying y 0 is greater than x 0 is the time component of this one y here and x 0 is the time component of x. So, it is saying that this one is rightly ordered that is what you expect from the definition if you recall what we wrote just here plus if the time ordering is not correct then it becomes phi x, phi y, theta of x 0 – y 0. This time is the access the one which is at a later time then this step function contributes and this one will not contribute then and because then this will give you 0.

This gives you non-zero only when y 0 is greater than x 0. So let me remind you theta of y 0 – x 0 is 1 if y 0 is greater than x 0, 0 otherwise, so this is the unit step function you have and

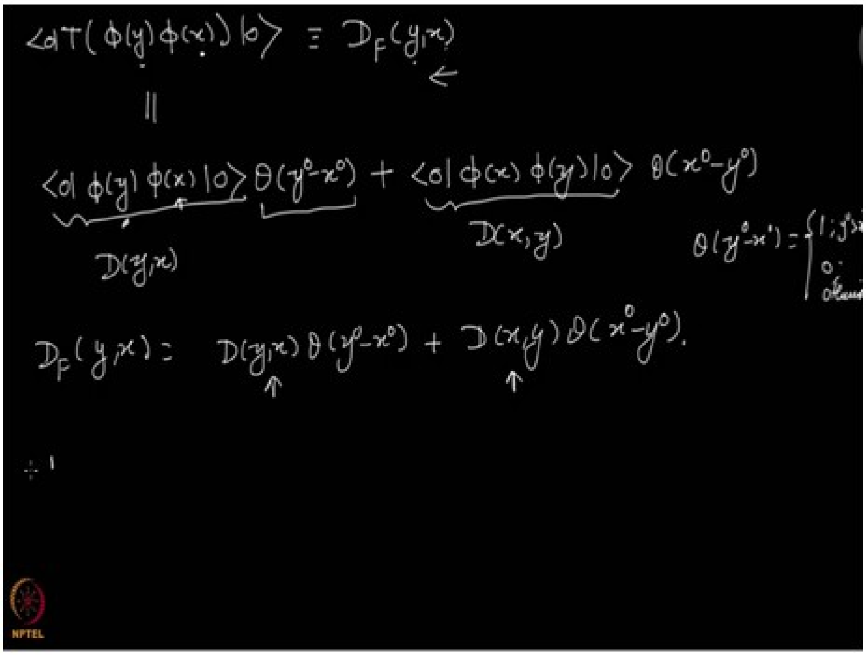

Figure 2: Refer slide time:07:27

I will denote this object by D of y x and that is a same object with the arguments reversed. So what we have here is the final propagator D F is equal to D of y x theta of y 0 - x 0 plus D of x y theta of x 0 - y 0 that is good. So I want to calculate final propagator is that is what is the probability amplitude for a particle to travel from point y to point y?

And I can do so by first calculating these objects so I will calculate xy now. So we will calculate D xy and let me see if I missing I want to say something. Let us go ahead.

$$
\begin{aligned}
T(\phi(t',\vec{y})\phi(t,\vec{x})) &= \phi(t',\vec{y})\phi(t,\vec{x}) \qquad \text{if} \;\; t' > t && (6)\\
T(\phi(t',\vec{y})\phi(t,\vec{x})) &= \phi(t,\vec{x})\phi(t',\vec{y}) \qquad \text{if} \;\; t' < t && (7)
\end{aligned}
$$

$$
\langle 0|T(\phi(t',\vec{y})\phi(t,\vec{x}))|0\rangle \leftarrow \qquad \text{Feynman propagator} \tag{8}
$$

$$
\langle 0|T(\phi(\vec{y})\phi(\vec{x}))|0\rangle = D_F(y,x) \tag{9}
$$

$$
\begin{aligned}
D_F(y,x) &= \langle 0|\phi(\vec{y})\,\phi(\vec{x})|0\rangle\;\theta(y^0-x^0) + \langle 0|\phi(\vec{x})\,\phi(\vec{y})|0\rangle\;\theta(x^0-y^0) && (10)\\
D_F(y,x) &= D(y,x)\,\theta(y^0-x^0) + D(x,y)\,\theta(x^0-y^0) && (11)
\end{aligned}
$$

Perfect, now this is easy to calculate all we have to do it substitute the expression for phi of x and if you remember let me go back and see if I can find it, yeah here. This object we have already phi x of acting on the vacuum this is this object. So, let me take this expression d cube k 2 pi 3 half 1 over root omega k and you have a and a dagger and a comes with the –ik dot x so that is what we will put there.

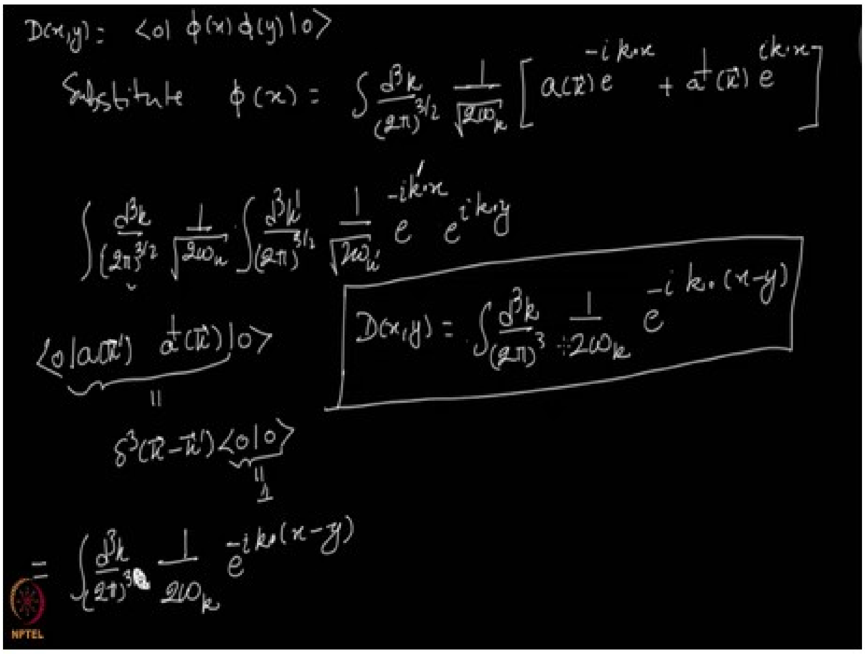

Figure 3: Refer slide time 11:22

So in this substitute phi of d cube k 2 pi 3 half 1 over 2 omega k and then we have a of k - i k dot x k ik dot x that is what we will substitute in here now. Let us check this is all current? Yes it is correct. Now the same thing we have been doing earlier this phi a dagger so the a will kill the vacuum only a dagger will survive and when this one is acting on the bra, the a dagger will kill the vacuum and only a will survive.

Let me write down then we have d cube k over 2 pi 3 halves 1 over to omega k and then a kills the vacuum and only a dagger k survives. So I put that a dagger k here a dagger k and then it will ket on the vacuum. I have the ik dot x remaining. So, let me write down here and not x but y because it is y here it is k dot y. Now phi acting on this one, so I will write as d cube k prime over 2 pi 3 halves 1 over 2 omega k prime then it will be the a of k and a of k will shift to the left here but not k but k prime.

And this piece will give us to the e to the –ik dot x so that is what we will have. Now remember what ak a dagger k prime is this will give you using the commutation relation you can interchange the order of these 2 and in that case a shift to the right and it will kill the vacuum here so that time goes away and what you will be left with is the delta cube of k – k prime. In case if you recall the commutation relation.

So, if you substitute that in here this piece will give you delta cube k - k prime and this is any way 1 and this is normalised to unity. So you get a delta cube k – k prime. Which means I can write this as the following; I will do an integral over k prime and use up this delta function which I have. So when I do integral over k prime I can use of the delta function. In doing so wherever k prime appears I should replace it by k.

So let us do that. So, d cube k over 2 pi 3 halves. This is this piece and I have 1 over 2 omega k then they have 2 pi 3 halves here and that makes a 2 pi 3 by 2 combined with this one and make to pi cube let me remove this and 2 omega k prime will turn into 2 omega k these 2 square roots will combined to give 2 omega k and something wrong here in this place. I should I put a prime here. And these 2 for k prime remember. I have to substitute k here it becomes e to the –ik dot x - y, this is the expression for D of xy.

So, let me write it down here. This is d cube k 2 pi cube 2 omega k. So using this I can easily calculate the Feynman propagator. Because Feynman propagator D F is just the same object with argument interchanged and 2 theta functions here that is good. Now we should be asking about everything here. How does the object D xy transform under Lorentz Transformation this one thing you should I ask ok and also note that another thing that it is not really a function of x and y independently as a function of x – y.

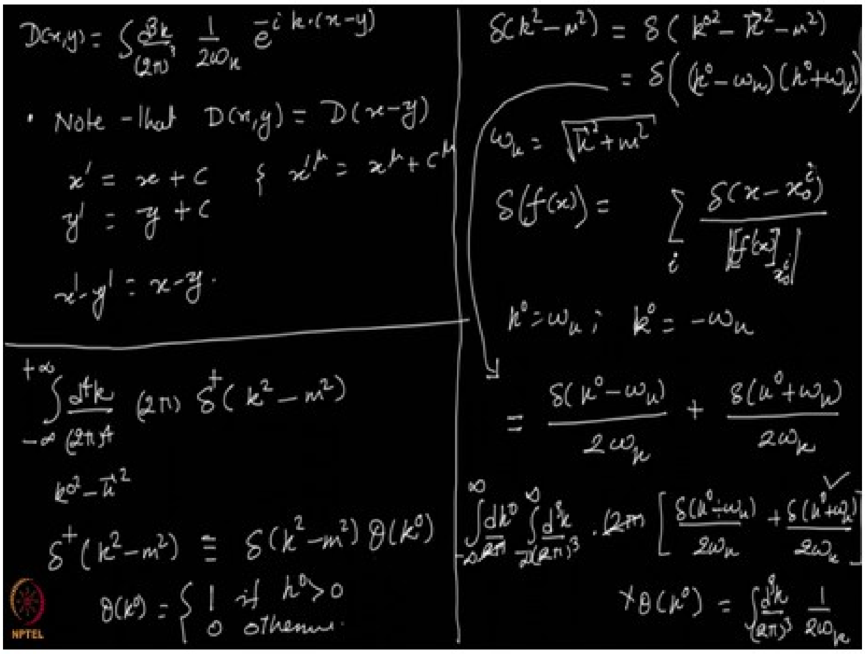

Figure 4: Refer Slide Time: 19:03

Where,

$$\begin{aligned} \theta(y^0 - x^0) &= 1; y^0 > x^0 \quad (12) \\ &= 0; \text{otherwise} \quad (13) \end{aligned}$$

$$D(x,y) = \langle 0|\phi(\vec{x})\, \phi(\vec{y})|0\rangle \tag{14}$$

Substitute

$$\phi(x) \;=\; \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \Big(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \Big) \tag{15}$$

$$\int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \int \frac{d^3k'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega'_k}} e^{-ik'\cdot x} e^{-ik\cdot y} \tag{16}$$

Using

$$\langle 0|a(\vec{k'}) a^\dagger(\vec{k})|0\rangle = \delta^3(\vec{k} - \vec{k'})\, \langle 0|0\rangle \tag{17}$$

which gives

$$\langle 0|\phi(\vec{x})\, \phi(\vec{y})|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-ik\cdot(x-y)} \tag{18}$$

Thus,

$$D(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-ik\cdot(x-y)} \tag{19}$$

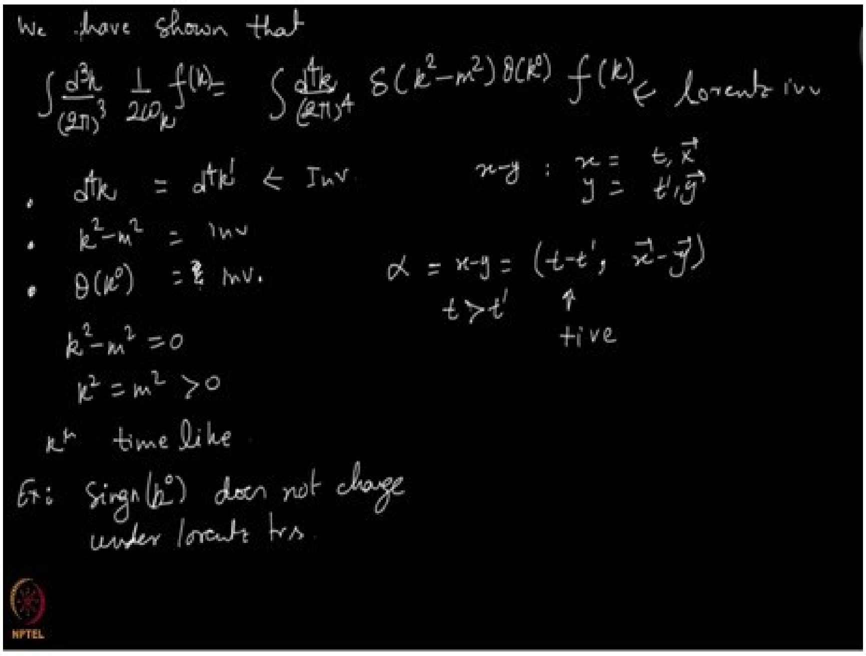

Figure 5: Refer Slide Time: 30:31

Let me say it in a little more words let me write here again the D xy as d cube k over 2 pi cubed 1 over 2 omega k let us check; that is correct. So, first note that D xy is actually not a function of x and y independently but it depends on the difference x - y. And which means that you have this result is invariant under translation. So that means if you instead of using x and y if you used x prime which is x + c what I really mean here is x prime mu is x mu + c mu where c mu is a constant.

See these are 4 numbers c 1, c 2 and c 3 and I take y prime and also shift by the same amount. So I am doing the translation and then you see the x prime - y prime is same as x – y. So does not matter whether you use x - y or x prime - y prime your result will be the same so that the function of x – y. Now what I will show you next is that this object is also invariant under Lorentz transformation not so manifest from here but will see that how to show this at least at some level and most I will leave it as an exercise, but I will sketch a proof. So this is fine. What I have now is so let now look at another object, so let us forget whatever I have done for a moment and look at a completely different and I will show you some connection with what we have here on the slide and object is the following. I want to look at the following integral d 4 k meaning I have integral not only over the three components but 4th is 0 the component also over 2 pi to the 4 and you have 2 pi here then you have a delta half k square – m square.

This K is K square is k 0 square - k vector square vector means 3 vector here. So, this object and I will put a plus symbol on this so I define the following define delta plus of k square - m square definition is this it is k square - m square the delta function times theta of k 0. By theta I mean the unit step function. So this is non-zero and equals to 1 when k 0 is positive if k 0 is negative this object is 0. So, theta of k 0 is equal to 1 if k 0 is positive 0 otherwise.

Integral over the integral over k 0 has support only when k 0 is positive. If k 0 is negative, but you doing the integral the integrand vanishes. So, here I will put plus or if you wish you can put k theta it is the same thing. So, I am interested in this integral and the limits of integration and running for all the 4 of them from minus infinity to plus infinity. So let us look at this subject now. So, we will first manipulate a little bit the delta function.

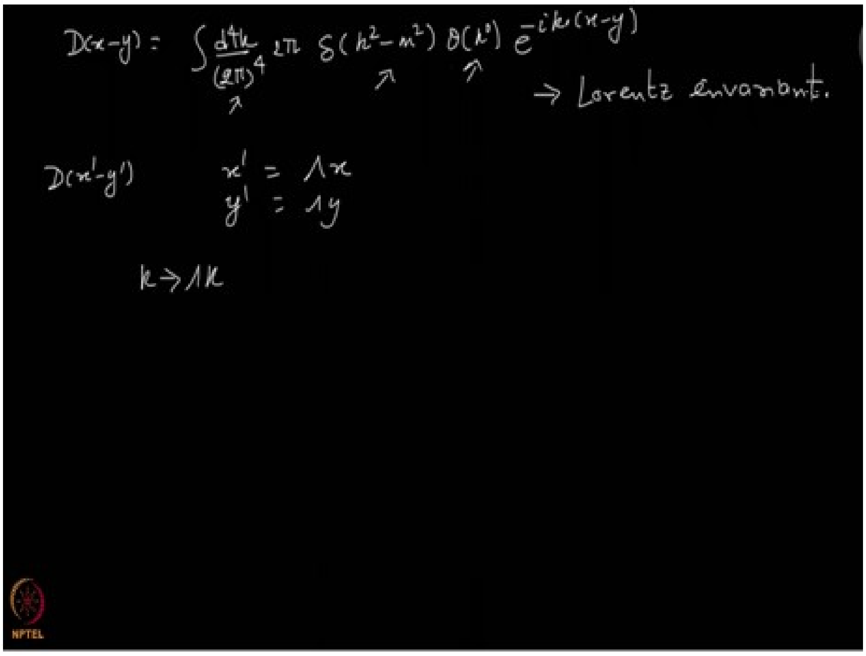

Figure 6: Refer Slide Time: 38:49

And before I do that let us locate delta of k square - m square this is same as delta of k 0 square – k 3 vector square - n square which you can write as delta half k 0 minus this write as k omega k and k 0 + omega k. Remember your Omega k is. So this is true. Now you will use the following take a delta function and if you are looking in the delta function of delta of this f of x then this is equal to you have to calculate delta of x - x 0 i.

So all the places where the function goes 0 where f of x is equal x i 0 are the solution of f of x and then you have to divide by the derivative of f and calculate it x i 0 and take the modulus x is modulo function and then put this to make it and then you have sum over all of i's. So here it is quadratic for us and it means that there are 2 solutions one is omega k and one is minus Omega k.

Note that $D(x,y) = D(x-y)$

$$\begin{aligned} x' &= x + c \qquad \{x'^{\mu} = x^{\mu} + c^{\mu}\} && (20) \\ y' &= y + c && (21) \\ x' - y' &= x - y && (22) \end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}(2\pi)\delta^{+}(k^2 - m^2) \tag{23}$$

$$\delta^{+}(k^2 - m^2) \equiv \delta(k^2 - m^2)\theta(k^0) \tag{24}$$

where,

$$\begin{aligned} \theta(k^0) &= 1 \text{ if } k^0 > 0 && (25) \\ &= 0 \text{ otherwise} && (26) \end{aligned}$$

$$\begin{aligned} \delta(k^2 - m^2) &= \delta\Big((k^0)^2 - \vec{k}^2 - m^2\Big) && (27) \\ &= \delta\Big((k^0 - \omega_k)(k^0 + \omega_k)\Big) && (28) \end{aligned}$$

where we have,

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \tag{29}$$

And we have used the property of δ fucntion,

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_0^i)}{|f'(x)|_{x_0^i}|} \tag{30}$$

$$k^0 = \omega_k \quad ; \quad k^0 = -\omega_k \tag{31}$$

So our k 0, so, k 0 takes the values so x k i 0 or k 1 0 and k 2 0 which are too many 0's let me not write it they are 2 solutions for us k 0 is equal to omega k and k 0 is equal to minus omega k.

And if I use this property of delta function I can write delta of this as sorry k and both of them will have plus sign. I have taken modulus there should be a minus sign but once you take the modulus of it the minus and goes away. So you have now. So you can write this integral equal to sign so then I can write this integral as d k 0 over 2 pi then d cube k over 2 pi cube and then we have 2 pi here and delta plus of this thing is delta of k 0 - omega k over 2 omega k plus delta of k 0 plus omega k over 2 omega k times theta of k 0.

Now let us have a look at this, this one this delta function click's only when k 0 is equal to omega k this delta function clicks only when k 0 is minus Omega k that is in k 0 is negative because omega k is positive but then the theta of k 0 is saying that the integral has support only when k 0 is positive. So if only k 0 positive is allowed and this one click's only when k 0 is negative then it means this one does not contribute to this we can drop.

This term this gives the vanishing contribution. We are left with only this piece and this is now redundant because this is saying k 0 is positive anyway both of them, but this is not adding any information. So if I look at this, what do I have 2 Pi cancels and I can do the integral over k 0 and that will give integral over k 0 and there is only this delta function. So that goes away and that gives you 1 and we are left with the following you have integral d cube k over 2 pi cube this piece.

Then you have 1 over 2 omega k and delta function has been used and there is no point in writing this because that the phi k 0 is positive is already insured by the argument of this Delta function because k 0 has to be omega k and that is positive. So we see that this object is equivalent to this object. So, let me write it on to the next slide. This object is equal into this object. Let me write it down on the next slide more nicely.

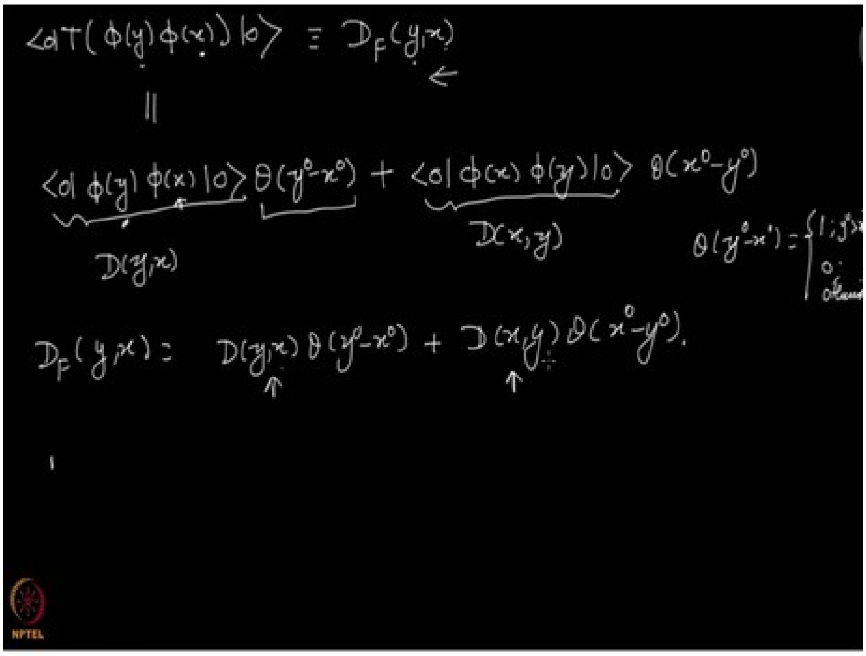

Figure 7: Refer Slide Time: 41:30

$$\delta\Big((k^0-\omega_k)(k^0+\omega_k)\Big) = \frac{\delta(k^0-\omega_k)}{2\omega_k} + \frac{\delta(k^0+\omega_k)}{2\omega_k} \tag{32}$$

$$\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\left[\frac{\delta(k^0-\omega_k)}{2\omega_k} + \frac{\delta(k^0+\omega_k)}{2\omega_k}\right]\theta(k^0) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} \tag{33}$$

We have shown that,

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}} = \int\frac{d^4k}{(2\pi)^4}\delta(k^2-m^2)\theta(k^0) \tag{34}$$

- $d^4k = d^4k'$
- $k^2 - m^2 =$ invariant
- $\theta(k^0) =$ invariant

What we have shown, we have show that as that integral d cube k over 2 pi cube 1 over 2 omega is equal to integral d 4 k over 2 pi to the 4 and then what delta of k square - m square but I should put a plus here and where the plus means I have a theta k 0 that looks good or maybe this time I will remove the plus and put it k theta 0 same thing. Now this one is manifestly Lorentz invariant and let us see how? They are three pieces one is your d 4 k, second is the argument of delta function, which is k square + m square 3rd is of k 0.

So what we should show is argue is this thing is invariant and which you can do in the following manner k square + m square is a Lorentz invariant quantity anyway because it is k mu k nu this is scalar anyway m square is a scalar this is a Lawrence invariant object. The d 4 k is also Lorentz invariant. You should do this exercise so that this is same as d 4 k prime. So, when you are doing Lorentz transformations you will get this by the time in proper orthognal Lorentz transformations.

And the theta of k 0 how about this one? Is this also Lorentz invariant? This is also Lorentz invariant because see here k square or the k is a time like vector why because the delta function will click only when k square - m square is equal to 0 or this is a positive number. So the k you have k mu you that you have is time like. Now show that for time like vector k mu. Sign of k 0 that does not change the Lorentz Transformation.

$$
\begin{aligned}
x &= t, \vec{x} && (35)\\
y &= t', \vec{y} && (36)\\
x - y &= (t - t', \vec{x} - \vec{y}) \text{ for } t > t', \;\; t - t' \text{ is positive} && (37)
\end{aligned}
$$

$$
\begin{aligned}
k^2 - m^2 &= 0 && (38)\\
k^2 &= m^2 > 0 && (39)
\end{aligned}
$$

k^μ is timelike

Exercise: Sign(k^0) does not change under Lorentz transformation

Means if cannot was positive it remains positive or if it was negative it remains negative. So, if it was positive and if it remains positive then with me is theta of k 0 which is 1 when k 0 is greater than 0 remains the same under Lorentz Transformation which theta of k 0 will also be Lorentz invariant object. So please show this but I will give you one way to think why should be this way.

So think of a vector let us call it; what is call it; so look at this x - y sum 2 space time points x and y and I am looking at the difference. And let us say my x is and y is t prime y. So if you look at the time component of the subject it will be so x - y will be t - t prime and x – y. Now if x - y, let us call it. Let is call it alpha, if Alpha is a time like vector meaning if the separation between the 2 x and y is time like then you know that you can if it is time like then the t and t prime there is an order the natural not natural there is a well-defined notion of which event happened earlier and which event happened later.

So there is a clear idea about which one is first and which one is second let us say t is the one which is bigger than t prime meaning the event x has happened later than the event y you know that if you have a time like a separation of intervals between 2 space time point which one has happened first and which one has happened later cannot be changed by doing Lorentz transformation. So, in all the frames in all frames of reference will always find that x happened later than y.

Which is not the case if you have a space like interval in that case you can change the order in which the events happened. So if t which means that if t is greater than t prime which means that this object is positive no matter to which frame you go to this always remains positive ok and if this is true for the vector the 4 vector Alpha then the same will be true for any 4 vector which is time like. Because in this we are not using any other property other than Lorentz transformation and you know that the difference of space time intervals is the prototype for defining vector.

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}f(k) = \int \frac{d^4k}{(2\pi)^4}\delta(k^2-m^2)\theta(k^0)f(k) \tag{40}$$

Lorentz invariant

So if this vector follows this all other also have to follow this. So, that is the way to think about it. But of course you can do an explicit competition and show that sign of k 0 will not change under Lorentz transformations or more properly under proper orthogonal Lorentz transformation. I will return to these formations in more detail later, but for now we have concluding that this object is Lorentz transformation invariant it is a invariant quantity.

So if you multiply in here any other object f of k which is Lorentz invariant and this integral will be Lorentz invariance. I should be multiplying f of k both sides and from here it is clear then hat this object D xy that is Lorentz invariant because this piece I have shown that this is Lorentz invariant. And this function will be Lorentz invariant. Get to see that note here what I am saying is take x let me write it. So you have D of x - y is now d 4 k over to pi 4 and then I have a 2 Pi. And then I have a delta of k square - m square then I have a theta of k 0 and then I have e to the –ik dot x - y that is the correct. Now already I shown you that this is Lorentz invariant, this is Lorentz invariant that object is Lorentz invariant. Now we are looking at this. So the way we have to look at this is the following. So you calculate the x prime - y prime where x prime is lambda x and y prime is lambda y where lambda denotes the Lorentz transformation.

So, when you do so here you will get lambda of x - y, then your k integral is dummy so you can choose a different; you can do a change of variables and go to k going to lambda k if you do so then the entire object you can imagine that it being calculated in the same frame of reference. And then because this is Lorentz invariant. This will turn out to be a scalar. This will turn out to be a scalar. And under the change of variables that you are doing that is equivalent to doing a change of frame of reference.

This we have already shown that this will remain invariant. This will still remain theta of k naught prime. This will also be the same way it will become delta of k prime square - m Square and you already know that the 4 k is m d 4 k prime. So this object is clearly now Lorentz invariant.

$$D(x,y) = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2-m^2)\theta(k^0)e^{-ik\cdot(x-y)} \tag{41}$$

is Lorentz invariant

$$D(x'-y') \qquad x' = \Lambda x \tag{42}$$
$$y' = \Lambda y \tag{43}$$
$$k \rightarrow \Lambda k \tag{44}$$

Now our next task is to look at the Feynman propagator, which is written in terms of this Dx and Dy that we will do in the next video.