Solid State Physics
Lecture 39

Electron Tunneling Through a Periodic Potential

Hello, we have been discussing Tunneling of Electrons through a Potential Barrier. (Refer Slide Time: 00:40)

So, today we are going to discuss the Electrons Tunneling through a Periodic Potential. In the context of periodic potential we have already discussed Kronig - Penney model, but in that model we have considered a delta function kind of potential barrier which is not quite realistic if you have an array of atoms in even in one dimension if you consider a one dimensional solid. You will have potentials that are not really like delta functions in the barrier rather the potential would go to the potentials will have a sharp dip wherever there are atoms and everywhere else it is it will behave like a barrier. So, in that context we better choose an arbitrary barrier to understand exactly what happens realistically and then we will understand whether Kronig - Penney model is realistic or it gives us some understanding far away from the real physics of a solid, let us discuss that. So, we can consider a set of periodic potentials like this say that extends to infinity on both sides if we consider this to be 0 in the x axis, the this is the x axis and this is the potential axis, V(x) if we consider this to be 0, this point is a x = a, here x = 2a, a is the lattice constant and so on. Now, if we consider one such profile of the potential say this one here, then just like earlier we can say that this is the potential axis V(x) this is the x axis 0, a, this barrier height let us say this is V_0. The incident electron wave it has an amplitude A_L from the left hand side, the one that moves towards the right in the right hand side is A_R and B_L moves this way in the left hand side, B_R moves this way in the right hand side of the potential profile. So, if we consider a one dimensional potential that is periodic in x and V_0 is the potential within a unit cell. So, if we have the potential if we consider a unit cell the unit cell is given by $0 \leq x \leq a$ this is the unit cell. Now we can consider the electron tunneling through the potential that is drawn here and it and. So, we are considering a region we have a lead at $x \leq 0$ and another lead at $x \geq a$ and in between we have the potential that we have drawn here. So, if we have this kind of a situation then the wave function in the left hand side of the potential that is the left hand lead would be given as $\psi_L(x) = A_L e^{iqx} + B_L e^{-iqx}$ this is for $x \leq 0$. So, we have discuss this kind of wave function earlier also now we are doing it in terms of the transfer matrix elements $\psi_R(x)$ in terms of transfer matrix elements can be written as $\psi_R(x) = (s_{11}A_L + s_{12}B_L)e^{iqx} + (s_{21}A_L + s_{22}B_L)e^{-iqx}$. Where we have this q which is a function of the energy of the electron is given as $q(E) = \sqrt{\frac{2mE}{\hbar^2}}$ this is the propagation wave number in the leads. So, here we look for a solution that satisfy the boundary conditions at x = 0 and x = a. So, what are the boundary conditions that we have? We have Bloch theorem now in mind. (Refer Slide Time: 07:42)

So, at those points what does the Bloch theorem require? The Bloch theorem required that $\psi_R(x = a) = e^{ika}\psi_L(x = 0)$ this is the requirement of the Bloch theorem. And if we consider the boundary condition for the first order derivative of the wave function we can write $\frac{d\psi_R}{dx}|_{(x=a)} = e^{ika}\frac{d\psi_L}{dx}|_{(x=0)}$. Note something interesting here the value of x = a, here the value of x = 0 we are equating them and recalling them to be so calling this condition to be a boundary condition because of the periodic boundary condition. So, we have periodic boundary condition at hand; that means, this point and this point are equivalent to this point equivalent to this point equivalent to this point and so on. And that notion of equivalence has this kind of a factor e^{ika} that is something we obtained from the Bloch theorem. Now if we have these two boundary conditions satisfied then we can obtain the linear homogeneous equations for the arbitrary coefficients A_L and B_L in these equations that can be written as $\begin{pmatrix} s_{11}e^{iqa} & s_{12}e^{iqa} \\ s_{21}e^{-iqa} & s_{22}e^{-iqa} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix} = e^{ika} \begin{pmatrix} A_L \\ B_L \end{pmatrix}$

This equation shows that the 2×2 matrix has an eigen value e^{ika} this must be an eigen value of this matrix. If that is true and if this matrix is unimodular; that means, the determinant of this matrix is 1; that means, the product of the eigen values is 1 then the other eigen value must be e^{-ika} then the product of these two eigen values would be one which is the determinant of the matrix ok. If we have that what is the trace of this matrix? The trace of the matrix is the sum over the eigen values, if we sum over these two eigen values e^{ika} and e^{-ika} we will obtain $trace = 2\cos ka$ because the trace is the sum of the eigen values. Now if we look at this matrix the trace is $s_{11}e^{iqa} + s_{22}e^{-iqa}$. So, the matrix the I trace and the determinant they remain invariant under a unitary similarity transformation and the unitary similarity transformation will give us the diagonal elements e^{ik} and e^{-ik} the off diagonal elements 0. So, we can still write that the $trace = 2\cos ka$ and we will have to satisfy a compatibility equation for everything to be consistent $s_{11}e^{iqa} + s_{22}e^{-iqa}$ that is the trace of this matrix, this has to be equal to the trace of the matrix that we have obtained from the other method $s_{11}e^{iqa} + s_{22}e^{-iqa} = 2\cos ka$ this is the compatibility equation that must be satisfied. So, this equation is similar to what we have obtained in the context of Kronig - Penny model. Now it is convenient to express that equation in terms of transmission amplitude. (Refer Slide Time: 14:26)

The transmission amplitude is given as $t = |t|e^{i\phi}$. So, this is the expression of t in its polar form if we do that we can write $s_{11}(E) = \frac{1}{t^*(E)} = \frac{1}{t(E)}e^{i\phi}$. Now, the compatibility equation by this kind of a transformation becomes $\frac{1}{|t(E)|}\cos\left[\phi(E) + q(E)a\right] = \cos ka$, $\cos ka$, $\phi(E)$ and t the —t(E)— these are the phase and the modulus of the transmission amplitude. The compatibility equation can be satisfied only for the values of energy for which the magnitude of the left hand side that is this part is ≤ 1 because $\cos k$ cannot be more than 1. So, if we have this quantity the absolute value of this $\frac{1}{|t(E)|}\cos\left[\phi(E) + q(E)a\right] \leq 1$ then this compatibility equation would be satisfied otherwise it would not be compatible. So, it is important to analyze the band structure produced by the periodic array of barriers from the point of view of electron tunneling. The transfer matrix that we have obtained earlier for a single barrier if we consider that in the present case the energy band would be given by the $Re[(\cosh\beta b + i\frac{q^2-\beta^2}{2q\beta}\sinh\beta b)e^{iqa-b}]$ real part of this quantity that has to be $= \cos ka$. Now, if we compare with the Kronig - Penney model we can write that a - b = the width w and if we put that in the above equation we get $\cosh\beta b\cos qw - \frac{q^2-\beta^2}{2q\beta}\sinh\beta b\sin qw = \cos ka$ So, with this equation we see that even for an arbitrary shape of the potential barrier in case of periodic potential we did not consider any chronic penny like delta function barrier here, we have considered this smooth shape of the barrier and that is arbitrary we did not really use the shape of the barrier anywhere in the mathematics that we have considered so far. We have seen that we can still obtain condition like the Kronig - Penney model. So, the gap energy gap the prohibited energy region that we have seen in the energy spectrum of electrons subject to a periodic potential weak periodic potential is obtained for any arbitrary shape of the potential barrier it is not really dependent on the shape exact shape of the potential barrier. So, this is a great lesson that we have learnt from this exercise.