Introduction to Quantum Field Theory

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Lecture 10 : Quantization of Klein-Gordon Theory continued (3)

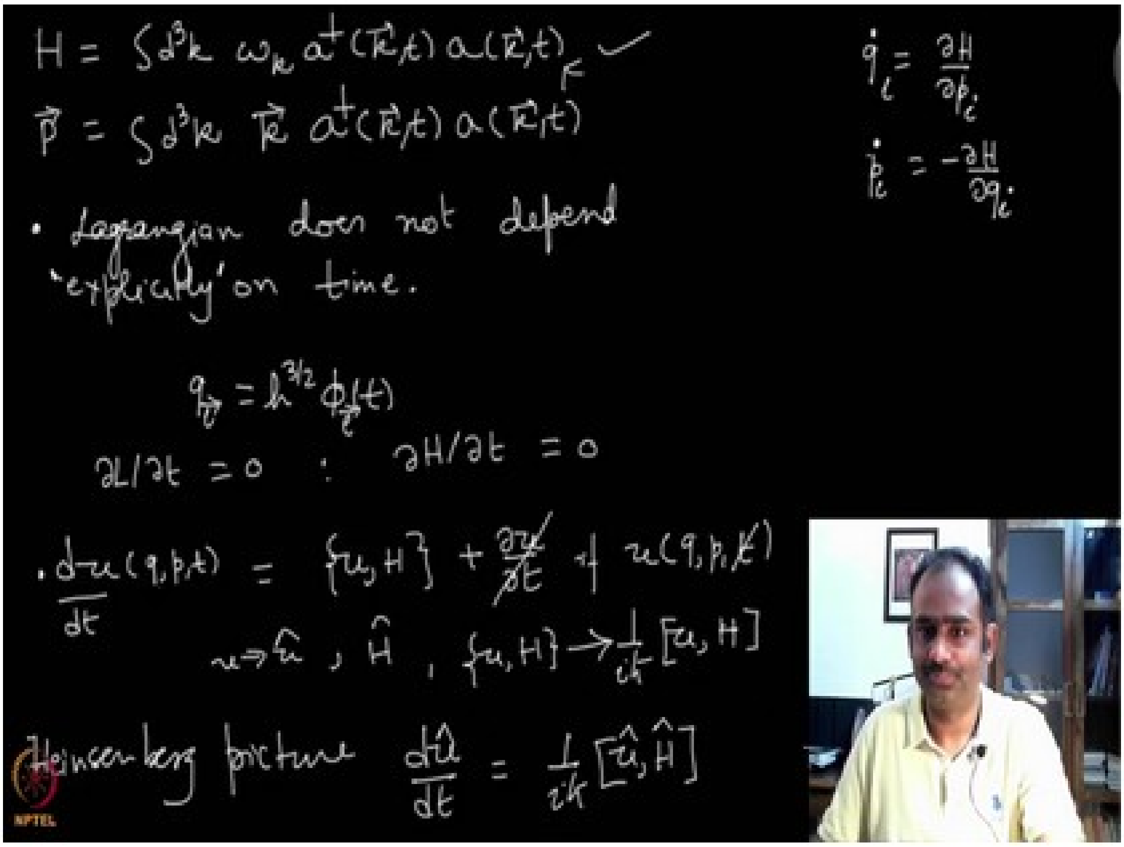

Figure 1: Refer Slide Time: 00:22

Last time we had written down the expression for Hamiltonian and also the momentum operator for this system. So, let me write it down again. So, we saw that we can write the Hamiltonian as d cube k omega k a dagger kt a kt and remember there was another piece which was an infinite constant which we have dropped can we I gave you the reason for that also in the last video and we have guessed that we should have an operator P we should have this form and of course I am going to come back to this in much more detail later.

But for now it is a good guess that we have. So, this is something familiar to us we saw already in the beginning of this course that we can write down the Hamiltonian as a system of independent harmonic oscillators. So, this is exactly what you get if you use the variables a dagger and a for writing down the Hamiltonian of a simple harmonic oscillator. And each oscillator comes with its own frequency omega k and now you have a continuous set of oscillators.

So, k is continuous. So, you are summing over all the oscillators k is continuous. So, omega k is also continuous and the different oscillators do not talk to each other. They are really independent because you see there is no term in this Hamiltonian which couples oscillator with k with another oscillator with k prime. So, these are all independent oscillators and this is what will happen in every field theory every free field theory which you will study.

You will always be able to write it as a as a sum of harmonic oscillators and that is always going to be your starting point when you are going to do perturbation theory for non-interacting fields sorry for interacting fields also. Anyhow I want to make a remark here mark or maybe too. Our Lagrangian that we wrote for this system that does not explicitly depend on time Lagrangian does not depend explicitly on time. So, here the emphasis is on the word explicitly see it depends implicitly on time through the coordinates.

So, to understand this, what you can do is you can again go back to discretized version of the Lagrangian in that case you will be able to write it down as a sum of the coordinates q i which we have seen that we can write as h 3 by 2 h is the volume sorry the length of a cube. So, three sides are h. So, that the h cube is the volume of each cell when you have quantize the space and then phi i is the discrete version of the field this is what we saw. And when you write down the Lagrangian using the q i's which i have written down here. So, this of course depends explicitly on time you will see that the time dependence is always through phi i there is no explicit time dependence anywhere phi i or pi i.

We defined

$$\begin{aligned} H &= \int d^3k\, \omega_{\vec{k}}\, a^\dagger(\vec{k},t)a(\vec{k},t) && (1)\\ \vec{P} &= \int d^3k\, \vec{K}\, a^\dagger(\vec{k},t)a(\vec{k},t) && (2) \end{aligned}$$

- The lagrangian does not depend explicitly on time

$$q_{\vec{i}} = h^{3/2}\phi_{\vec{i}}(t) \tag{3}$$

$$\frac{\partial L}{\partial t} = 0 \quad ; \quad \frac{\partial H}{\partial t} = 0 \tag{4}$$

If you are looking at the Lagrangian of course there is no pi it is just phi's and phi dots. So which implies that this is 0 there is no explicit time dependence and we also know that if Lagrangian does not depend explicitly on time then the Hamiltonian also does not depend explicitly on time that is good. Now we have the Hamiltonian here which has a dagger kt a of kt. Now I am going to I want to find out how these operators evolve with time the operator a for example.

So, let me first treat it classically and then I will quantize it. So, if you recall; you have learned in classical mechanics that if you have any function u of coordinates and momenta and time possibly. So, let me write q p and t. So, p is the conjugate momentum you could have several q i's let me; so, I am suppressing the indices p i and q i. Then if you ask for a total time derivative d u over d t then if you use equations of motion the canonical equations of motion which are let me write down here which are q you know q dot.

If you use these canonical equations of motion using this you can write down the total derivative of u as the Poisson bracket of u with h that is correct plus a partial time derivative of u with t because you could possibly depend explicitly on time. Now if I assume that there is no explicit time dependence then this drops out if u is not explicitly depend on time if that is the case. Now let us go to quantum version of this.

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{5}$$

$$\frac{du(q,p,t)}{dt} = \{u, H\} + \frac{\partial u}{\partial t} \tag{6}$$

$$\frac{du(q,p,t)}{dt} = \{u, H\} \tag{7}$$

So, if u is an operator if I promote u to an operator. So, it goes to u hat h becomes the operator h hat then you know that the Poisson bracket u h it has to replace to be replaced by the commutator with a factor of 1 over ih bar sorry here I am sorry u is the commutator here 1 over ih bar that is the prescription right see the Poisson bracket is ih bar times the commutator is ih bar times the Poisson bracket.

So, that is what I have written here. So, if I go to quantum theory and I work in Heisenberg picture where we the entire time dependence is carried by the operators. And if you work in Heisenberg picture which I want to work with. And if you have an operator u then you will have du hat over dt is equal to this Poisson bracket gets replaced by 1 over ih bar times the commutator of u and h. So, I will use this and try to calculate what a k of t is how it evolves with time. Now see your a is not explicitly dependent on time the operator a.

Only when $u(q,p,t)$ is not a function of time i.e $u(q,p)$.

$$u \rightarrow \hat{u} \quad , \quad \hat{H} \quad , \quad \{u, H\} \rightarrow \frac{1}{\iota\hbar}[u, H] \tag{8}$$

Heisenberg picture

$$\frac{d\hat{u}}{dt} = \frac{1}{\iota\hbar}[u, H] \tag{9}$$

$$\frac{da(\vec{k},t)}{dt} = \frac{1}{\iota\hbar}[a(\vec{k},t), H] \tag{10}$$

So, if I write the same equation above. So, I will not put a hat it is obvious that we are looking at the quantum version because I have a bracket to remind us that. Now our a is not explicitly time dependent because it depends on the pi's and phi's and the time dependence is through them. So, I do not have a del a over del t piece here and I can just calculate this commutator to find out what should be the right hand side.

And that is easy because the h you have is in terms of a and a daggers and we know the commutation relation of a and a daggers. The only point here is that when you are substituting for h in this expression let us be a little bit more explicit here. Let me put the indices properly not in this is the arguments kt is equal to; now this h if you go back here has a dagger kt akt but this t could be different in spell you could put t prime.

But because this is time independent I can choose the time which you have here same as what you have here then all is good. So, you can just now use the commutation relation which we have seen earlier let me show you again if I find it easily here a a dagger is h bar del cube. Now I will put h bar to be 1. So, that one you can use and arrive at the following.

From commutator algebra

$$[a\,b, c] = a\,[b, c] + [a, c]\,b \tag{11}$$

Example

$$\frac{da(\vec{k}, t)}{dt} = \frac{1}{\iota\hbar}\omega_k\, a(\vec{k}, t) \tag{12}$$

Where

$$a(\vec{k}, t) = e^{-(\iota/\hbar)\omega_{\vec{k}}t}a(\vec{k}) \tag{13}$$
$$a^\dagger(\vec{k}, t) = e^{(\iota/\hbar)\omega_{\vec{k}}t}a^\dagger(\vec{k}) \tag{14}$$

$$H = \int d^3k\,\omega_k a^\dagger(\vec{k})a(\vec{k}) \tag{15}$$

So, here is a very simple exercise for you can show that da kt over dt is 1 over i h bar omega ka kt that you can show all you will have to use is this thing.

This if you use you can arrive at this result and this is easy to find the solution it is easy to find the solution for this equation remember a is an operator here. So, a k of t I can write down as e to the -i over h bar omega k t and we have a constant piece a k of not looking nice k of 0 and this thing I will define to be a k. So, that is the solution and we will not keep the h bars in fact I notice that I have not been very careful in or being consistent in keeping h bars.

So, somewhere here yeah somewhere here I was not very careful but anyway at least at the places where I was writing commutators I was writing h bar properly and at other places I was putting h bar to be 1 but if you put all h bars to be one it will be good. So, there is no mistake as such but it is not nice that I missed at some places that is good. So, we have a k of t is e to the minus e to the -i omega k t and a dagger I will put h bar right now.

But not later omega k t a dagger k where a dagger k is just the Hermitian conjugate of this one that is good. Now it is clear that if you write down the Hamiltonian there will be no time dependence left because this and these are complex conjugates. So, they will just cancel you can you are left with only a dagger k ak. So, its manifestly there is no time dependence here. I can also use this result these two to write down the expression for phi.

I am putting x without a vector which means it has both the x component and the t time component. So, when I write x without a vector I mean the following or even better this is what I mean. Now let me go back and show you yeah here. So, we have d cube k over 2 pi 3 halves 1 over root 2 omega k. The a kt we have now written as this piece is now a of k and the time dependence is what e to the minus i omega kt.

Now if i combine it with this one this exponential in the exponent you will get e to the -i and in the brackets omega k t - k dot x. So, here you will get this combining this e to the –i omega kt - k dot x from here and this piece it has e to the -i k dot x with three vectors and it has a dagger which will give you e to the i omega kt. So, if you pull out i omega kt you will have here e to the i omega kt - k dot x that is what you will have.

So, you see these two are just complex conjugates of each other. So, I will have an a k e to the -i omega kt - k dot x and here you will have plus a dagger k e to the i omega kt - k dot x. So, that is what I am going to write there. Now yeah let us see if I can remember correctly all the expression 2 pi 3 halves then I had a 2 omega k in the square root that is correct and then I have a and a of k e to the -i omega k t - k dot x looks like everything is fine till. Now a dagger k e to the i omega kt - k dot x.

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2\omega_k} \left[a(\vec{k}) e^{-\iota(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} + a^\dagger(\vec{k}) e^{\iota(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} \right] \tag{16}$$

We write

$$k^\mu = (\omega_k, \vec{k}) \tag{17}$$
$$x^\mu = (t, \vec{x}) \tag{18}$$
$$k \cdot x = \omega_{\vec{k}} t - \vec{k}\cdot\vec{x} \tag{19}$$

So, I have used equations of motion in writing this phi of x right because this relation is coming because I have used equations of motion remember I use canonical equations of motion in writing down what the time derivative of a is and that is what I have plugged in here. So, equations of motion have been used in writing down the phi that is good all c and h bar have been put equal to 1 and also remember that here in this expression omega k is as before k square plus m square that is also good.

Now I hope you have noticed this but let me say anyway. So, note that we have e to the -i omega k t - k dot x that is with a and the other one has e to the i omega k t - k dot x both have the same apart from the minus sign they have the same thing. Now see if you write k mu as k omega k which is the k 0, k vector this is a 4 vector right. Now x mu is also 4 vector. So, both these quantities transform as a vector.

I can construct a scalar out of them k dot x I am not putting vectors because I imply a 4 vector and the k dot x would be omega k t - k dot x right which means that what you have in the exponent is a scalar a Lorentz scalar right because that is what a dot product of 2 4 vectors is it is a scalar. So, you have e to the -i k dot x when I am writing k dot x without any this thing arrows on the top it means a dot product of four vectors and the other one is with e to the -ik dot x it nicely shows that these exponents they are large scalars which they should be.

So, phi of x is now d cube k over 2 pi cube omega k and then we have a of k e to the -i k dot x + e k dagger e to the ik dot x. This is fine that is good you should also be thinking like I mean I have been saying that phi is a scalar phi is a scalar this part is transforming properly how about the other pieces do they transform correctly to form a scalar that is one question which should be on your mind. Now but I will leave it for.

Now or maybe I will most likely I will give it as an exercise. So, this is one thing I wanted to write and let us also record what pi of x will be the momentum density and remember pi dot pi x you can of course substitute in the expressions that we wrote earlier but if you recall that pi x is just phi dot then it is easy to see that if i take a time derivative it will pull out a -i omega k here right where it is it has omega kt.

So, that will pull out -i from this one and if I take that factor out I will get a minus high here this one when I take the derivative it will give you i omega k and because I have pulled out a -i there will be a relative minus sign. So, let me write here a of k then of course you get back the exponential and you have a minus sign and because you are pulling out minus i omega k this gets changed to omega k over 2 the square root.

And then this is a dagger k and e to the ik dot x let us see whether everything is correct here; all is good here. This is nice this will be for our later use. Now we will in the next video start looking at the states in this free theory. So, see you in the next video.

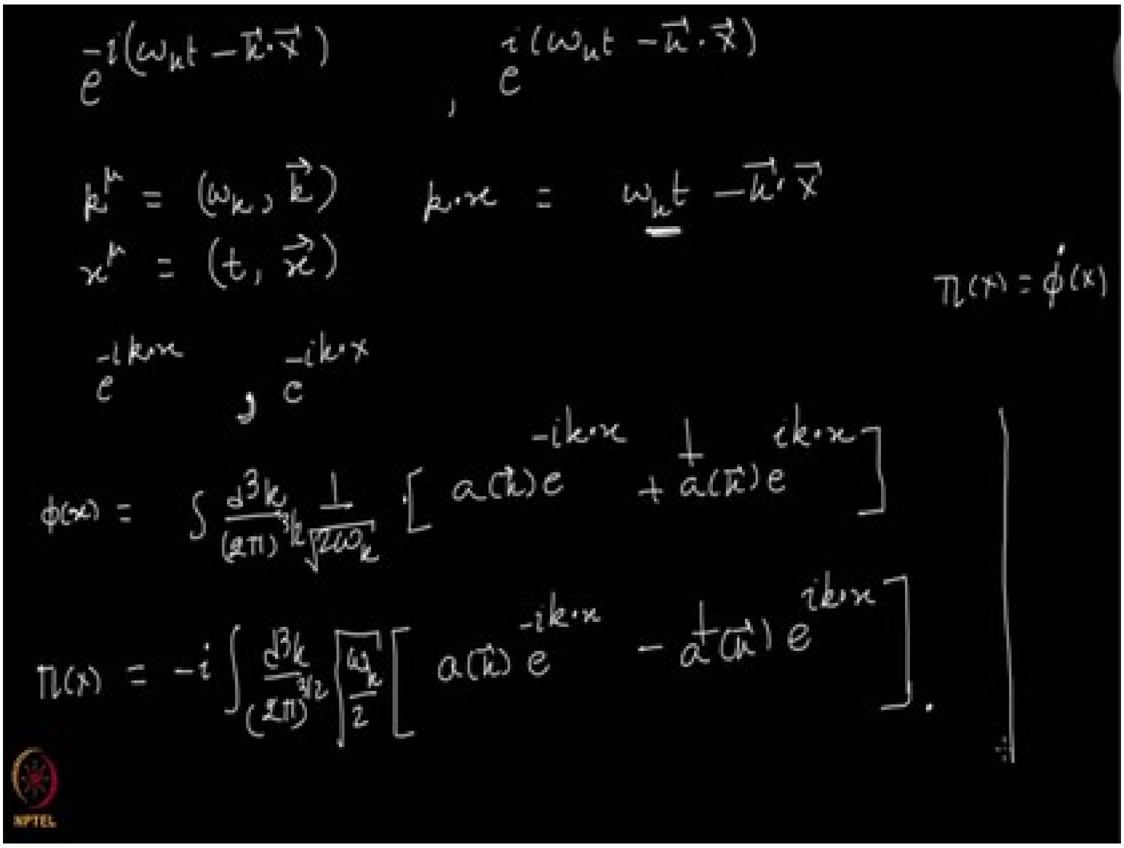

Figure 2: Refer Slide Time: 09:28

In terms of these operators

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2\omega_k} \left[a(\vec{k}) e^{-\iota k\cdot x} + a^\dagger(\vec{k}) e^{\iota k\cdot x} \right] \qquad (20)$$

$$\Pi(\vec{x},t) = -\iota \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\omega_k]}{2} \left[a(\vec{k}) e^{-\iota k\cdot x} - a^\dagger(\vec{k}) e^{\iota k\cdot x} \right] \qquad (21)$$

Where we have used,

$$\Pi(\vec{x},t) = \dot{\phi}(\vec{x},t)$$