Introduction to Quantum Field Theory

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Lecture 11 : Quantization of Klein-Gordon Theory continued (4)

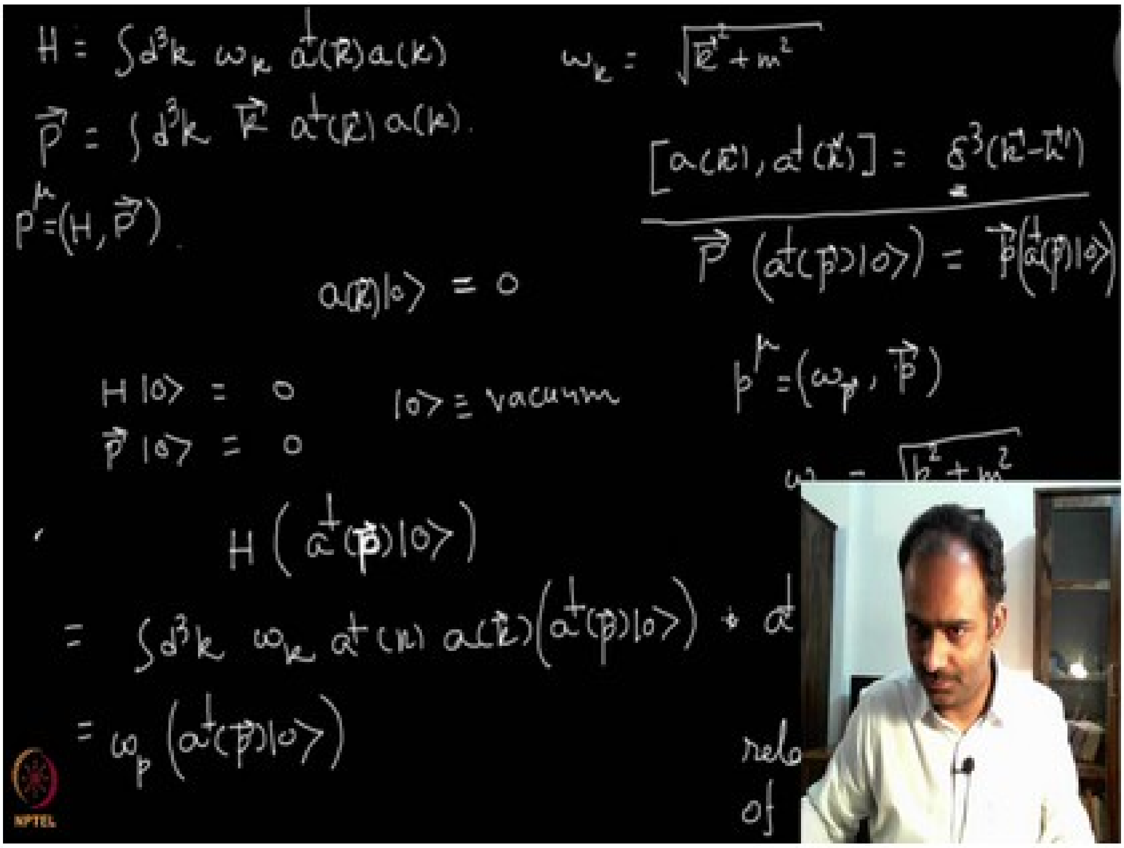

Figure 1: Refer Slide Time: 00:16

It is nice day because we are going to see today finally after long wait particles in this theory; the theory which we have been studying. So, I have written down here the Hamiltonian which we have expressed using the operators a and a daggers and remember omega k is k square + m square again square root. And I also wrote down the momentum operator in this theory and I said that it will have this form.

And I also said that the Hamiltonian together with this momentum operator they will form a 4 vector. Now you see the Hamiltonian is telling you that the system can be described as a sum of infinite number of harmonic oscillators and the frequencies are continuously changing because your k is continuous. So, the entire machinery of harmonic oscillators that can be imported, so, you know you have been building states with operating a daggers on ket 0 which you have done in quantum mechanics and also in this course in the beginning.

So, we will just repeat those steps. So, let us define as before a ket 0 and it has a property that if you take any of the a k's and act on it you get 0. So, the operator a annihilates the ket 0. Now if you take the state ket 0 and act on it the Hamiltonian you get that it is an eigenstate of the Hamiltonian and it gives you a 0 here because you see the a here when x on the ket 0 will kill that ket 0.

And also if you take the momentum operator and act on this state you get again 0 and this ket 0 is what is what is called the vacuum of this theory. Maybe I will save some space here this is defined this is called the vacuum of free theory because we are looking at free theory. So, it is a state of 0 energy and 0 momentum because the momentum operator acting on this state is supposed to give you the momentum and the Hamiltonian acting on this is supposed to give you energy.

And if I had retained a constant piece here you would have got an infinity but we said that we can shift the 0 point energy and set it to 0 that is fine. Now take an a dagger and hit it on the vacuum. Now is this an eigenstate of the Hamiltonian. So, we can check. So, take the Hamiltonian H and act on this state and it is easy to check I think I want to show how to do the derivation I believe I have done it before.

So, all you have to do is you have to use the commutation relation of a of k and a dagger of k and remember that this is delta cube sorry I should put a k prime here k - k prime. If I use this commutation relation then I can do the following. So, I take d cube k omega k a dagger of k a of k let me put here instead of k let me put a p otherwise it is not nice and then you have a dagger of p ket 0. So, that is the state you have.

Now what I want to do is bring this a to here, so, that I can kill the vacuum and in doing. So, I will pick up the delta function delta cube k - p. So, you I will have an a dagger sitting here and a delta function and when I do the integral over k the delta function will set omega k to be omega p and you will have this is anyway this side. So, you will get a dagger p get 0 right this is what you will get with the omega p.

So, it is clear that this is an eigenstate of the Hamiltonian with energy omega p that is good but then also I can see that the only difference between the Hamiltonian and the momentum operator is just this factorize whether it is omega k or k. So, whatever manipulation I have done just. Now goes through completely because that involves only using the commutation relation and I will get the following.

So, if I take the operator p act on a dagger p ket 0 this state then as before instead of omega p here I will get p a dagger of p ket 0 not beautiful. So, I will a dagger of p ket 0. So, that is what I am going to get means that this state is an eigenstate of the momentum operator and it has a momentum p. So, that is nice. Now this state a dagger p ket 0 has an energy omega p and momentum p.

$$H = \int d^3k \, \omega_{\vec{k}} \, a^{\dagger}(\vec{k}) a(\vec{k}) \quad (1)$$

$$\vec{P} = \int d^3k \, \vec{K} \, a^{\dagger}(\vec{k}) a(\vec{k}) \quad (2)$$

So, if I write down then if I can write down that it has an associated 4 momentum p mu which is given by this where omega p is as before p square + m square. Now see this relation which we have here is the relation between energy and momentum of a relativistic particle which means that a dagger p ket 0 indeed represents a relativistic particle of mass m that should be not difficult. Now here when I am saying particle I am not thinking of a particle which is localized in space. You see this part this stated aggregate p ket 0 has a very precise momentum it has its 4 momentum it is it is three momentum it is 4 momentum they are very precisely defined. So, this cannot represent a particle which is localized in space because you know if you fix the momentum then the particle then it spreads out in the x space in the coordinate space. So, this is not localized in space but it is localized in momentum.

And of course if you want to construct a particle which is localized in space then you will have to create a spread over the momentum and that is what we will do sometime later. But it is clear that this indeed represents a particle with momentum p mu with 4 momentum p mu. Also note, so, that is one point let me write down here.

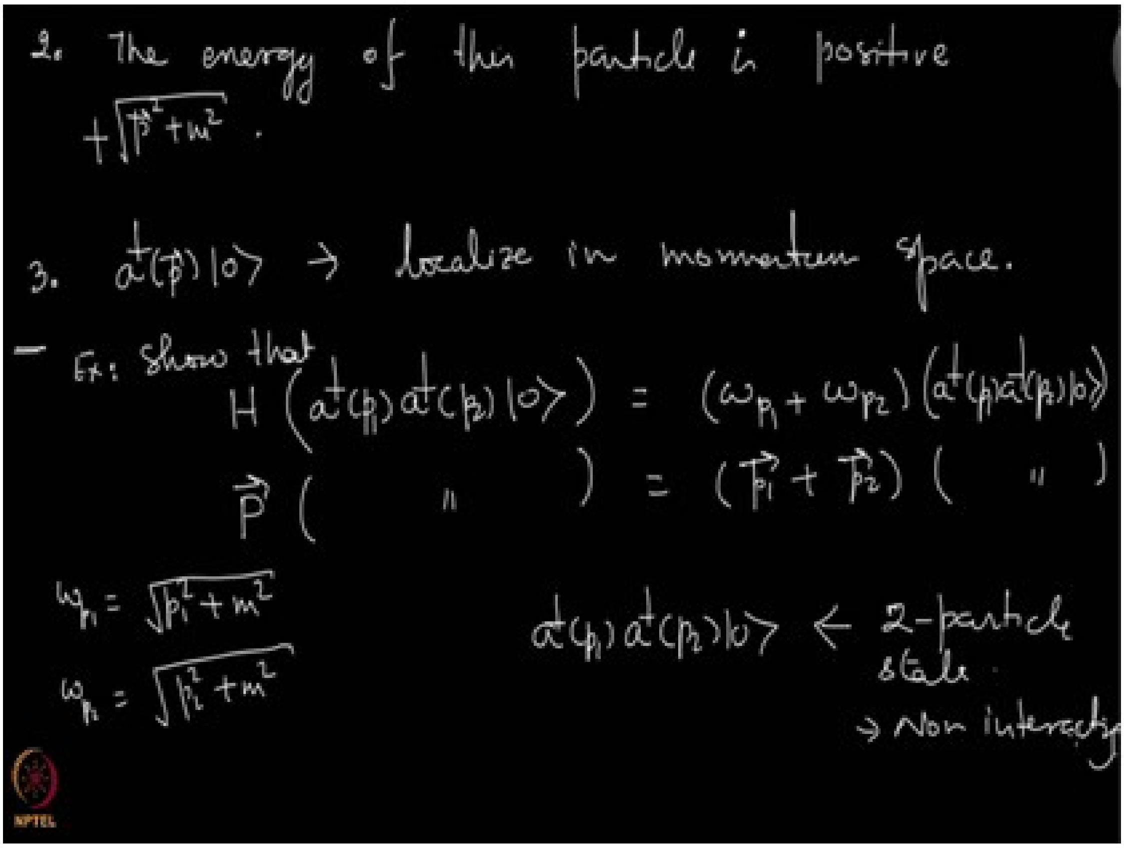

Figure 2: Refer Slide Time: 10:10

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \tag{3}$$

$$P^\mu = (H, \vec{P}) \tag{4}$$

Defining the ground state,

$$a(\vec{k})\,|0\rangle = 0 \tag{5}$$

$$H\,|0\rangle = 0 \tag{6}$$

$$\vec{P}\,|0\rangle = 0 \tag{7}$$

$$
\begin{aligned}
\vec{P}(a^\dagger(\vec{p})\,|0\rangle) &= \vec{p}(a^\dagger(\vec{p})\,|0\rangle) && (8)\\
p^\mu &= (\omega_p, \vec{p}) && (9)\\
\omega_p &= \sqrt{\vec{p}^2 + m^2} && (10)
\end{aligned}
$$

Where $|0\rangle$ is the vacuum, now we want to calculate, $H(a^\dagger(\vec{p})\,|0\rangle)$

$$
\begin{aligned}
H(a^\dagger(\vec{p})\,|0\rangle) &= \int d^3k\,\omega_k\,a^\dagger(\vec{k})a(\vec{k})\Big(a^\dagger(\vec{p})\,|0\rangle\Big) && (11)\\
&= \omega_p(a^\dagger(\vec{p})\,|0\rangle) && (12)
\end{aligned}
$$

In the last line we have used the commutation relation,

$$
[a(\vec{k}), a^\dagger(\vec{k})] = \delta^3(\vec{k} - \vec{k}') \qquad (13)
$$

- $a^\dagger(\vec{p})\,|0\rangle$ represents a relativistic particle of mass m.
- The energy of the particle is positive $+\sqrt{\vec{p}^2 + m^2}$
- $a^\dagger(\vec{p})\,|0\rangle$ is a localize state in momentum space

Also note that the energy of the state of this particle is positive which is p square + m square and we put a plus sign to emphasize that you recall we had difficulty when we were trying to interpret the Klein-Gordon theory as a single particle theory. When we were treating phi as the wave function describing the particle then we had the issue that the energy operator when it acts on this it gives you both the possibilities of having positive energy and negative energy.

But now we are not treating phi as a wave function phi is just a classical field we have quantized it and now we have constructed a particle by acting a dagger ket 0 and when I look at the energy of this state it comes out to be positive. So, there is no issue of having negative energies which was a problem when you interpret phi as a wave function. So, that problem is automatically gone from this.

I mean this problem is not present in this formalism that is good. Let me just write one more remark although I have said this already p ket 0 is a particle localized in momentum in momentum space and of course to construct localized state and localized state in the position space you will have to spread out the momentum. Now what do I get if I take a dagger p 1 and one more factor of a dagger p 2 and then act on the ket 0 that is something which we can ask and I leave it as an exercise to do.

Check that show that this state is an eigenstate of the Hamiltonian and what you get here is that it is omega p 1 + omega p 2 and the same state again. So, it is an eigenstate and the energy is omega p 1 + omega p 2 and of course if you take the momentum operator act on the same state you get p 1 + p 2 and what does that mean let me let me just write down again. So, this means that the state a dagger p 1 a diagram p 2 acting on ket 0 or acting on the vacuum is a state which contains 2 particles they have momentum p 1 and p 2 their energies are omega p 1 and omega p 2 and they are not interacting with each other.

So, this part should be clear because if you have 2 particles and if their momenta and energies just add up then the only way this will work out is if they are not interacting I mean if they interact then this this will not hold true. So, you have 2 particles which do not interact and this system represents a 2 particle state. So, a dagger p 1 a dagger p 2 acting on ket 0 is a state is a 2 particle state and they are not interacting. Please convince yourself that indeed they are non-interacting. Now can we represent this state as a single particle state. So, can I think that this is a state that carries momentum p 1 + p 2 and has energy omega p 1 + omega p 2 but it is a single particle state it is not like there are 2 particles but it is just having energy which is sum of those omega p 1 and omega p 2. So, let me write it down can a dagger p 1 a dagger p 2 represent a single particle state. So, let us see single particle state of momentum p 1 + p 2 and energy omega p 1 + omega p 2 good.

So let us see what happens if we try to interpret it this way. Now if indeed this represents a particle of mass m with these energy and momenta then it should satisfy this relation right which means if for p I put p 1 + p 2 whole squared minus for energy I should put omega p 1 + omega p 2 and then I this should be satisfied this relation should be satisfied. Now I will just instead of solving it in checking this thing in general what I will do is I will take p 1 and p 2 to be the same.

I mean I can act on the vacuum with a dagger p a dagger p that I can do and then I can check easily that what happens in that case. So, let me take p 1 equal to p 2 is equal to p then you have 2 p square which makes a 4 p square here minus now omega p 1 will become p square + m square square root and the same thing here. So, you have a factor of 2 which makes a 4 p square + m square correct.

Exercise: Show that

$$H(a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle) = (\omega_{p_1}+\omega_{p_2})(a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle) \tag{14}$$
$$\vec{P}(a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle) = (\vec{p}_1+\vec{p}_2)(a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle) \tag{15}$$

$a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle$ represents a 2-particle non-interacting state. Where

$$\omega_{p_1} = \sqrt{p_1^2+m^2} \tag{16}$$
$$\omega_{p_2} = \sqrt{p_2^2+m^2} \tag{17}$$

Question : Can $a^\dagger(p_1)a^\dagger(p_2)$ represent a single particle state of momentum $(\vec{p}_1+\vec{p}_2)$ and energy $(\omega_{p_1}+\omega_{p_2})$?

$$E^2-\vec{p}^2 = m^2 \tag{18}$$
$$(\omega_{p_1}+\omega_{p_2})-(\vec{p}_1+\vec{p}_2) = m^2 \tag{19}$$
$$\vec{p}_1=\vec{p}_2 = \vec{p} \tag{20}$$

The LHS,

$$4(\vec{p}^2+m^2)-4\vec{p}^2 = 4m^2 \tag{21}$$

So, $a^\dagger(p_1)a^\dagger(p_2)$ does not represent a single particle state.

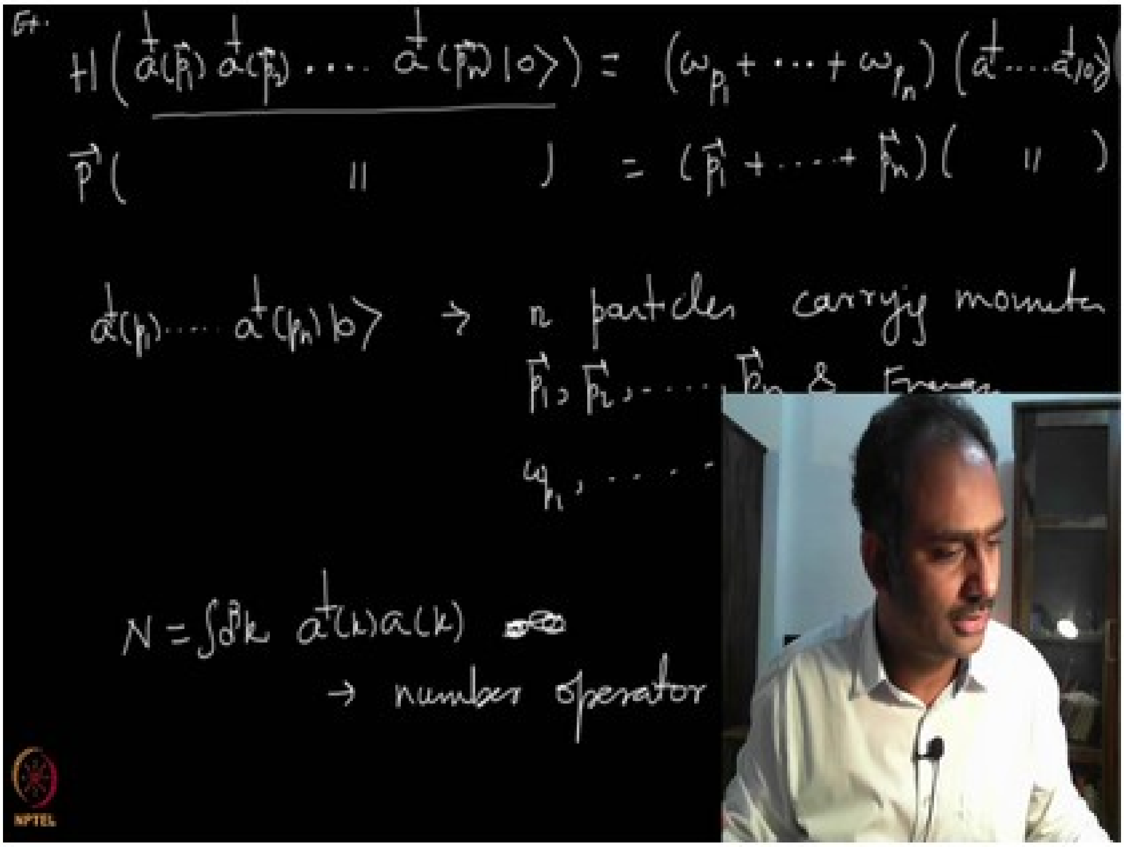

Figure 3: Refer Slide Time: 15:21

And so, I am taking the left hand side here that is this piece and what is this is 4 p square - 4 p square and you get 4 m square you see you do not get m square you get 4 m square. So, clearly this state does not represent a single particle state. So, that is fine which we anyway know that this is a state with 2 particles not one particle. So, this was anyway not going to work out. Now that we have acted on the vacuum 2 way daggers how about acting on the vacuum with n number of diagrams?

So, let us say I take ap 1 dagger ap 2 dagger up to ap n dagger and hit on the vacuum, vacuum of the free theory then what do I get. So, it is an exercise please check that if you find out I mean that this state is the eigenstate of the Hamiltonian and the eigenvalue is this which I am sure you have already guessed in the same state again we will with all these a daggers I am sorry that I am omitting the arguments here it is the same thing.

And if you take the momentum operator and act on the same state you get p 1 + p n and again the same straight back. So, which means that; this state represents; n non-interacting particles each of mass m and carrying a momentum. So, it shows that a dagger p 1 a dagger p n ket 0 represents carrying momenta p 1 p 2 p n and energy omega p 1 omega p n. So, we have seen now particles in this theory.

And you saw that the h and the p operators they are just giving you these factors. So, if instead of having omega k in here if you just put 1 here instead of omega k then in this algebra here you would not have got omega p you would have got just 1 or when you operate it twice you got omega p 1 omega p 2 and instead of this you would have got 1 + 1 which means if you look at the operator d cube k a dagger k a k when this acts on this state here which I have written here it will just give you n times the same state back right it will give you n times this.

So, which means it is going to count for you the number of particles that you have in your theory. So, we will call it as a number operator I am sorry I said number of particles in your theory what I meant is the number of particles in your state which you are looking at that is good. I think I have almost said everything that I wanted to say.

$$H\left(a^\dagger(p_1)a^\dagger(p_2).....a^\dagger(p_n)\,|0\rangle\right) = (\omega_{p_1}+\omega_{p_2}...+\omega_{p_n})\left(a^\dagger(p_1)a^\dagger(p_2).....a^\dagger(p_n)\,|0\rangle\right) \quad (22)$$

$$\vec{P}\left(a^\dagger(p_1)a^\dagger(p_2).....a^\dagger(p_n)\,|0\rangle\right) = (\vec{p}_1+\vec{p}_2...+\vec{p}_n)\left(a^\dagger(p_1)a^\dagger(p_2).....a^\dagger(p_n)\,|0\rangle\right) \quad (23)$$

Thus $a^\dagger(p_1)a^\dagger(p_2).....a^\dagger(p_n)\,|0\rangle$ represents n particles carrying momentum $\vec{p}_1, \vec{p}_2...., \vec{p}_n$ and energy $\omega_{p_1}, \omega_{p_2}...., \omega_{p_n}$

Now just 2 minor remarks actually 1 minor and one non-minor remark. You can of course construct states which have several a daggers acting on the vacuum and they all have the same momentum that you can do right you can have particles which all carry the same momentum. So, n number of particles each is carrying the same momentum that is allowed. So, this will correspond to a state carrying n particles each having the same momentum that you can do.

Also if you take any state like this one let me just use a 2 particle state then this is symmetric under the interchange of k 1 and k 2. So, this state is same as this state sorry k 2 k 1. And this is so, because a daggers commute, a is commute and a dagger is also commute and this means that this state is symmetric under the interchange of these 2 which is a property of Bose Einstein statistics. If there was a minus sign here then that would mean that these particles correspond to these particles are fermions.

But here because it is symmetric it means that they represent bosons and of course here you see that you can put any number of particles in the same state carrying the same momentum p. So, we see that the system which we are looking at the system of real scalar fields that system upon quantization gives you bosons and this is what we have seen here that this is perfect good. So, I have completed whatever I had planned for this video and we will start developing further in the next video.

And please make sure that you have full clarity about why it represents a state with 2 non-interacting particles and not something else that when you have 2 a daggers acting on ket 0. So, if all there is no confusion about these things it will be easier to follow what we do next. See you in the next video then.

$$N = \int d^3k\, a^\dagger(k)\, a(k) \quad (24)$$

N is the number operator

- $[a^\dagger(p)]\,|0\rangle$
- $a^\dagger(k_1)a^\dagger(k_2)\,|0\rangle = a^\dagger(k_2)a^\dagger(k_1)\,|0\rangle$

System of Bose-Einstein statistics

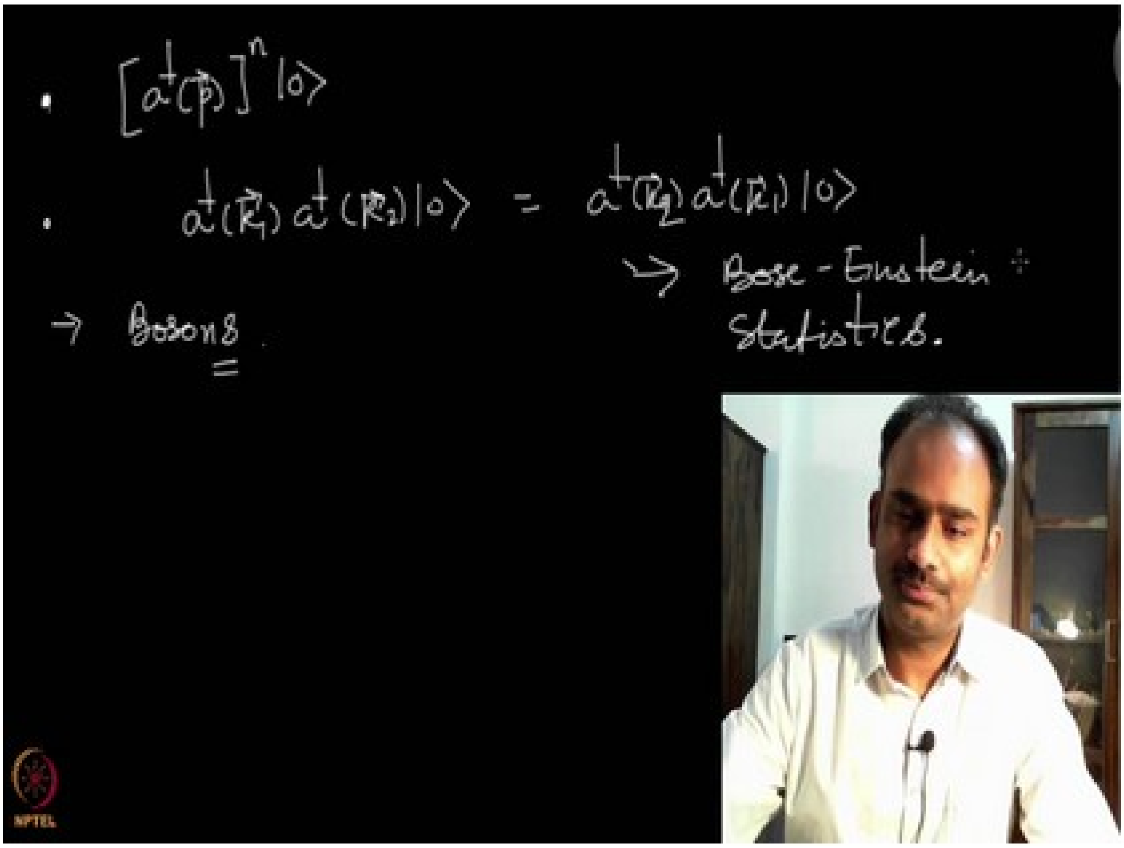

Figure 4: Refer Slide Time: 23:01