Introduction to Quantum Field Theory

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Lecture 12 : Quantization of Klein-Gordon Theory continued (5)

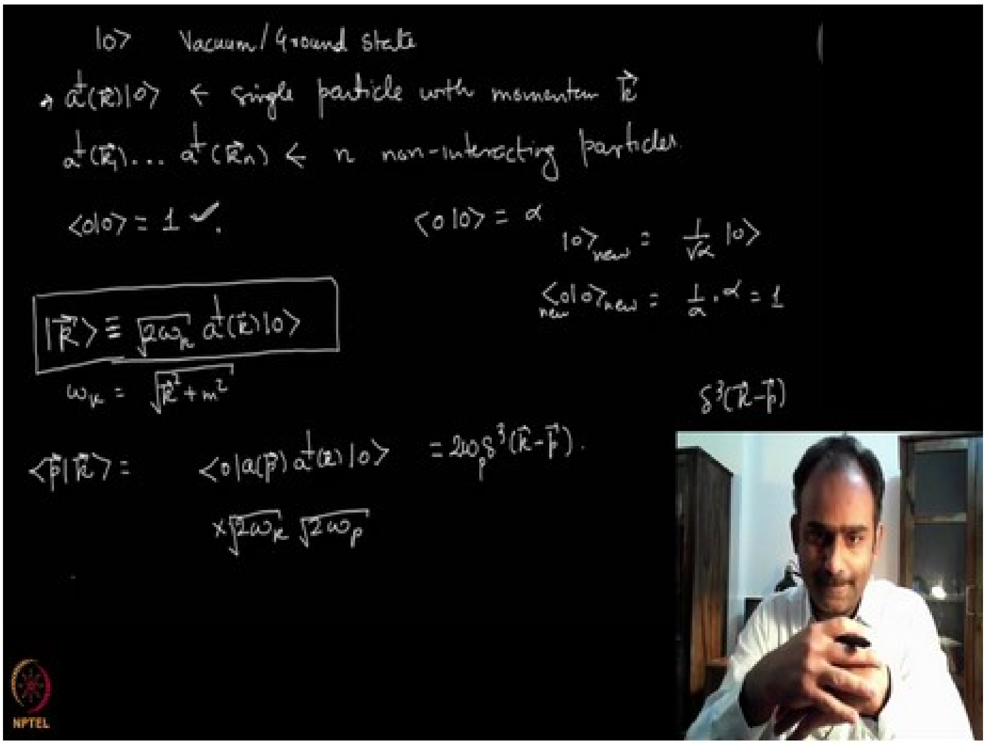


Figure 1: Refer Slide Time: 00:25

So last time we were discussing the following we had defined the vacuum of the free theory or the ground state of the free theory which has 0 energy and 0 momentum that we saw. And we also saw that if we act with a dagger k on this vacuum I get a single particle state with momentum k. And if I act with the string of a daggers let us say n of them then I generate n particles. Now I will normalize this state to 1.

So, I will say that this vacuum is normalized such that it is if you calculate the inner product you get 1 this you can always do. So, if let us say this was not one but some other constant let us call it alpha then you can always define a vacuum which let us call new is one over square root of alpha of the old one. So, that when you take the inner product of this new vacuum that you have defined this will be one over alpha times this will give you an alpha and which gives one.

So, you can always do that thing and which means that we are free to choose to normalize the vacuum like this. Now if you look at this thing which I am saying it is a single particle state if you multiply this thing with any function of k it still remains a single particle state with momentum k because what counts is this operator acting on the vacuum. What other function you multiply in front of it is immaterial and that freedom we can use to define what the normalization of single particle states we want to choose. So, I will choose the following. So, I will choose to define a single particle state with momentum k by this notation. So, if I write a k in this ket that means a single particle state and of course it has to have this and for the normalization I can choose some function here and that I choose 2 omega k square root of that where omega k is as before k square + m square that is a choice and not everyone makes this choice different authors will use different normalizations.

But you are free to choose whatever you like and you have to just consistently follow your normalization. So, this is what my definition of a single particle state would be with proper normalization. So, this we should remember. Now let us see if we choose such a normalization what do we get for an inner product of these 2 single particle states. So, if I take this then ket k is a dagger k acting on ket 0.

So, I have forgotten a 2 omega k let me write it down in the next line. Then if I take the p here it will give me a 2 omega p in the square roots and you will have instead of a dagger a of p and the vacuum again. So, this is what the expression is and if you take this quantity I want to bring a dagger to the right and that I can do by using the commutation relation and when it comes to the right it hits the vacuum on the right and kills it.

Equivalently the a dagger when it goes to the left it hits the vacuum to its left and kills it. So, whichever way you see it is going to kill it. So, I bring it here but in doing. So, I have to use the commutational relation and that gives me delta function another term with the delta function and only that delta function term survives and gives you the following. So, this will lead a delta cube of p - k delta q of k - p and only that term survives and that delta function you can use to set k is equal to p or p is equal to k up to u.

So, you will have this delta cube k - p and then you will have the vacuum from the left and the right and that gives you 1 because of this condition. So, that is 1 times 2 omega p that is what you get if you wish you can also write it as 2 omega k delta cube k - p you can do so, because you have a delta function here. So, whether you write p here or you write k here for the argument of omega it does not matter.

So, that is how our states are normalized. So, let me write it down here maybe I will later collect all these results. So, this is good these are single particle states with precise momentum defined. But we also want to see particles that are localized in space and of course you know if you try to localize the particle in space it will spread out in the momentum space. So, it will not have a fixed momentum but its momentum will be spread out.

So, we want to search for a particle which is localized at one point in space now how are we going to find that thing.

$|0\rangle \rightarrow$ Vacuum/Ground state

$$a^\dagger(\vec{k})\,|0\rangle \quad \leftarrow \text{Single particle with momentum}\,\vec{k}$$

$$a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_n)\,|0\rangle \quad \leftarrow \text{n non-interacting partciles}$$

$$\langle 0|0\rangle = 1$$

$$\langle 0|0\rangle = \alpha \tag{1}$$

$$|0\rangle_{new} = \frac{1}{\sqrt{\alpha}}|0\rangle \tag{2}$$

$$\langle 0|0\rangle_{new} = \frac{1}{\alpha}\cdot\alpha = 1 \tag{3}$$

$$|\vec{k}\rangle = \sqrt{2\omega_k}a^\dagger(\vec{k})|0\rangle \tag{4}$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \tag{5}$$

$$\langle\vec{p}|\vec{k}\rangle = \sqrt{2\omega_k}\sqrt{2\omega_p}\,\langle 0|a(\vec{p})a^\dagger(\vec{k})|0\rangle \tag{6}$$

$$= 2\omega_p\delta^3(\vec{k}-\vec{p}) \tag{7}$$

- Particle localised in space-time
  - Sum over momenta of single particle states of definite momentum
  - Transforms nicely under Lorentz transformation

$$\phi(x)|0\rangle = \int\frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right)|0\rangle \tag{8}$$

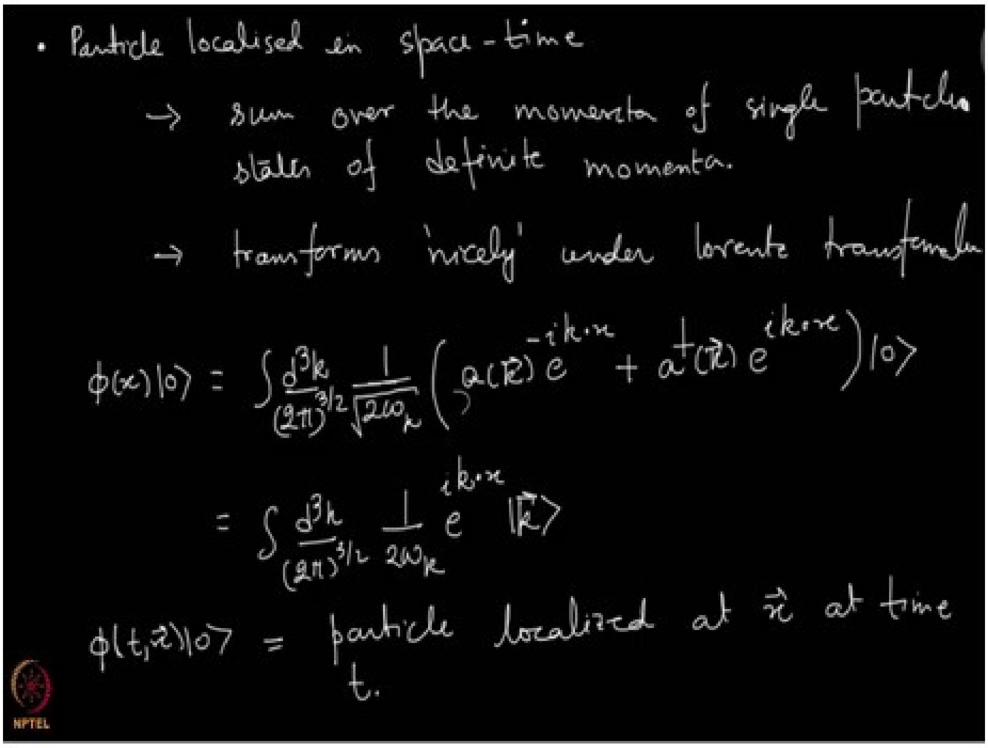
$$= \int\frac{d^3k}{(2\pi)^{3/2}}\frac{1}{2\omega_k}e^{ik\cdot x}|\vec{k}\rangle \tag{9}$$

So, let me write down what we are looking for is a particle localized in space and time also you may ask that the particle is here at this time. So, that is what I am asking now recall that you already know from quantum mechanics and what we are doing is quantum mechanics it is just quantum mechanics of the fields is that if you try to localize the particle at one place its momentum will completely spread out that we know which means that I will have to integrate over all the momenta.

So, I take single particle states of precise momentum but I have to sum over all of the momentum that is what we know and. So, this will let me write it down this will be a sum or basically integral over the momenta of single particle states single particle states this is what we will have to do also note that the state that you define should transform nicely under Lorentz transformation it should have a proper transform.

So, let me write that down also transforms nicely under Lorentz transformations. And so, we are looking for such a thing and of course the second point transform nicely in the largest transformation is a guide and of course the first point is also a guide to searching for how to define the particle localized in space time. So, let us look at what we have already seen if you look at phi x.

Let me act the operator phi on the vacuum of the theory and if you substitute the expression for phi of x which was d cube k over 2 pi 3 halves 1 over 2 omega k I hope I am writing the expressions correctly here and then we have a of k e to the - ik dot x remember this is a dot



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product of 2 four vectors. So, I am not putting arrow here these are 4 vectors + a dagger of k e to the ik dot x and this acts on get 0 that is what we have on the left.

Now of course this quantity what you have on the right transforms nicely under Lorentz transformations because phi of x is a scalar and also we assume that the vacuum remains unchanged under Lorentz transformations. So, this quantity on the left has a nice transformation property. So, our second criteria is definitely met but let us see whether this is a sum of single particle states of definite momentum and that is indeed the case because if you take this a and act on the vacuum it kills it.

So, that term is gone from this expression and what you are left with is just this a dagger k and a dagger k acting on vacuum is what gives you single particle states of definite momentum of momentum k. So, let me write this down y 2 now I will use the normalization that we have already given for single particle states here. So, a dagger k acting on vacuum gives you single particle state k divided by 2 omega k or whichever way you see.

I can then multiply and divide this thing by root 2 omega k which will give you 1 over 2 omega k and that root 2 omega k combines with a dagger k and k 0 to give you the state k single particle state of momentum k and you have e to the ik dot x. Now if you see that this is satisfying our both the requirements that we wanted and clearly this is representing a a particle that is located at point x at time t right.

So, this I mean if you are still not feeling very happy about what I am saying you can try to map it to what you know from single particle quantum mechanics and see that this is indeed in the limit where in the non-relativistic limit this indeed gives you what you have learned in your quantum mechanics course. So, it goes down it gives you the same thing apart from some overall factors of this omega k which you did not have there.

So, anyhow this is clearly representing a particle that is localized let me write x in or t, x. So, I hope it is clear that this indeed represents a particle localized at x at time t. So, that solves our search and this completes our search for how to write down the particle localized at a particular space time point. So, this also we should remember.

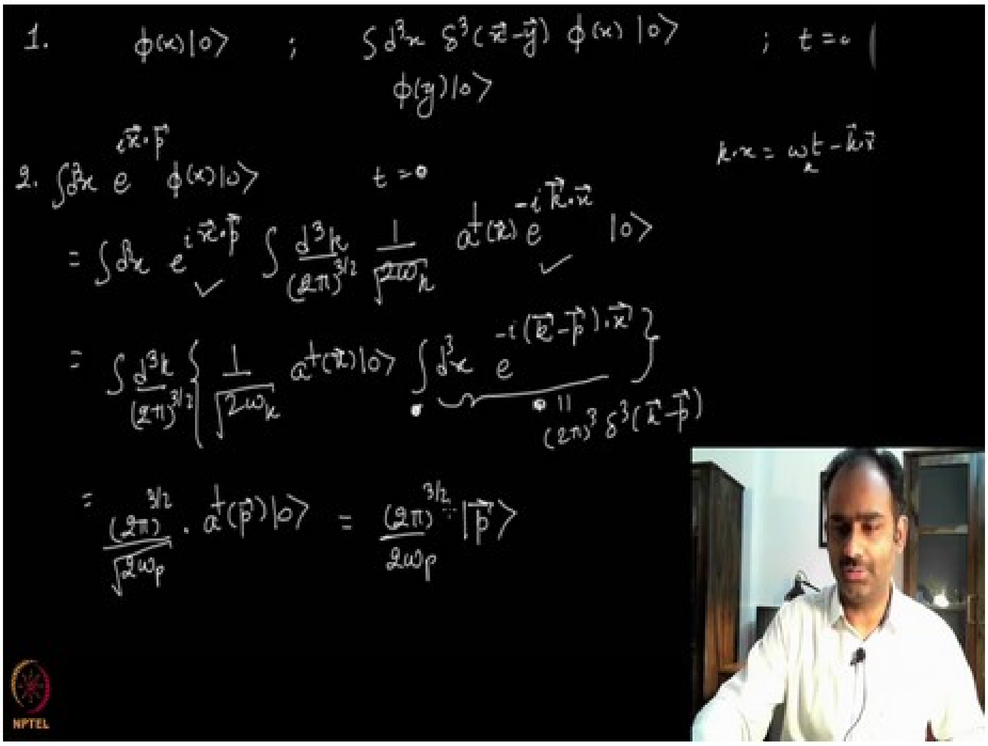
$\phi(\vec{x},t)\,|0\rangle$ = Particle localised at $\vec{x}$ at time t

1.

$$\phi(x)\,|0\rangle = \int d^3x\ \delta^3(\vec{x}-\vec{y})\phi(x)\,|0\rangle \qquad ;t=0 \tag{10}$$
$$= \phi(y)\,|0\rangle \tag{11}$$

2.

$$\int d^3x\ e^{i\vec{x}\cdot\vec{p}}\phi(x)\,|0\rangle = \int d^3x\ e^{i\vec{x}\cdot\vec{p}}\int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}a^\dagger(\vec{k})e^{-ik\cdot x}\,|0\rangle \tag{12}$$
$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left\{\frac{1}{\sqrt{2\omega_k}}a^\dagger(\vec{k})\,|0\rangle \int d^3x\ e^{-i(\vec{k}-\vec{p})\cdot\vec{x}}\right. \tag{13}$$
$$= \frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}\cdot a^\dagger(\vec{p})\,|0\rangle \tag{14}$$
$$= \frac{(2\pi)^{3/2}}{2\omega_p}\,|\vec{p}\rangle \tag{15}$$



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Let me do some very trivial and simple calculations to get some familiarity with now manipulating these quantities. So, something really simple suppose you are given a particle at point x and time t and you want to turn it into a particle at point y how would you do that you can just do an integral over d cube x insert a delta cube of x - y and then you have phi of x get 0 let us put t equal to 0 for the moment it does not matter.

So, that this is representing a particle located at x and when I do the integral it will turn it into phi of y that was too simple actually you could have just put y here that is the same thing. Now suppose I am given phi of x ket 0 and I want to from here I want to go to a particle which is localized in momentum. So, right now the particle is localized in space it is at x t I want to do something to it and turn it into a state which gives me a state of well defined momentum.

Let us say that momentum is p then I can do the following I can take the phi x 0 phi extracting on the vacuum and integrate over all of the space if I am integrating the single particle state over all of the space. So, I am taking it and you know delocalizing it in space and when I delocalize in space I expect to get a localized particle in momentum and us see what whether that has happened. So, here this is d cube x e to the ix dot p and let me again put t equal to 0.

So, that I do not have to carry around the omega k t term this and you have for from the phi d cube k over 2 pi 3 by 2 1 over 2 omega k and only the a dagger term survives from here. So, you have a dagger k e to the ik dot x is correct and I have put t equal to 0. So, I am putting arrows here and this is correct. Now I will combine this exponent exponential with this exponential and do the integral over x because I want to see how it looks like in the momentum space.

Now when I do that integral there should be a minus sign because here it has e to the ik dot x when you have four vectors and k dot x is omega t - k dot x let me write this as omega t - k dot x. So, you get a minus n because of this. So, this is fine. So, these 2 exponentials combine and give you delta function you know and then you still have the integral dx d cube x and e to the i or - ik - p dot x and this is what gives you 2 pi cube delta cube of x k - p.

Now I can integrate over yeah now I can integrate over k that will turn all the k's into p's. So, I will get and this of course 2 pi cube gets cancelled here. So, I have 2 pi 3 halves divided by 2 omega k but k is turned into p because of the delta function 2 omega p times a dagger k turns into a dagger p ket 0 which is if I multiply and divide by 2 omega p then I get 2 omega p times this of course I could have just instead of starting with this e to the ix dot p.

I could have multiplied inverse of this constant factor this factor which I am getting here in here and then I would have ended up with p anyway it does not matter it is the same thing. So, you see that by doing this you can get us a single particle state with precise momentum. So, I have been till now either talking about completely localized state in position space or completely localized state in momentum space but we would also like to construct states which are representing wave packets which have which mimic the real particles.

But which we see I mean there is nothing unreal about what I am saying about these things but typically when you are doing an exp doing an experiment you have an understanding that the particle is here roughly and has momentum this much roughly within the limits of uncertainty principle. So, that kind of notion we want to see here that is we want to construct the wave

where we have used

$$k \cdot x = \omega_k t - \vec{k} \cdot x \quad (16)$$

$$\int d^3x \; e^{-i(\vec{k}-\vec{p})\cdot\vec{x}} = (2\pi)^{3/2}\delta^3(\vec{k}-\vec{p}) \quad (17)$$

3.

$$\int d^3x\ e^{i\vec{x}\cdot\vec{p}}e^{\frac{-(\vec{x}-\vec{x}_0)}{\sigma^2}}\phi(x)\,|0\rangle \qquad\qquad t=0 \tag{18}$$

$$= \int d^3x\ e^{i\vec{x}\cdot\vec{p}}e^{\frac{-(\vec{x}-\vec{x}_0)}{\sigma^2}}\int d^3k e^{-\vec{k}\cdot\vec{x}}a^\dagger(\vec{k})\,|0\rangle \tag{19}$$

$$= \int d^3k \underbrace{\int d^3x\ e^{i\vec{x}\cdot(\vec{p}-\vec{k})}e^{\frac{-(\vec{x}-\vec{x}_0)}{\sigma^2}}}\, a^\dagger(\vec{k})\,|0\rangle \tag{20}$$

$$= \int d^3k \# e^{(\vec{p}-\vec{k})^2\#}\,|\vec{k}\rangle \tag{21}$$

the underbraced part is the gaussian which will be,

So, it will be of course something in between what I have done till now. So, this one when I took this particle which was completely localized and I integrated over entire space with a factor which was you know if you look at the mod square of this it is one everywhere right. So, you are really spreading it out uniformly throughout the space and that is why you got completely localized state in the momentum space.

So, instead of spreading it out iuniformly over the entire space I could spread it in such a manner that it has most of it is localized around some point and then the spread decays the moment you go slightly away from that position. So, that is what I want to do. So, this time instead of integrating e to the i what was that x dot p. Now I will not integrate with this factor but I will integrate with x - x naught squared over sigma square sigma you need something here to absorb the dimensions of a length or length that you have in the numerator because the exponent has to be a dimensionless quantity.

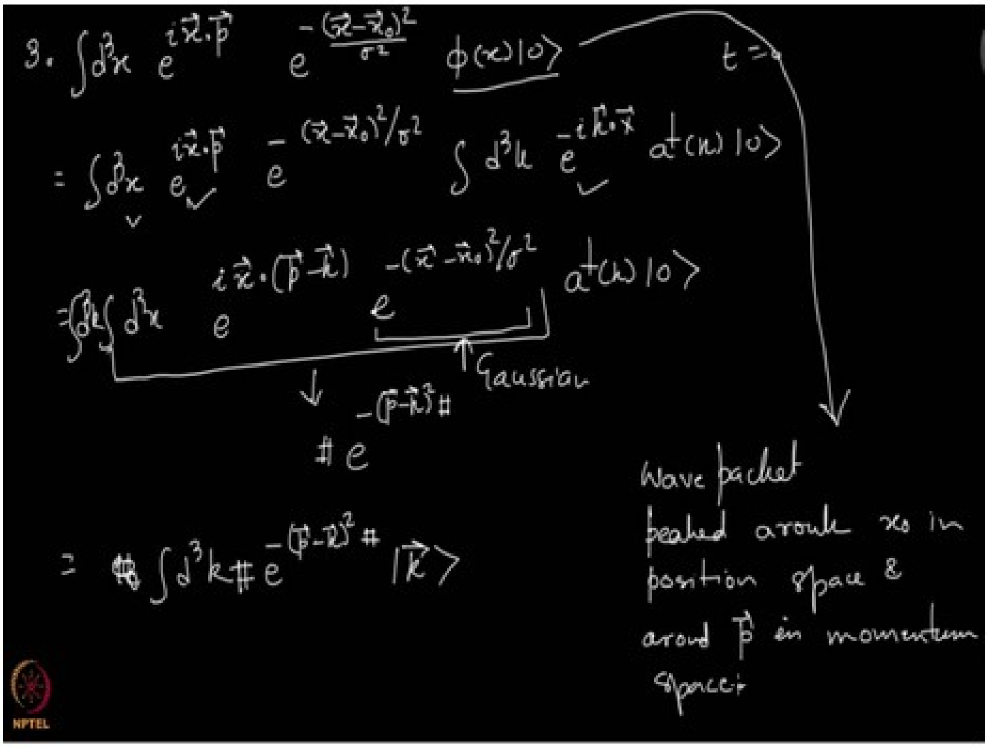
And this of course as you go away from x naught this piece is providing exponential damping. So, I take this factor and I integrate over all the space this quantity right that is what you will do to localize not look like but slightly spread around x naught because this one this piece was anyway localized. Now you are spreading it around now that is good. Now let us see what happens what I want to see now is that this indeed gives me a state which is also having a peak in momentum around some point and then it falls off that is what we want to see.

So, that is not difficult this guy is cube k let me not write 2 pi cubes and all those things I will also put t equal to 0 just to avoid writing many things then you get e to the - ik dot x that we have seen and you have a dagger k 0 now again I will do the same thing I will combine these 2 and then do the integral over x. So, you have integral d cube x e to the ix dot p – k but you also have x here unlike the previous case where it was not there e to the - x - x naught square over sigma square this is the Gaussian which you have its is this bell shaped curve.

And then you of course have d cube k let me write here and a dagger k ket 0. Now look at this space the x dependence is solely here this is just a Fourier transform from this is just a Fourier transform of this Gaussian function and you know that the Fourier transform of a Gaussian function is again a Gaussian function. So, this integral d cube x of this Gaussian is going to give you apart from some constants another Gaussian but this time it will be p - k whole square and of course some constants will appear here.

So, what you get is d cube k you can supply all these constants which I am omitting right now d cube k e to the - p - k whole squared over some factors which you can determine a dagger k ket 0 let me instead write k because anyway I am omitting some factors. So, I can just have supply of factors of omega k or root omega k and turn it into k. So, you see what you have here is again a sum of single particle states which is again a single particle state.

But then here because you're summing over all the momenta it is clearly spread out but because of the Gaussian which is peaked at the value p momentum p you the state which you have is really representing a wave packet which is peaked around momentum p and it spreads around and also from here in the beginning what we saw what the way we constructed it was that it is peaked around x naught



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$$\int d^3x \; e^{i\vec{x}\cdot(\vec{p}-\vec{k})} e^{\frac{-(\vec{x}-\vec{x}_0)}{\sigma^2}} \quad = \quad \#e^{(\vec{p}-\vec{k})^2\#} \tag{22}$$

$$\tag{23}$$

Wave packet peaked around x_0 in position space and around $\vec{p}$ in momentum space. So, this function this quantity it represents a wave packet peaked around or predefined around x naught in position space and around p in momentum space. That is I think all I wanted to say this time yes exactly. So, we will continue further in the next video.