Introduction to Quantum Field Theory

Dr. Anurag Tripathi,
Assistant Professor,
Indian Institute of Technology, Hyderabad

Lecture 1 : Schrodinger field

Hello, I am Anurag Tripathi I am a faculty member in the department of Physics at IIT Hyderabad and I will be the instructor of this course. It is Introduction to Quantum Field theory. We will be mainly covering here theory of scalar field. I would not be doing Quantization of Fermionic fields or gauge physics in this course and this will be mostly in fact set solely Canonical Quantization which will be covered in this course. And this is 30 hours course which will run for 12 weeks and the formulation that we are going to develop here. This framework of Quantum Field theory that we develop here it can be applied to a theory of Fermionic field as well. So you can construct for example electrodynamics using it but we will restrict to scalar fields as said. And there are few prerequisites for this course, let me write it down. So here are the prerequisites. Not so sure about the spelling is correct. So you should have encounter quantum mechanics in your master course. And then of course special relativity and then you have also studied classical mechanics in your master course. Mainly you should be familiar with the concept of action how to obtain equations of motion using variational principles from the action and the principle of least action that you should know. How to construct Hamiltonian and other related things to the sequences so these 3 courses you must have taken and to have some idea of fields it will also be useful if you have taken a course on electrodynamics. These three together form the prerequisite for this course. Then you should know how to construct a Quantum theory given classical theory. So it might have been covered in your Quantum Mechanics course, but let me nevertheless mention here.

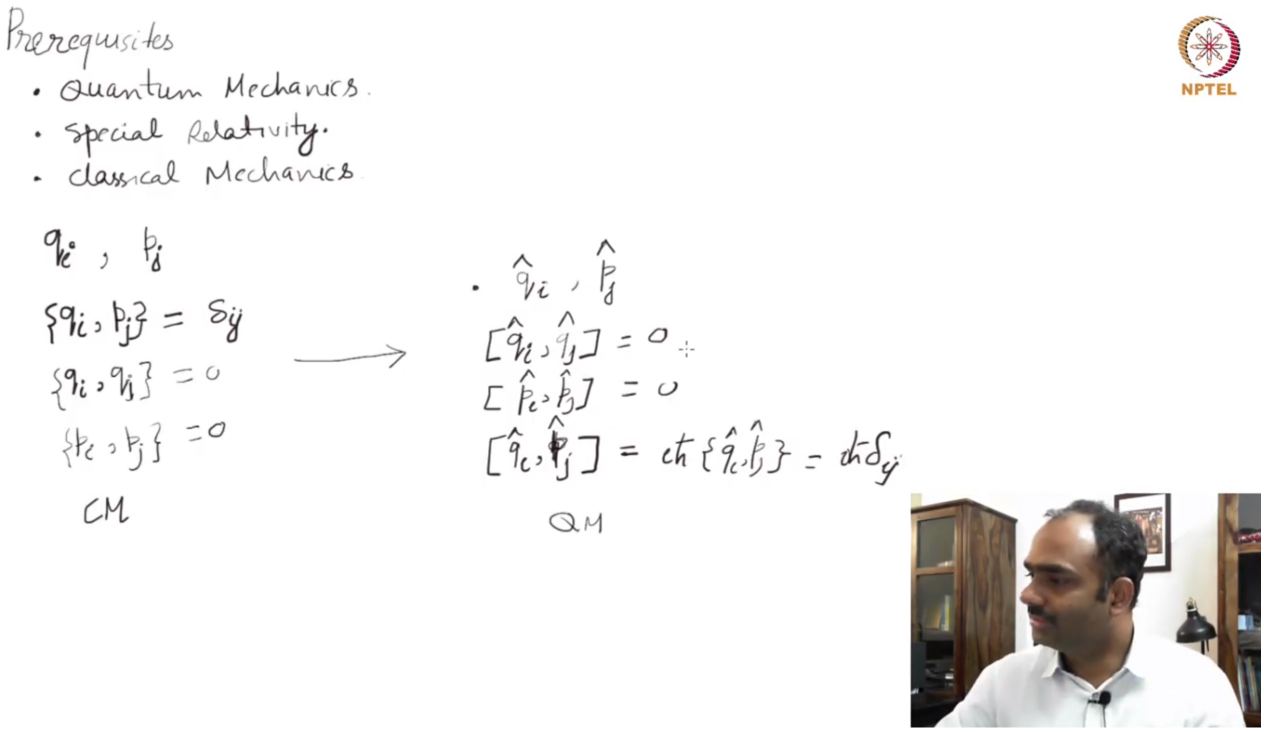
Pre-requisites

- Quantum Mechanics
- Special Relativity
- Classical Mechanics

Promoting observables to operators

$$\begin{aligned} CM \qquad & \qquad QM \\ q_i, p_j \qquad & \qquad \hat{q}_i, \hat{p}_j \\ \{q_i, q_j\} = 0 \qquad & \qquad [\hat{q}_i, \hat{p}_j] = 0 \\ \{p_i, p_j\} = 0 \qquad & \qquad [\hat{p}_i, \hat{p}_j] = 0 \\ \{q_i, p_j\} = \delta_{ij} \qquad & \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\{q_i, p_j\} = i\hbar\delta_{ij} \end{aligned}$$

So suppose you have some system which is described by generalized coordinates which are available as q_i where i runs from 1 to n where n is the degrees of freedom that system has. And the p_j's are the momentum conjugate to the variable q_i to the coordinates q_i. So these are your



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canonically conjugate variables. This is at classic level. Now to make a Quantum theory for the class to system which is described by these coordinates.

What you have to do is replace the strangePoisson brackets by commutator and you have to throw in a $i\hbar$ so you should have also studied Poisson brackets in your course. So if you construct Poisson bracket of q with p this curly brackets I am using to denote Poisson brackets and if you do so you get δ_{ij} and of course the Poisson brackets of q with itself is zero, and let me write down this is one relation and of course the other ones are these.

So Poisson bracket of q with another q or with itself is 0. Similarly for the conjugate momentum p is 0 and q with p is non-zero when q and p are conjugate of each other. So, if the index are different then of course it is 0. So the prescription for going from the classical description to quantum mechanical description is this. Let me write it down here. So this is classical mechanics and you want to make a quantum theory here.

So, what you do is first you promote all the coordinates all the q's and p's to operators so they become operators now. That is what the hat is indicating that these are all operators and the Poisson brackets are replaced by commutator. So you define $\hat{q}_i$ $\hat{q}_j$ commutator is 0 that you would have already studied in your quantum mechanics. Similarly $\hat{p}_i$ and $\hat{p}_j$ is equal to 0. So what we are doing here is; let me right down this one and it will be clear what I am doing p_j.

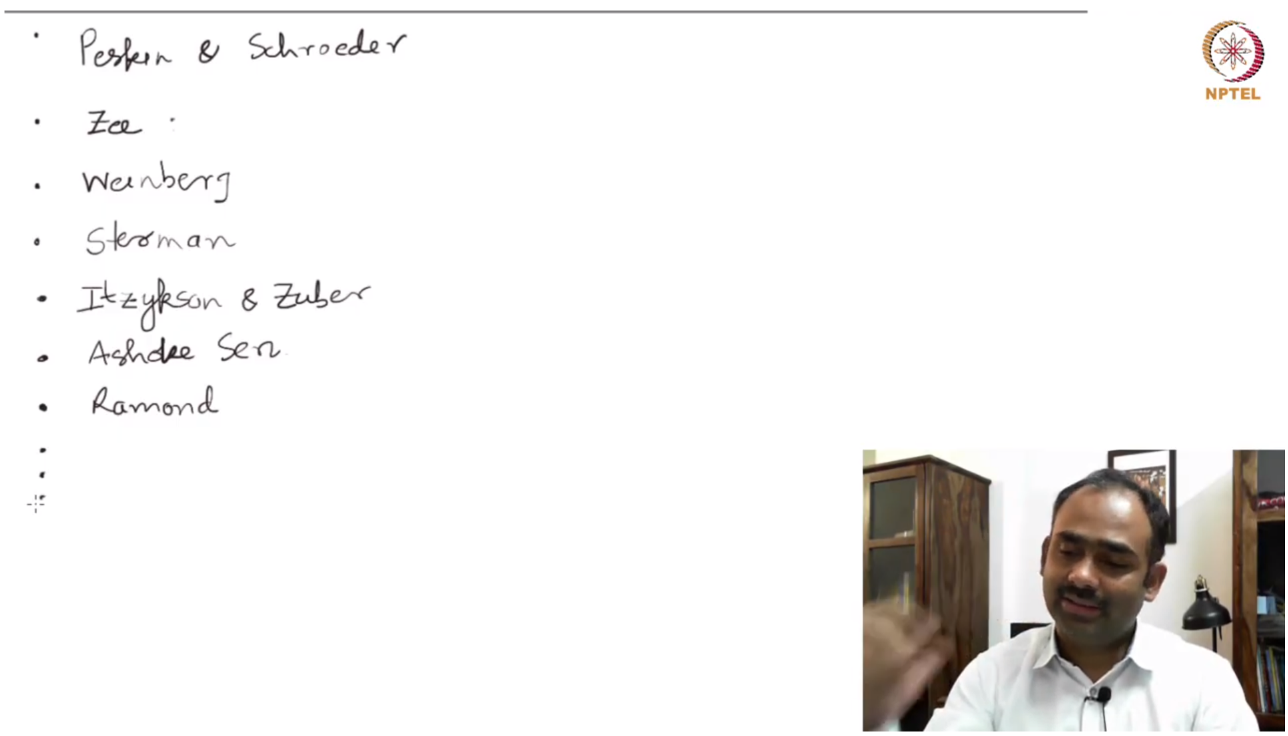
So you found this commutator p_j in this thing it is replaced by or it is equal to $i\hbar$ Poisson bracket is calculated in the classical theory. Let us check, it is correct. And what did you calculate? $\hat{q}_i$ $\hat{p}_j$ was delta hat so it becomes $i\hbar\delta_{ij}$, delta is the kronecker delta. So if you are looking at q, p where q and p are conjugate variables and just becomes $i\hbar$ and thus the commutator, you know, you have in quantum mechanics.

So thus the prescription, how you make a Quantum theory starting with classical theory, it take the coordinates and the conjugate momenta calculate the Poisson brackets and to make the theory in quantum theory you promote them to operators. Define a commutator and commutator will be the same as your Poisson bracket the classical theory $i\hbar$ that is how you make quantum theory out of a classical theory that is a prescription.

Let me mention some of the books that you can refer to and which I will also be using in this course. So I will not write the title I will just write the names. So, you have a book by Peskin and Schroeder and the title is an introduction to Quantum Field theory. Another book is by Anthony Zee very nicely written book its title is Quantum Field Theory in a Nutshell. Then you have Weinberg. So there are three volumes to this series and the one which will be most useful for this course is the quantum theory of fields volume 1.

You have book by George Sterman and is called an introduction to Quantum Field theory that is also very, very nice book. It goes into details which are usually not covered in other books. So I would also recommend this book. You have Itzykson and Zuber that is also very nice book and of course you have lectures by Ashoke Sen you can visit his web page and you will find his lectures there. And this book by Raymond this is called Field Theory of Modern Primer. And of course, there are many other books which have come recently in last few years and you can also refer to them. So these are all the references that I have.

- An Introduction to Quantum Field Theory by Peskin and Schroeder
- Quantum field theory in a nutshell by Anthony Zee
- The Quantum Theory of Fields by Steven Weinberg
- An introduction to quantum field theory by George Sterman
- Quantum Field Theory by Itzykson Zuber
- Field Theory: A Modern Primer by Ramond
- Ashoke Sen notes and lectures from his website



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To set the stage for this course imagine this, you have an electron and a positron, positron is an anti particle of the electron and these two particles are rushing towards each other at very, very high speed, speed that are very near to the speed of light.

So they are coming with enormous speeds this lot of energy in this system and they collide. This is something which was done at the lab. For example large electron proton collider when this collision happen both the electron and positron they disappear they do not exist anymore. What is produced is a variety of particles many, many things are produced. You can producing higgs boson, you can be producing photons and W boson, Z boson even quark anti-quark. So you could be producing all these things which were not there to begin with and what was there to begin with that electron and positron they disappeared. If you want to describe the system its time evolution the quantum mechanics of single particle quantum mechanics that you have learnt is not going to work for the simple reason. Because when you set up those equations Schrodinger equation for example, the wave functions ψ in there describes the probability of finding a particle.

Let us say electron and positron which one to choose in space and time and you remember if you take the main function ψ you have to normalize that wave function of course but then the interpretation is that it represents a probability density and if integrated over the entire universe I get 1, meaning let me emphasis this little more by putting this argument. Meaning this particle exists somewhere in the universe at all times, but the process that are described there, that electron disappears is not there anymore and not only that new particles were created.

But here you have this requirement that the particle has to exist at all times. So you cannot deal with creation of particles or annihilation of particles using the single particle quantum mechanics, for example, Schrodinger equation. So that is not going to work and here you need a different formalism to deal with such situations and that Formalism is that of Quantum Field theory so idea is the following.

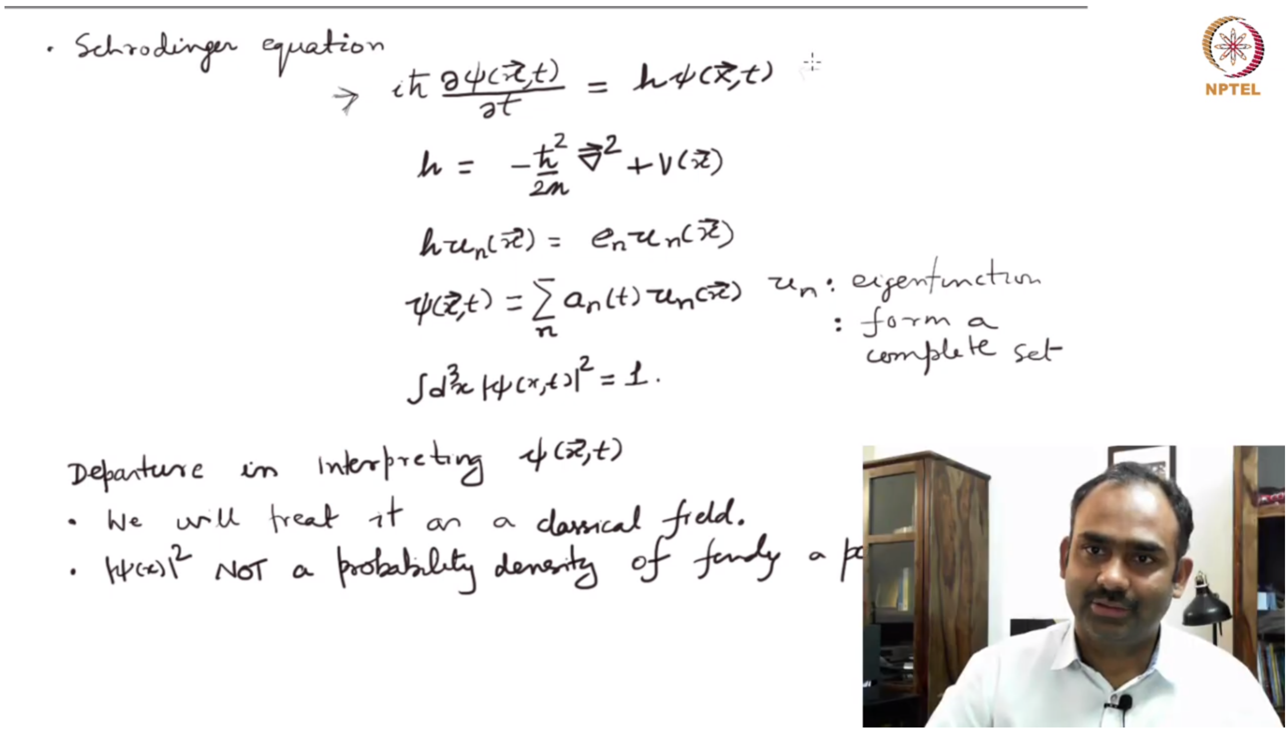
What we will do is we will create that ψ you get here. That is the wave function which satisfies Schrodinger equation we will treat it as a classical field even though we got it because you doing quantum mechanics but now good I do not want to treat it as a quantum mechanical thing anymore. It is a field which satisfies the equation which is the field equation, but I will treat it as if it is something classical.

And then I will quantize that field and it works it does what we need and you will see that happens in the course that how to get an idea is a separate issue, but let us accept that this strangeis really working we can check then let us proceed with quantizing of fields rather than quantizing the coordinates, again quantizing the coordinates using your f(q), f(p) and you put commutator relation, but here you will now be quantizing the fields. So that is what we are going to do now so let us begin.

So first very quick review of what you already know. So even though we want to have a theory which is relativistic because these particles were moving at great speed so you need relativity. Let us start with something which is familiar to everyone let's start with Schrodinger equation we will worry about relativity later and bring it later. Let us start with our very familiar Schrodinger equation.

Schrodinger equation because I am going to use capital H later. So small h is the Hamiltonian of the system which we are studying so you know what h is. Let me write down here may be here itself. Gradient square that is a scalar quantity remember this dot product here so it is scalar under rotations plus the potential and this h is your operator.

So we are imagining system which is described by some potential V can you have some particle in there. The wave function of the particle is given by $psi(x)$ of t. So I am doing the old thing which you all know and here let me also write down the eigenfunctions of h. You should take h its eigenfunctions let us label by $u_n(x,t)$ because it is Hamiltonian the eiganvalue will be energy



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and denoted by e, $e_n u_n(x)$.

And n runs typically from 1, it takes infinite values 1, 2, 3 upto Infinity. And let me write down your u_n they are eigenfunction of the Hamiltonian h and they form a complete set they form a complete set because they are eigenfunctions of the hermitian operator h because the operator is hermitian that is why this is fine. And because they form a complete set I can write ψ as a sum of these functions is a linear sum of this function.

So I can decompose ψ as $a_n(t)$ u n of x this is a summation over all the n that is what we can do because u_n forms a complete set. And as I said before this cannot handle creation and annihilation of particles because of this reason Psi Square x it has to be somewhere at all the times it cannot be somewhere. Now as I was saying we will make a departure from the way we interpret Psi. We are going to make a departure in interpreting Psi.

So I do not want to treat it as a wave function of anything. I do not want to say that Psi describes the wave function of a particle which is moving around in potential V. I want to do that. I just want to say Psi is a classical field which satisfies this equation. This is what I want to do. One remark I wanted to make here was Psi, remark not really falling in the line of argument. But note that this equation cannot describe an electron for example.

You see a electron, you know has spin up and spin down. It has spin so Psi only gives you a number or actually 2 numbers because it is complex it has real path and complex path. Each space time points at x and at this value of x, at any value of time Psi has this value. So, with just this information we cannot tell; there is no information on there is no provision here in this way written up to distinguish between whether that electron has a spin up or spin down or its in some linear combination that information is not present here.

So this you can think of describing a particle which does not have any internal degrees of freedom. And electron has internal degrees of freedom in addition to where it can also be in those other states which are permissible. So it could be here with spin up or spin down or whichever way. So that internal degree or degrees of freedom they are not present in this equation but anyway it is beside the point.

Interpretation of wave function in quantum mechanics,

$$\int d^3x |\psi(\vec{x},t)|^2 = 1 \tag{1}$$

During creation and annihilation of particles Schrodinger equation is not going to work,

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = h\psi(\vec{x},t) \tag{2}$$

Where,

$$h = \frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}) \tag{3}$$

$$hu_n(\vec{x}) = e_n u_n(\vec{x}) \tag{4}$$

u_n is the eigenfunction and forms a complete basis, thus

$$\psi(\vec{x},t) = \sum_n a_n(t) u_n(\vec{x}) \tag{5}$$

Departure in interpretation of $\psi(\vec{x},t)$, now we will treat $\psi(\vec{x},t)$ as a classical field and $|\psi(\vec{x},t)|^2$ is not the probability density of finding particles. Now our aim is to find action that gives Schrodinger equation as the equation of motion.

$$\psi(\vec{x},t) = \sum_n a_n(t) u_n(\vec{x}) \quad ; n = 1, 2...$$

So I was saying we will make departure in interpreting Psi of x,t and we are going to treat it as a classical field. So we will treat it as a classical field. Now, of course because we are not going to treat it as a wave function this is not a probability density of anything. Good then so what should we do? I want to write down the classical theory which has Schrodinger equation as the equation of motion, which means I should be able to construct an action from which I can derive Schrodinger equation as an equation of motion.

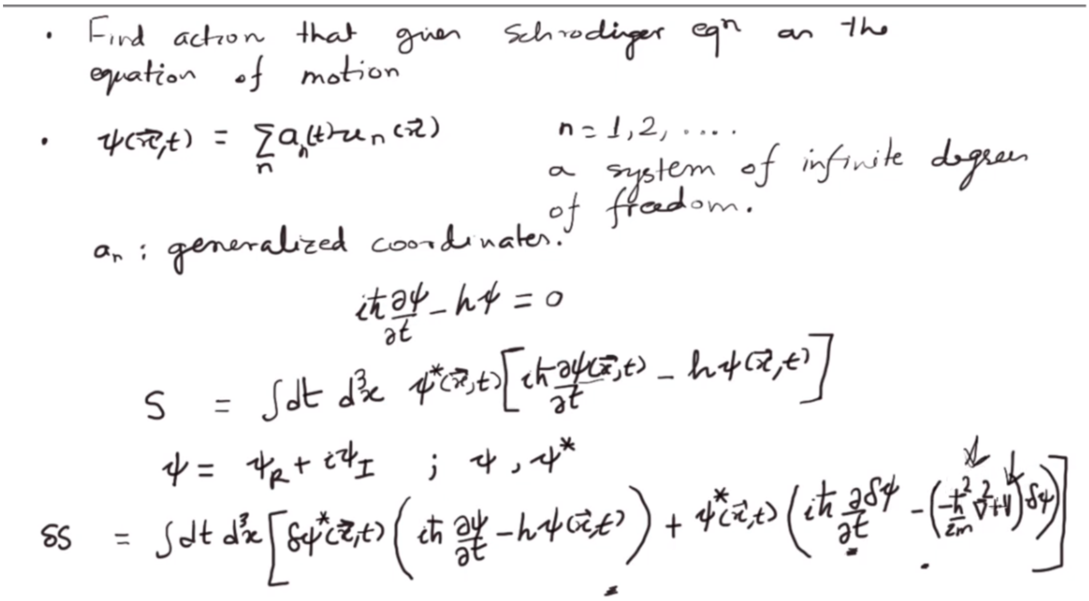
What does it mean to specify a system u it means that you give the action of that once you give the action everything else follows. You can use the variational principles. You can extremise the action find out the equation of motion from there, construct the Hamiltonian whatever you wish and that is what means to specify a system. So let us do that. So the goal now is to write down the action which will give this as the equation of motion so let us see.

A system of infinite degrees of freedom, and a_n are the generalized coordinate. When we do variations in action we should get,

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t) = 0 \tag{6}$$

So here is the goal. I am looking at a; I am searching for an action of a system. Find action that gives Schrodinger equation as the equation of motion that is what we want to do. Now that the system which we are going to construct will have infinite degrees of freedom because your Psi describes that the system and Psi is the field. Let us see why it is an infinite; it is a system of infinite degrees of freedom.

So if you write Psi x of t as u and x and this coefficients a n of t that is the decomposition used before also. So if your system is described by Psi you see it effectively described by a. The u n's



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are wave functions that are fixed to solve; you have found the eigenfunctions of the Hamiltonian that small h. These are fixed. So you know, what is u1 and then you know, what is u2 what is u1 billion, you know, those they are there.

The freedom that you have been choosing an so they are the generalized coordinates for the system now because there are infinite of them so you have a system of infinite degrees of freedom. That is good and your, a n's are generalized coordinates. That is good. Now, let us go back to our desire to have any action for this theory. So we do the following, I want to construct the action which will give this as a equation of motion.

So what did we have? You had i h bar del Psi over del p – h Psiz equal to 0. So this is what I want to get from my action. So let me construct my action. This is what I should get when I do a variation. Let us do this. I construct S as integral dt over Lagrangian and that Langrangian I write this d cube x times this Psi star xt and I put this thing which you have here. This entire p's which you have multiplying Psi star.

And you will see that why I am doing that; make a square bracket ih bar del Psi over del t Psi of xt – h hat h is a operator so h maybe next time I will provide more but here it is fine + Psi xt times ih bar delta Psi xt over del t minus I started doing something wrong let me see that I was having strange feelings at What am I doing? So here is your action. Now, you can see why I have constructed in this way when I take a variation with respect to Psi and Psi star I will get this equation of motion.

You see your Psi is a complex field so it has a real part and an imaginary part. So you can; at each point you have 2 number Psi r and Psi i and they both are independent of each other. So you can use them as independent fields and do variation or you can choose to instead work with Psi and Psi star, is identical whether you work with Psi r and Psi i or whether you work with Psi and Psi star because you can always write Psi r using Psi and Psi star, Psi i using Psi and Psi star.

So both are equivalent, we will work with Psi and Psi star. So if I am looking at delta S a variation of S, I get the following f dt d cube x and then do a variation of Psi star and then do a variation of Psi. So this is what we get when we do this variation. So your delta Psi star I am

doing exactly what you do in your classical mechanics and then you have ih bar delta Psi over delta t - h Psi xt plus now, you take the term Psi star xt and I am suppressing x and t and other parts.

The action,

$$S = \int dt d^3x \left[\psi^*(\vec{x}, t) \left(i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} - h\psi(\vec{x}, t) \right) \right] \tag{7}$$

We can write

$$\psi = \psi_R + i\psi_I \tag{8}$$

We can either work with (ψ_R, ψ_I) or (ψ, ψ^*), we will work with the second pair. The variation will be

$$\delta S = \int dt d^3x \left[\delta\psi^*(\vec{x}, t) \left(i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} - h\psi(\vec{x}, t) \right) \right.$$
$$\left. + \psi^*(\vec{x}, t) \left(i\hbar \frac{\partial \delta\psi(\vec{x}, t)}{\partial t} - \left\{ \frac{-\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{x})) \delta\psi(\vec{x}, t) \right\} \right) \right] \tag{9}$$

And then you have variation in this piece ih bar delta over delta t of delta Psi you to suppress other arguments minus h the Hamiltonian maybe, I should be here in this; here it is Hamiltonian is - h bar square over 2m gradient square and this is acting on the +V of x and this entire thing is acting on Psi and that has become delta Psi for us, now, if they put delta s to 0 and because delta Psi star and delta Psi are independent of each other.

I will be able to find the equations of motion. So first term is good, everything is fine here because you have delta Psi star. The second term is not so nice because your delta Psi on which several derivatives are operating here delta t time derivative is acting. Here you have differential operator del square acting on delta Psi. So what we want to do if we want to a free up the delta Psi of all this operator, so we want to pull them away.

And it is easy to do using the method of integration by parts. So, those integration by parts, which you have learnt in your school. I think so, that is what I am going to do. So nothing has to be done here. Only I should do something here. Remember what integration by parts does. When you integrate by parts, you take derivative from one factor and put it on the other factor and you pick up a minus sign.

Every time you put the derivative and put on the next one next factor. You pick up a minus sign and of course, there are boundary terms which you generate which I will leave for you to figure out why I am not writing those. So let me write it down now, maybe I can tell you here itself. You see you have a delta over delta t so doing integration by parts this delta over delta t will shift to Psi stars so this will become delta Psi star over delta t and you will get a minus sign.

Then this term is fine V delta Psi. This one which has a delta square, delta square is 2 derivatives it is del dot del, del is basically del over del x. So when you pull one derivative and put on Psi star, you pickup 1 minus sign but you are still left with one more del when you pickup one more del than put on the Psi star you have another minus sign. So that minus and minus makes a plus. So neither this term is going to generate sign not this term this anyway remains the same.

So these two are not going to generate any negative signs only the derivative gets transferred to Psi star. This one will generate a minus sign because there is only one derivative here. So the answer is this.

$$\delta S = \int dtd^3x \Bigg[\delta\psi^*(\vec{x},t)\Bigg(i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} - h\psi(\vec{x},t)\Bigg) + \delta\psi(\vec{x},t)\Bigg(- i\hbar\frac{\partial\psi^*(\vec{x},t)}{\partial t} - \Bigg(\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}))\psi^*(\vec{x},t)\Bigg)\Bigg)\Bigg] \tag{10}$$

Let me write down, I hope I can, let us see delta S is integral dt d cube x first term was easy I can remember delta Psi star and what you have in the bracket is basically what you will eventually get as equation of motion. So that is ih bar delta psi over delta t - h which is Hamiltonian h Psi plus the other term. Let us go back. How do I go back? So this will become minus because of being transferred minus ih bar del over del t acting on Psi star and you have you have a delta Psi.

So let us go, plus delta Psi which I pulled out -ih bar del Psi star over del t that is fine. And for this one you will have delta Psi will be freed up which we have already written as a factor outside and you will have is minus sign so that we should take care and then this place, so you will have minus what was that h bar square over 2m del square acting on Psi star and plus a V of Psi star + V of x Psi star these things are now acting on Psi star that is good and I hope everything is correct here. It is fine.

Where we have used integration by parts

$$\int_a^b dx\frac{d}{dx}\Bigg[f(x)g(x)\Bigg] = \int_a^b dx\frac{\partial f(x)}{\partial x}g(x) + \int_a^b dx f(x)\frac{\partial g(x)}{\partial x} \tag{11}$$

Putting variations to zero,

$$\delta S = 0 \tag{12}$$

We get two equations of motion,

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = h\psi(\vec{x},t) \tag{13}$$

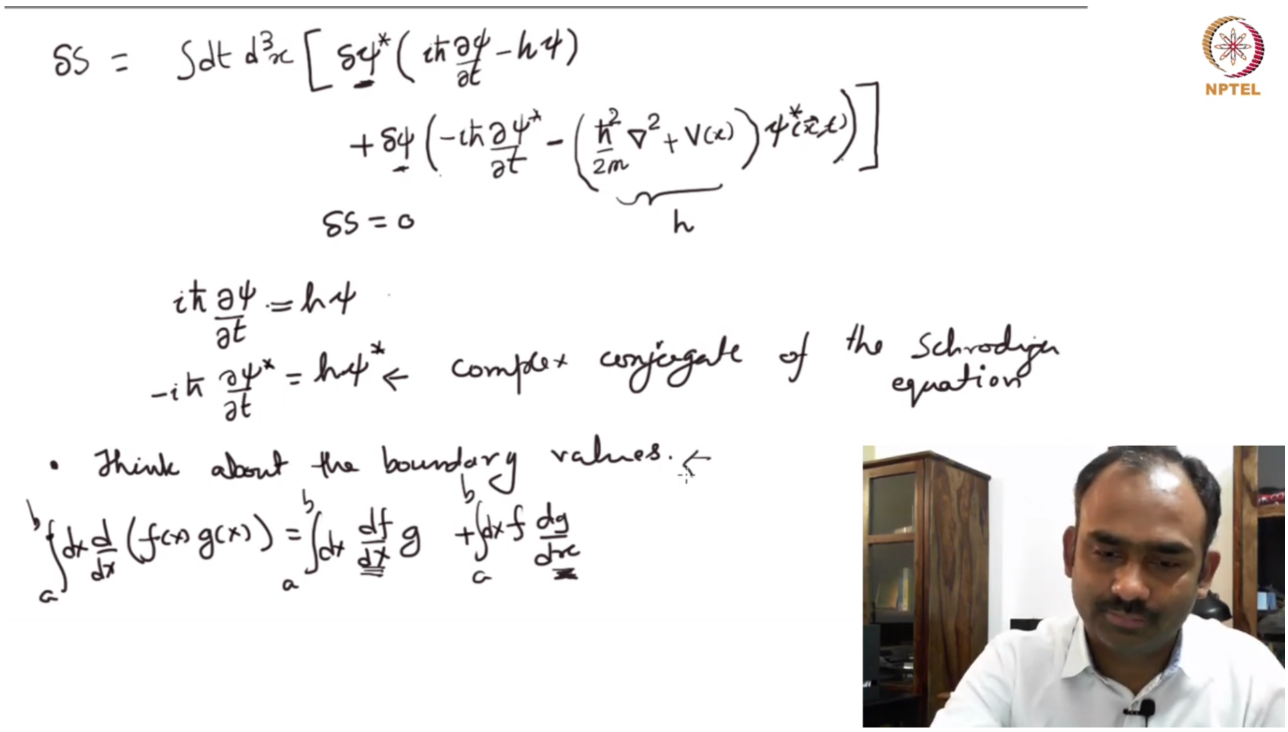
$$-i\hbar\frac{\partial\psi^*(\vec{x},t)}{\partial t} = h\psi^*(\vec{x},t) \tag{14}$$

Now, I have freed up delta Psi star and delta Psi here and because these are independent variations of two independent variables. The only way delta s can be 0 is if these two factors from the brackets they vanish by themselves, so I get the equations of motion ih bar delta Psi over delta t − h Psi which is your Schrodinger equation, right is equal to 0 I can put all there and taking to right side.

And other equation I get from here, I get 2 equations of motion -ih bar delta Psi star over delta t is equal to x Psi star this is it. And as you realise this equation the second equation is just the complex conjugate of the first one. It is not a new equation. Coming from this one and when you are taking Complex conjugate that i becomes minus i and Psi become Psi star Hamiltonian is this, no complex conjugate is involved here and Psi from Psi star.

So I got the equation of motion starting with this action, which means that the action and what we wrote is correct. Because it is giving me correct equation of motion and see how we have built it. I have just taken this piece this left hand side of the equation and put it here and multiplied with Psi star this you can always do. And you will be able to control the action if you know the equations of motion.

So where we are now we have the action. So let us see what else? Nothing. So I was asking you to think about the boundary, bounded values. Let me just tell you very quickly about integration



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by parts in case you have forgotten. It is the simple thing. If you have a function f times g and you look at d over dx the derivative of the product. Then it is df over dx g + f dg over dx. If I integrate, integrating both sides you see that here the derivative is on the first factor f which you can write as a derivative on the second factor that is what I have done.

And because this term we have to take to the other side is a minus sign which we have used and this is the one we generate the boundary terms. In case if you put some a to b that is the one which is generating boundary terms and that is what I am asking you here to think about it. So I will continue further looking into the system in the next video.