Solid State Physics
Lecture 21
The Drude Theory of Metals

Hello. So, far we have discussed the crystal structure, determination of crystal structure using X ray diffraction or neutron diffraction and the binding of solids. After discussing all these, let us move on to the third module that is elementary transport theory of the solids. (Refer Slide Time: 00:49)

For discussing the elementary transport theory of the solids, the first thing we will discuss is the Drude model, so the Drude Theory for the Metals. Why shall we start our discussion from this model? Because this happens to be the first model for metals, historically also this is the simplest model of metals; of course it involves certain drastic assumptions and within those drastic assumptions, we get certain results. Some of which agree with experimental findings, some of which do not agree; but analyzing that will give us a nice understanding of what act actually happens in solid. If we can predict something correctly using this model, good; if we cannot predict something correctly, then we learn for what reason we cannot predict that. So, this model is pretty useful. So, the fundamental thing that we have for this model, the Drude theory of metals is here the electrons are considered like gas particles. So, in case of gas particle, we consider the that the atoms or molecules in that gas they are free to move around. Here electrons are treated similarly and in gas they collide with each other the gas molecules; electrons are also considered to be doing something like that, they collide with other electrons or ions. Here we consider that the positive charges are heavy, that means proton and neutron, the nuclei those are heavy and they do not move; because of being heavy, they stay immobile, rather Drude theory is valid within the limit when the nuclei can be considered can be assumed immobile, otherwise it is not valid. When atoms of a metallic element are brought together to form a metal, the valence electron become detached and they can move freely. So, how can we represent an atom? An atom may be represent like represented like this, this is the nucleus, having a charge; e is the protonic charge and Z_a is the number of protons in that atom. Here we have the core electrons. If we have Z number of valence electrons, then the number of core electrons would be $(Z_a - Z)$, Z is the number of valence electrons. So, the charge of these core electrons would be $-e(Z_a - Z)$ this and then we have the outermost region of valence electrons, like this. And that will have a charge of -eZ. Now many such atoms come together to form a solid, form a metal. Drude considered this kind of atoms and applied the kinetic theory of gas for conduction electrons; that means the green electrons that we have considered here, those are the conduction electrons. And each electron have mass m, that move against subject to a background of heavy immobile ions. So, what are ions? This black region plus the blue region, that is the ion. And the density of the electron gas can be easily estimated; a metallic element it will contain Avogadro number of atoms per mole and since each atom contributes to Z number of electrons; Z number of electrons that is the number of conduction electrons, then the number of conduction electrons per unit volume can be given as, $n = \frac{N}{V}$ n is the number of conduction electrons per unit volume small n and capital N is the total number of conduction electrons and V is the volume. So, this is how you define any density, this is the electron number density; this is Avogadro number $6.023 \times 10^23 \times \frac{z\rho m}{A}$, ρm is the mass density over A, where A is the atomic mass. This is how we can obtain the electron density n. And these densities are typically thousand times more than that of a classical gas; number density of a classical gas is thousand times less than the number density of free electrons in a metal. Now, Drude made certain assumptions and those assumptions are very important in order to develop the theory. So, we will discuss those assumptions with some importance and we will write down all the assumptions. And I would advise you to go through these assumptions by pausing and restarting and going back in in the video and then you will get a good feeling about the assumptions, then it will be easy for you to follow what happens next, ok. Let us discuss the assumptions of the Drude model. (Refer Slide Time: 07:33)

The first assumption is that between collisions, the interaction of a given electron both with the other electrons and the ion that is neglected. What do we mean by this? This means that in the absence of externally applied electromagnetic field, the electrons are assumed to move in a straight line and Newton's law is applicable for describing that motion of the electrons. This approximation is known as the free electron approximation. The second approximation is that collisions in the Drude model as in kinetic theory of gas that, these collisions are instantaneous events and they abruptly alter the velocity of an electron. And Drude attributed them to the electrons bouncing off the impenetrable ion cores; that means electron hits an ion core and it bounces back or bounces and scatters to some other direction, it may not come back exactly on the same trajectory, it may arbitrarily go along any direction that is pretty random. The third assumption is that, electron experiences, an electron experiences a collision with a probability per unit time $\frac{1}{\tau}$, where τ is the relaxation time or the mean free time; τ represents relaxation time, also called the mean free time in similarity with the phrase mean free path. And now comes the last assumption; listen to it carefully the last assumption is the most difficult to comprehend. Electrons are assumed to achieve thermal equilibrium with their surroundings only through collisions. What does this statement mean? The collisions are assumed to maintain local thermodynamic equilibrium in a simple way that, immediately after each collision an electron is considered to emerge with a velocity that is not related to its velocity just before the collision; but it is randomly directed and with a speed that is appropriate to the temperature of the system. So, one electron comes collides with something else and after the collision, it emerges with certain velocity in a particular direction; this direction is pretty random, it has nothing to do with its previous direction of motion and the speed with which the electron emerges after the collision that is commensurate with the temperature of the surroundings, temperature of the system that is the electron is in, it has nothing to do with its velocity before the collision. And that is how it attains a thermal equilibrium with the entire system. Now, these are the assumptions of Drude model. Here many of these assumptions are pretty much drastic and it there is it is very difficult to justify these assumptions; but within these assumptions, it is easy to work out some mathematics. And that is the reason we would make these assumptions, work out the mathematics and test our model, that is Drude model against experiment and see what we have achieved; have we achieved something useful have, could we reproduce some of the experimental results, could we explain the physics that is what we are going to understand. And in our understanding, the first thing that we will study is the DC electrical conductivity of a metal, DC stands for Direct Current. (Refer Slide Time: 16:26)

You all know that in this context of dc electrical conductivity the famous law is Ohms law, that is the potential the $V = I \times R$ If we have an electric field $\overrightarrow{E}$ and if we consider a current density $\overrightarrow{j}$, volume current density; then we can write Ohms law as $\overrightarrow{E} = \rho \times \overrightarrow{j}$, where ρ is the resistivity. And $\overrightarrow{j}$ can be represented as I/A, this is the magnitude of $\overrightarrow{j}$ and the direction is the direction of the motion of the current; that is opposite to the direction of the motion of the electrons, because the electrons are of negative charge and we consider the motion of the positive charge, that direction to be the direction of the current according to our convention. Now, resistivity can be give, the total resistance can be expressed in terms of resistivity as $\rho \times L/A$ the length of the wire over the area of the wire; here capital A is area, not the atomic mass. Now, if we have n number of electrons per unit volume small n, that is the number density of electrons and if we consider all of them to move along one direction with velocity v; then the current density it will give rise that will be parallel to v, parallel to the direction of the motion. And if we consider time interval dt, within this time interval the charge density crossing the a given cross section A in time dt can be written as; we are writing it as the charge transported within this time short time of dt would be $-ne\ v\ A\ dt$, because minus e is the electronic charge v A dt, we are assuming that all the electrons are moving along one particular direction. And if this happens then the current density, volume current density can be written as $\overrightarrow{j} = -ne\overrightarrow{v}$, this is the volume current density. All conduction electrons in a metal do not move along the same direction as you all know. What happens? In the absence of any external electric field, they would the electrons would still move; but in such a way that the average velocity in the absence of electron electric field, this quantity $\langle v \rangle = 0$. And if you apply some electric field, then this $\langle v \rangle \neq 0$; there would be some drift velocity of the electrons as dictated by the electric field, externally applied electric field and that will lead to some current. (Refer Slide Time: 21:18)

So, in time t, if we consider that a typical electron at time $t = 0$ had velocity $\overrightarrow{v_0}$. It had velocity $\overrightarrow{v_0}$ at time t=0 and let us consider that in time t, it did not undergo any collision. Then it would it was subjected to the external electric field and it will have some additional velocity, because of the acceleration due to the electric field. So, the additional velocity maybe expressed as $-e\overrightarrow{E} \times t/m$ mass of the electron. And $\overrightarrow{v_0}$ has no contribution to be average to the average velocity; because $\overrightarrow{v_0}$ is random after the collision and it is along the along. So, the direction of $\overrightarrow{v_0}$ is random; if you consider many electrons, so it will average out to 0. So, the average of time t, that is the relaxation time in which there is no collision; if we take the average of t in which there is no collision for different electrons, then we will get the relaxation time τ. Therefore, we can write that the average velocity $\overrightarrow{v}_{avg} = -\frac{e\overrightarrow{E}\tau}{m}$. And if this is the average velocity, we can calculate the average current density that is the current density $\overrightarrow{j}$, which can be given as $(\frac{ne^2\tau}{m})\overrightarrow{E}$. And this result can be expressed in terms of the conductivity. We introduce a new term σ that is the conductivity, which is nothing but $1/\rho$. So, we can write that $\overrightarrow{j} = \sigma\overrightarrow{E}$, which gives us $\sigma = \frac{e^2\tau}{m}$. Now, this establishes the linear dependence of the volume current density $\overrightarrow{j}$ and the electric field $\overrightarrow{E}$ and gives an estimate of the conductivity σ in terms of the quantities that most of which we know, except this relaxation time that we do not know. Now, if we can measure the conductivity, we can estimate the relaxation time as $\tau = \frac{m}{\rho ne^2}$ from the Drude model. (Refer Slide Time: 25:24)

Now, let us consider something interesting, at any time t the average electronic velocity. Let us say $\overrightarrow{v} = \frac{\overrightarrow{p}(t)}{m}$, this is the average velocity of an electron. Therefore, the current density can be given as $\overrightarrow{j} = \frac{-ne\ \overrightarrow{p}(t)}{m}$. Now, if we calculate the momentum at time $t = t + dt$, we want to calculate this quantity. How do we calculate that? Let us understand the probability of the collision of an electron. So, if there is no collision of an electron, then we can just subject the electron to the external electric field and then we will find the increment in the potential or change in the potential. And if there is a collision, then the situation is pretty different. An electron taken at random, picked up at random at time t. For example; it will have a collision before time $t+dt$ with probability $\frac{dt}{\tau}$. If we pick up at time t, we have picked up an electron and we now go to time $t + dt$; within this time interval, this electron will have a collision, at least one collision with the probability $\frac{dt}{\tau}$. Now, if this is the probability of having a collision, then the probability of having no collision within this time interval t to (t + dt); that means surviving without any collision in this time interval the probability of that, that becomes $1 - dt/\tau$ simple. And if there is no collision, then the particle that is the electron evolves under the external uniform electromagnetic field that we have applied and it acquires an additional momentum. Now, if we consider f(t) as the external force on the electron; this is a uniform kind of a force. And the additional momentum that the electron would acquire over time dt, that would be $\overrightarrow{f}(t)dt + O(dt^2)$. And the contribution from all those electrons that do not collide between t and t + dt to the momentum per electron would be; can be calculated. If we neglect the contribution from the electrons that undergo a collision, then we can write the p(t + dt); we are considering only those electrons that survived without a collision, those went through a collision we are not considering here. It can be written as the probability of surviving without a collision, that is $(1 - \frac{dt}{\tau})[\overrightarrow{p}(t) + \overrightarrow{f}(t) + O(dt^2)]$. Now, something of the Odt^2 squared would be small and the correction due to electrons that went through a collision within this time that would also be of this order. So, if we consider those electrons also our calculation does not change, this remains the expression. Now, if this remains the expression then the number of electrons colliding in this time that is $\propto \frac{dt}{\tau}$. (Refer Slide Time: 30:55)

So, the change in momentum that would be $\propto \overrightarrow{f}dt$. If this is the situation, the number of electrons is proportional to this number, the change in momentum is proportional to this number; then the contribution from this, the number of electrons and the change that they make that would be a product of these two, which is of the Odt^2. So, we can absorb this term, that is the contribution from the electrons that went through a collision into this term itself. So, this happens to be the change in the momentum within certain order of magnitude. Now, with this we can write that, the change in the momentum that is $\overrightarrow{p}(t+dt)-\overrightarrow{p}(t) = (\frac{dt}{\tau}\overrightarrow{p}(t)+\overrightarrow{f}(t)dt+Odt^2$. Now, if we divide this expression by dt and put the $dt \rightarrow 0$, so dt is very small. At this kind of a situation, the left hand side is just the time derivative of the momentum; so the left hand side can be written as $\frac{d\overrightarrow{p}(t)}{dt} = -\frac{\overrightarrow{p}(t)}{\tau} + \overrightarrow{f}(t)$. So, what we have learnt from here, we can see that this equation simply tells us that the effect of individual electron collision is to introduce a functional damping term into the equation of motion; the time derivative of momentum according to Newton's law, Newton's equation of motion was proportional to the force. And now we have found in addition to being proportion, in additional addition to this force term; we also have a $-\frac{\overrightarrow{p}(t)}{\tau}$ kind of a term, this is the damping. And this damping term comes, because we are considering collisions of the electrons; they collide with other electrons as well as the ionic core regions. And so, the collision brings in this damping term, this is our damping term coming due to the collision. Now, this equation deviates from the Newton's law; that means you can clearly see that because of the collision, Newton's law overall is no longer obeyed. Although between two collision Drude has assumed it to be valid, Newton's law is valid; but if you consider the effect of collisions, Newton's law is no longer valid.