Statistical Mechanics
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Lecture - 65
Ultra Relativistic Bose Gas Stefan Boltzmann Law

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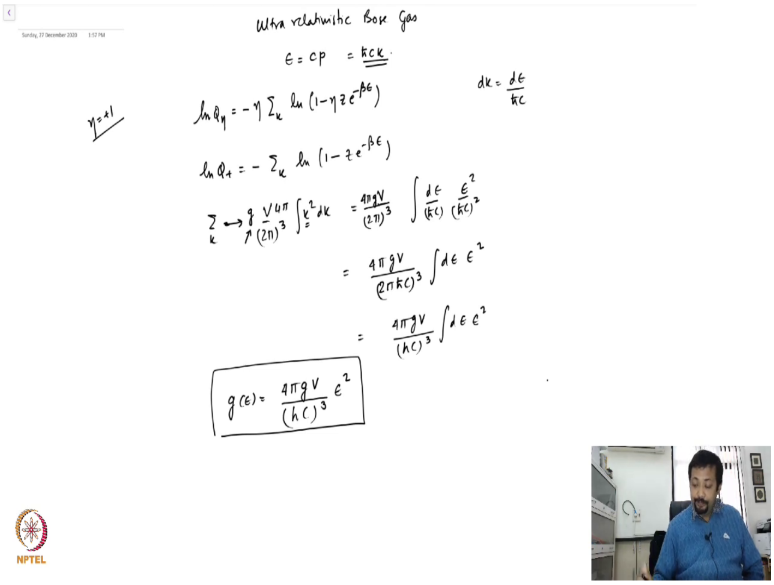

So, now, that we have looked at an ideal Bose gas confined Bose gas, we will take up the final example in this system. So, what we want to look at is the Ultra Relativistic Bose Gas where my epsilon the energy is given by C times p, which is h bar C k. And I start with the partition function ln Q eta; the general expression was minus eta sum over k ln 1 minus eta z e to the power minus beta epsilon.

Now, here eta is plus 1, eta is equal to plus 1 for a bosons. So, I have Q plus is going to be minus sum over k ln 1 minus Z e to the power minus beta epsilon. The sum over k as usual is converted to this density of state, as V over twice by whole cube integral k square over dk in three dimension and since I have this expression, I can write down dk, as d epsilon over h bar c.

So, that I have V over there has to be a gV here and a 4 pi over here. We will substitute the value of g a little later, let us keep it 4 pi times gV divided by 2 pi whole cube and then I have integration of d epsa over h bar c. And I have h bar C Square in terms of replacing k sorry, this has to be epsa square over h bar C whole square.

So, that I have 4 pi gV divided by 2 pi h bar C whole cube integration d epsa epsa square, which gives me 4 pi gV divided by hC whole cube integration d epsa epsa square. So, that the density of state, we will straight forward write down as 4 pi gV hC whole cube epsa square.

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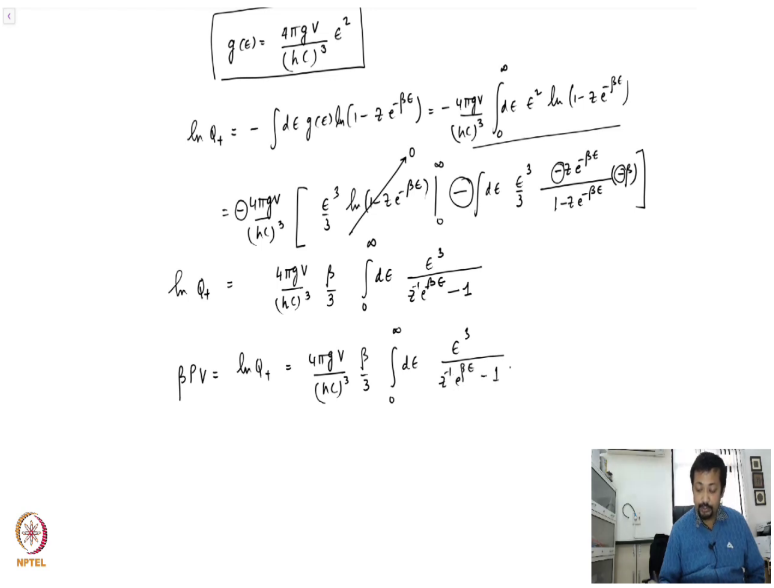

Now, this means, that ln of Q plus is going to be minus integral, let us write down the factors ok. So, d epsilon, g epsilon 1 minus Z e to the power minus beta epsilon log of this, which is going to be minus 4 pi gV over hC whole cube d epsa epsa square ln 1 minus Z e to the power minus beta epsa.

We carried out these integrals, the way to go ahead is to use integration by paths and we will do it over here also, this is epsa cube by 3 ln of 1 minus Z e to the power minus beta epsa, 0 to infinity this is 0 to infinity minus; I am going to have integration d epsa epsa cube by 3 derivative of this log is going to give me Z e to the power minus beta epsa minus Z e to the power minus beta epsa.

And then d d epsa of the minus beta epsa, which is going to give me a minus beta factor. So, that this minus and this minus is going to make it a plus. And this minus and this minus is going to make it a plus. And I know that this first term is going to vanish, I am going to have 4 pi gV hC whole cube beta no; yeah, beta over 3 integral 0 to infinity d epsa epsa cube Z inverse e to the power beta epsa minus 1.

This is ln of Q plus and I know that beta PV is going to be ln of Q plus, which is going to be 4 pi gV hC whole cube beta over 3, 0 to infinity d epsilon epsa cube Z inverse e to the power beta epsa minus 1.

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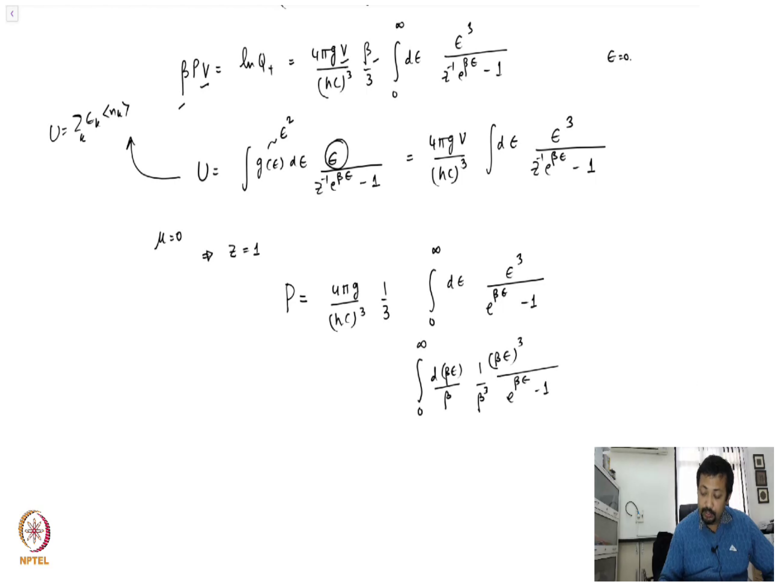

The total energy is going to be g epsa, d epsa and epsilon divided by Z inverse e to the power beta epsa minus 1. We have seen it several times, because this follows from the relation the total energy is sum over k epsa k n k. This gives me 4 pi gV hC whole cube. Integration d epsa; g epsa behaves as epsa square and I have an additional epsa factor here so, this gives me epsa cube Z inverse e to the power beta epsa minus 1.

Now, if you turn your attention to photons, it is something is very very special. So, what you see over here, that for low excitations like; photons the chemical potential mu is equal to 0, because it does not cost you energy to create particles at epsa equal to 0 right. So, once mu is equal to 0 this implies that Z is equal to 1.

So, essentially, where the thermodynamic pressure becomes 4 pi g; the volume factor cancels over here, the beta factor cancels over here, hC whole cube one-third 0 to infinity d epsa epsa cube e to the power beta epsa minus 1. And I can substitute for this 0 to infinity d of beta epsa divided by beta beta epsa whole cube 1 over beta cube e to the power beta epsa minus 1.

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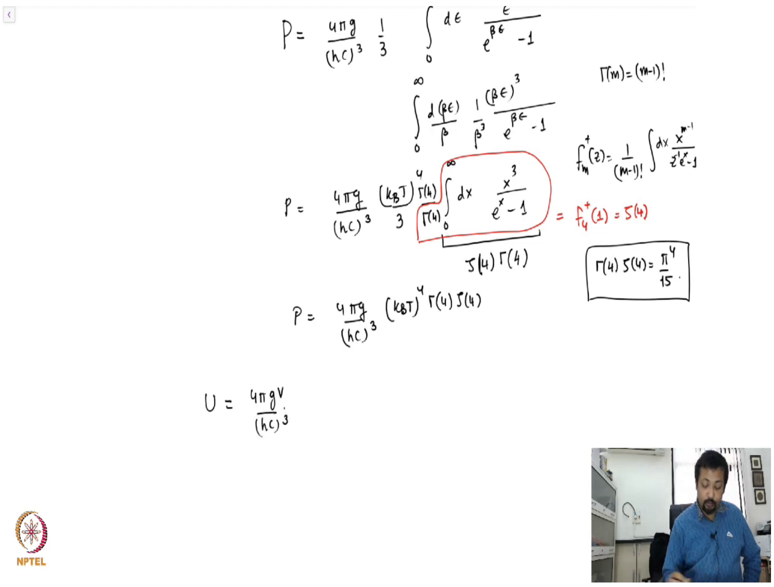

So, that the thermodynamic pressure I can write down; 4 pi g over hC whole cube k B T raised to the power 4 by 3, 0 to infinity dx x cube over e to the power x minus 1. Now, the value of this integral I know. The value of this integral is zeta 4 times gamma 3 am I right, sorry this has to be gamma 4.

Because I know, from our earlier discussion that F m plus of Z is 1 over M minus 1 factorial dx x to the power m minus 1 z inverse e to the power x minus 1. And gamma of m is going to be m minus 1 factorial. So, essentially I multiply this by gamma m sorry, gamma 4 and divided by gamma 4 and I realize, that this quantity that I see over here is equivalent to F 4 plus of 1, which is zeta times 4.

So, that the thermodynamic pressure is 4 pi g hC whole cube k B T raised to the power 4 gamma 4 zeta 4. You should know, that gamma 4 times zeta 4 is pi 4 over 15. We will later on use this one. Now, the internal energy, it follows is 4 pi gV hC whole cube. Once again if you look at the structure of this internal energy I am going to have 0 to infinity.

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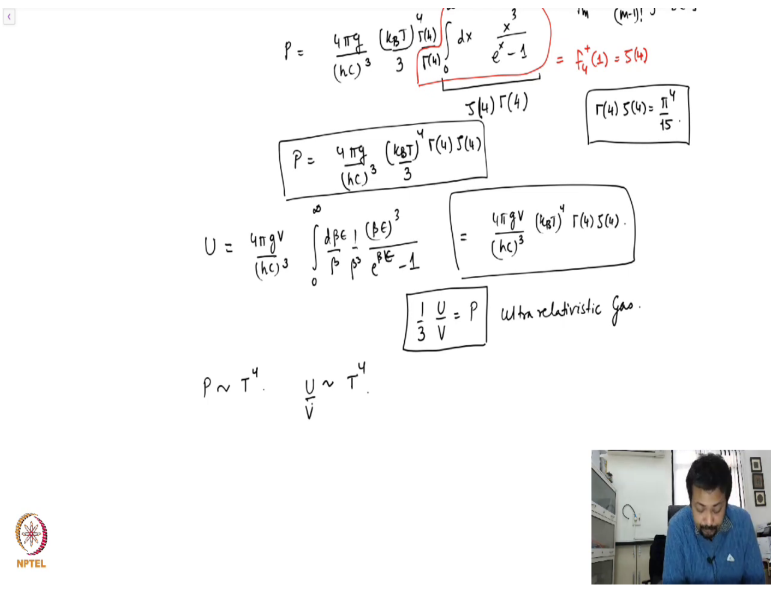

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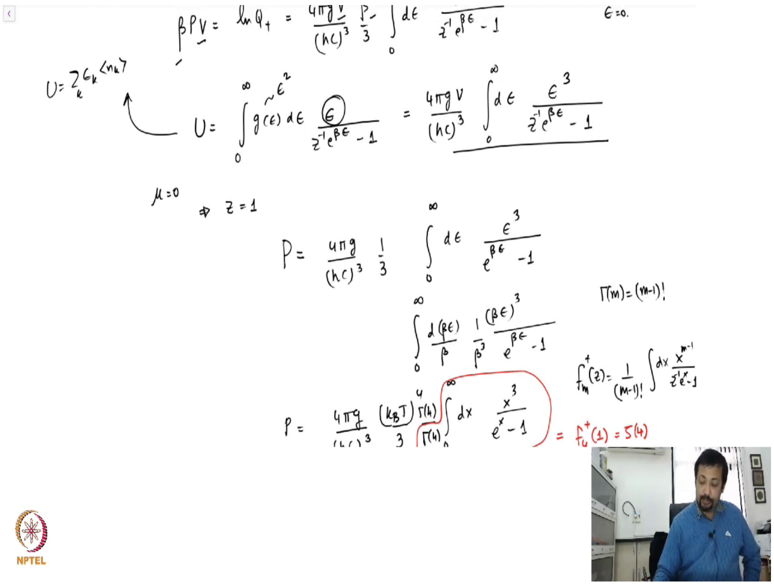

So, I have always missed out, this integral limits. So, please note that, they always run from 0 to infinity; d of beta epsa the standard trick is apply over here, 1 over beta cube e to the power beta h epsilon minus 1 and that is going to be 4 pi gV over hC whole cube k B T raised to the power 4 again it is going to give you gamma 4 zeta 4.

So, if you compare this expression over here and the last expression over here, you see that; U by V I have missed out a factor 3 over here. One-third of U by V is the pressure, which is the case for an ultra relativistic gas, which we have done several times in this course. Additionally, you must note that the pressure goes as T to the power 4 and U goes as T to the power 4 in this case. Now, that I know the energy density even I can write down U by V, the energy density goes as T to the power 4.

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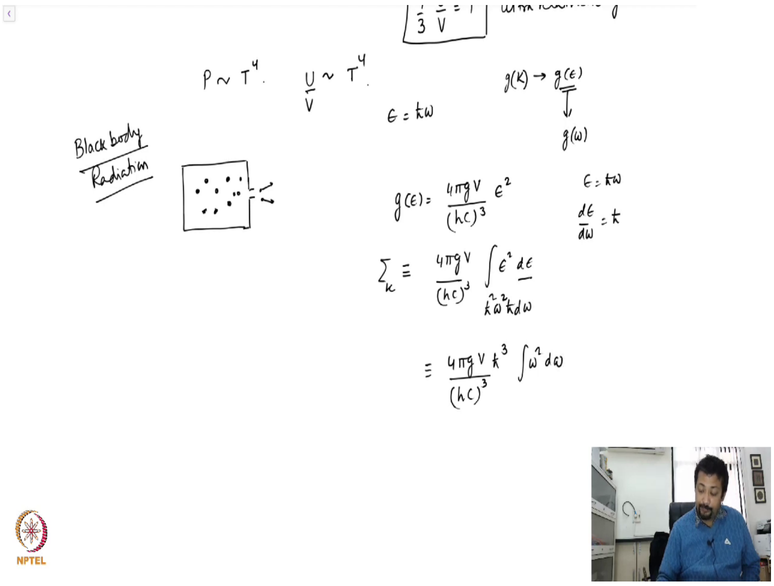

Now, suppose I have a container, which has a tiny tiny hole and this is filled with photons. So, that these photons come out from this hole. So, essentially what I am trying to look at is; what is called a black body radiation right. For this, I want to write down epsilon as h bar omega.

So, I started off with K, I went to g epsilon. So, g of K the density of states and from this, I want to go to g of omega. So, g of epsilon was 4 pi gV divided by hC whole cube and then, I had epsa square. So, that this sum over K, I had it was equivalent to 4 pi gV over hC whole cube integration epsa square d epsa, do not forget that.

Now, epsa is h bar omega and therefore, d epsa d omega is going to be h bar. So, this becomes h bar d omega and this becomes h bar square omega square. So, that this is; 4 pi gV over hC whole cube times h bar cube omega square d omega.

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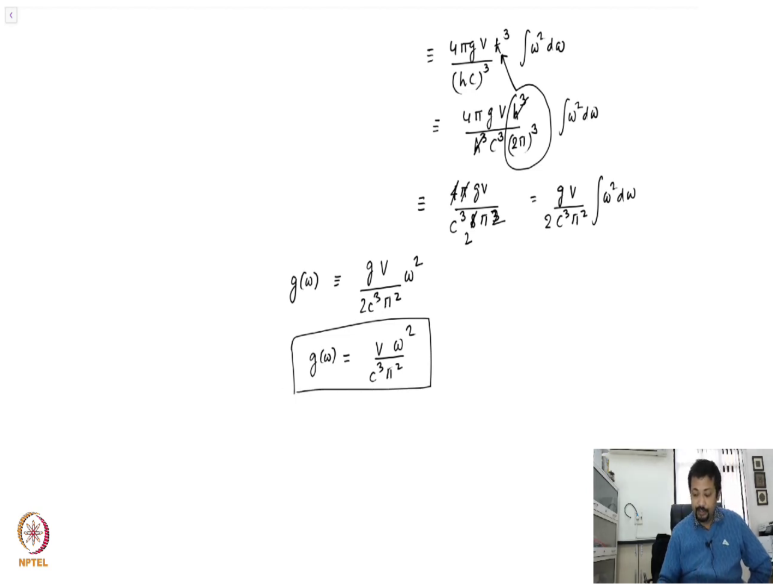

Which is equivalent to 4 pi gV divided by; I have h cube, h cube C cube and 2 pi whole cube. This is the part, which follows from h bar. So, that and then I have omega square d omega. This becomes 4 pi gV, the h cube h cube cancels out, C cube 8 pi cube; this is 2 and I have pi square. So, I have gV over 2 C cube pi square integration omega square d omega.

So, that the density of state in the frequency spectrum is equivalent to gV over 2 C cube pi square, omega square. Now, for a photon the degeneracy factor is 2. So, that I can write down this as; v omega square over C cube pi square a very nice expression. For this density of state right.

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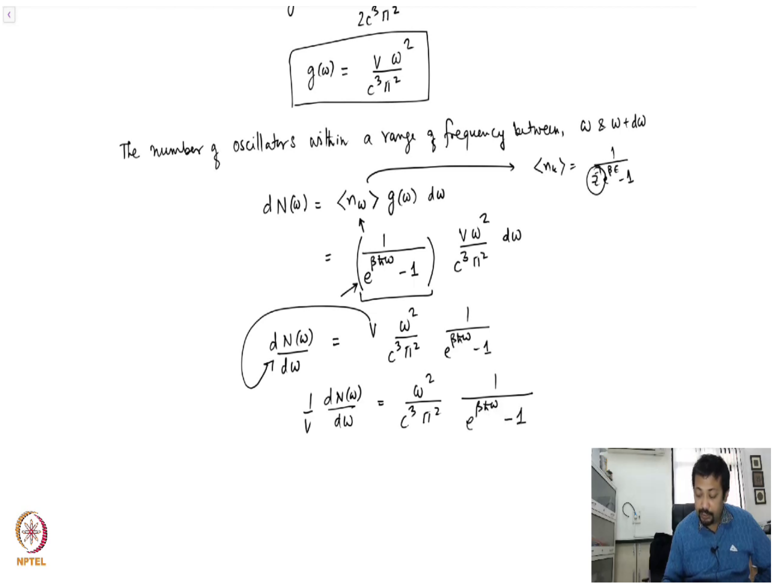

Therefore, it follows that a number of oscillators within a range of frequency between omega and omega plus d omega is going to be d of N omega, which is going to be average of n omega, g omega, d omega. And average of n omega, I know is e to the power beta h bar omega minus 1, because Z is 1; g omega is V omega square C cube pi square times d omega.

So, one has d N omega d omega as v omega square over C cube pi square 1 over beta h bar omega minus 1. This part the 1 in the bracket essentially follows from this average of n omega, which I know average of n k is 1 over Z inverse e to the power beta epsa minus 1 and Z is 1 for a photon gas, epsilon is h bar omega so, I have this expression right.

Now, I can bring the volume in the denominator. So, that I have 1 by V dN omega d omega is going to be omega square C cube, pi square 1 over e to the power beta h omega minus 1 correct.

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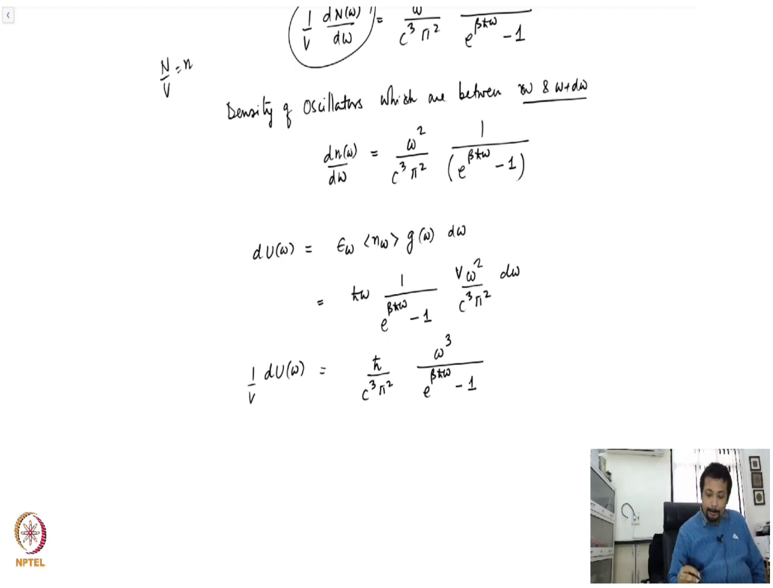

Let us see, then look at the left hand side. N by V is the number density small n. So, therefore, the density of oscillators, which are between so, density of oscillators, which are between omega and omega plus d omega, which has frequencies between omega and omega plus d omega is dn omega d omega is given by this expression omega square C cube pi square 1 over e to the power beta h bar omega. Good.

Now, I want to calculate the amount of energy that is contained within this frequency interval. So, now, I want to know the energy density contained within this frequency interval omega and omega plus d omega. So, one can also write down dU of omega, which is that amount of energy, which is contained within that is going to be; epsa omega, average of n omega, g omega and d omega.

Epsa omega is h bar omega and n omega is 1 over e to the power beta h bar omega minus 1 and then, you have g omega, which is omega square C cube pi square d omega. There is a V factor over here omega square V. So, the energy density; so, 1 by V dU of omega is going to be h bar C cube, pi square omega cube divided by e to the power beta h bar omega minus 1.

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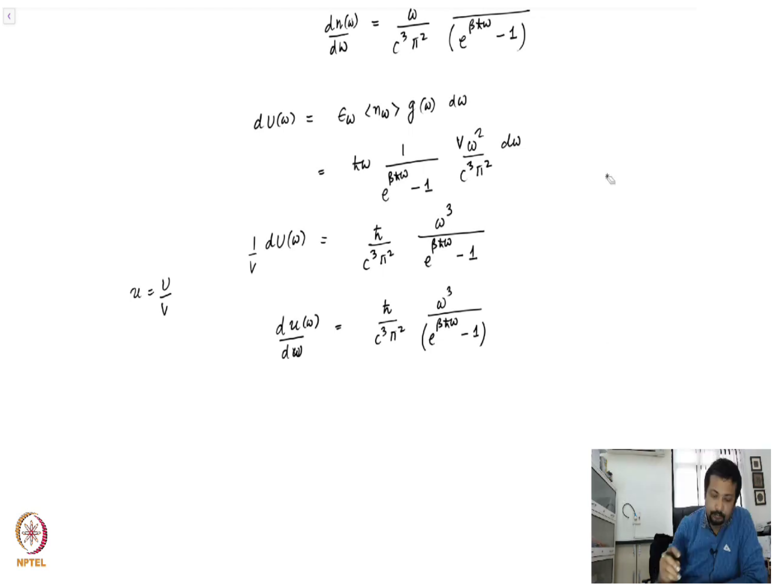

So, clearly this means; that the energy density small u of omega, where small u is defined as capital U divided by V the density, divided by d omega is going to be h bar C cube, pi square, omega cube e to the power beta h bar of omega minus 1. Now, one has to go back to the original problem and look at this; that I have a box, which is filled with photons and I have a tiny tiny hole here from which the photons emerge.

So, I am going to want to know the flux, and the flux; if these photons move with the velocity C, the flux is going to be n over one-fourth n times C right where it is understood; that this is going to be n omega times. So, now this is the energy density. So, if you want to ask, how what is the amount of energy which is contained within a frequency interval omega and omega plus d omega; the answer would be du d omega times d omega.

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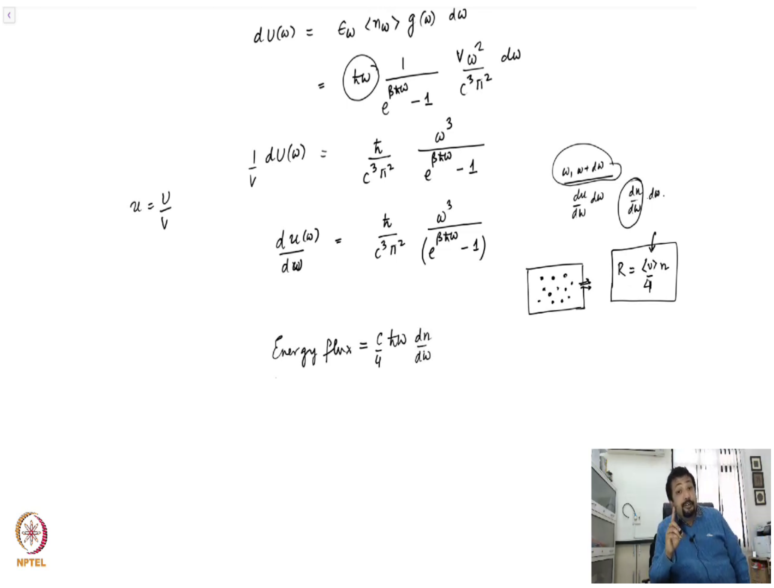

And similarly, the number of oscillators you have within the frequency interval omega and omega plus d omega, you have dn d omega times d omega. Now, the original question that we wanted to ask was that; we have this problem where we have a box and there is a small hole from which photons are coming out, right.

So, this is the radiation black body radiation, which you are looking at. So, I want to know the energy flux correct. So, here this is the classical problem of effusion and the answer to that is the number density, the number flux is going to be average V divided by 4 times n. This is for the classical result.

In our case, we adopt this relation and identify that we are looking at the frequency interval, omega and omega plus d omega. And therefore, we have number of oscillators, which is dn d omega and each of these oscillators carry an energy h bar omega, which we have already identified.

Therefore the energy flux du d omega; so, we will write down this as, the energy flux that is coming out of this small tiny hole is going to be h bar omega sorry, the average velocity is the velocity of light, which is C by 4 and I have dn d omega. This is per frequency interval right.

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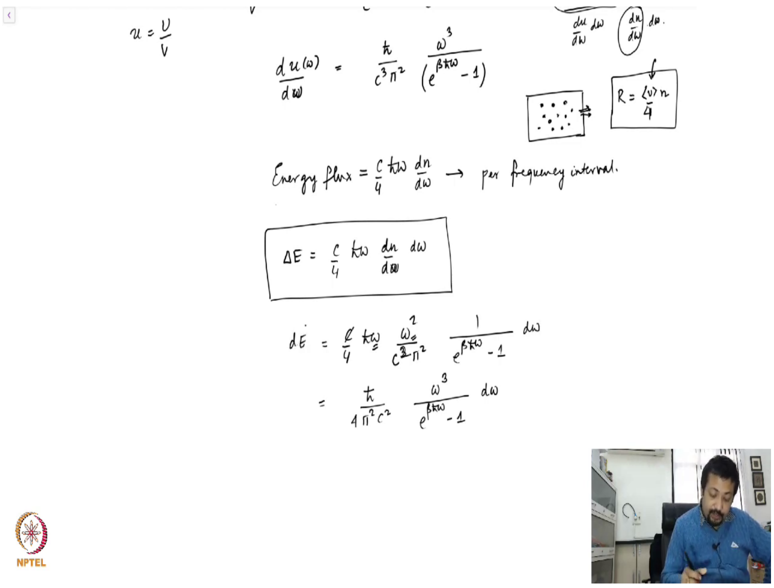

So, this is per frequency interval. Once again, if you want to know how much energy is coming out within this; so, let us say delta E is going to be C by 4, h bar by omega dn d omega times d omega.

This is the amount of energy, which is coming out within a frequency interval of omega and omega plus d omega which we write down as C by 4, h bar omega and then I have dn d omega, which we had before which was this expression becomes omega square C cube and I have pi square 1 over e to the power beta h bar omega minus 1 and this gives me omega cube over ok; so, first we write down as h bar, h bar 1 factor of C cancels with 1 factor also t give you C square.

So, you have 4 pi square C square and then you have omega cube, this omega and this omega square here and then you have e to the power beta h bar omega minus 1. So, if you were to measure the amount of energy contained, which is coming out of this tiny pore within a frequency interval of omega and omega plus d omega, then you will see that this is the 2 amount delta E times d omega.

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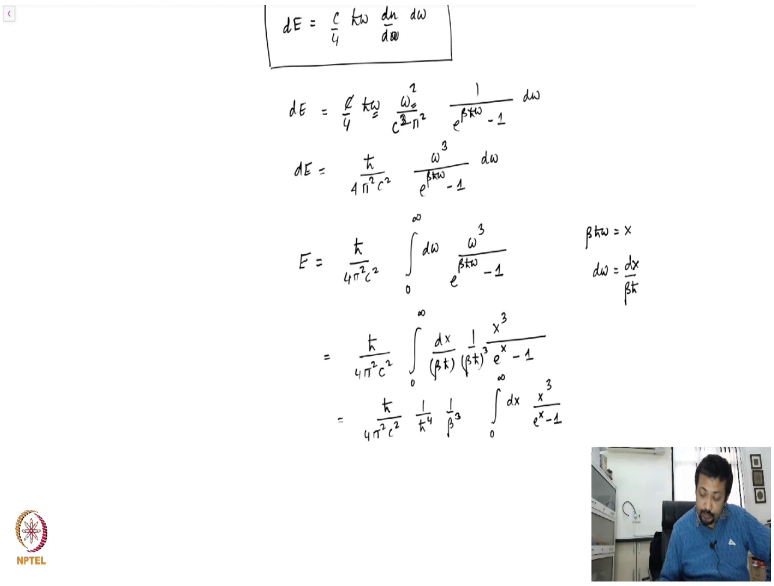

So, we will write down dE instead of writing a delta, we will write down dE. Hence the total energy throughout the whole spectrum is going to be E, which is going to be h bar 4 pi square, C square 0 to infinity d omega omega cube e to the power beta h bar omega minus 1.

If you substitute beta h bar omega is equal to y or let us say x, this would be a preferred choice then you see, d omega is going to be dx over beta h bar. So, that you have h bar over 4 pi square, C square integration 0 to infinity dx over beta h bar and you have x cube over e to the power x minus 1, 1 over beta h bar whole cube which gives you h bar 4 pi square, C square 1 over h bar raised to the power 4; you will have beta cube, 0 to infinity dx x cube e to the power x minus 1. This is once again familiar to us.

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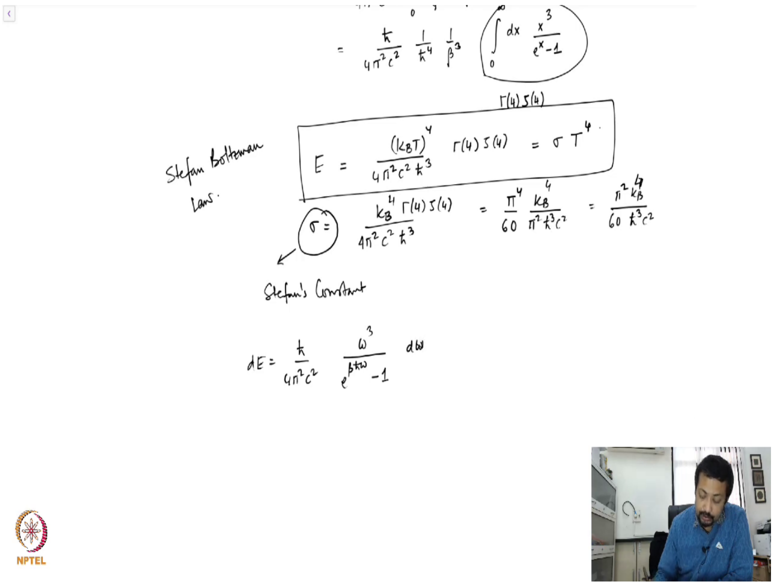

This is gamma 4 zeta 4. So, that you have this expression, the total energy, which is radiating out of this hole is going to be 1 over 4 pi square C square h bar cube k B T raised to the power 4, gamma 4, zeta 4.

And this is sigma T to the power 4, where sigma is going to be k B raised to the power 4 divided by 4 pi square, C square, h bar cube times gamma 4 zeta 4. which is going to be pi 4 over 60 and then you have k B raised to the power 4 pi square, h bar C whole cube so, that you have pi square k B raised to the power 4 divided by 60 h bar C whole no, sorry, this is wrong here. So, this is going to be h bar cube C square, h bar cube C square.

This law that you see over here is called the Stefan Boltzmann law. And this constant is called Stefan's constant. One can plug in the values of k B h bar and C to determine its value. As a concluding remark, we come back to this expression, we come back to this expression that we have over here. And we look at; so, we have dE, which is going to be h bar over 4 pi square, C square and then I have omega cube e to the power beta h bar omega minus 1 d omega.

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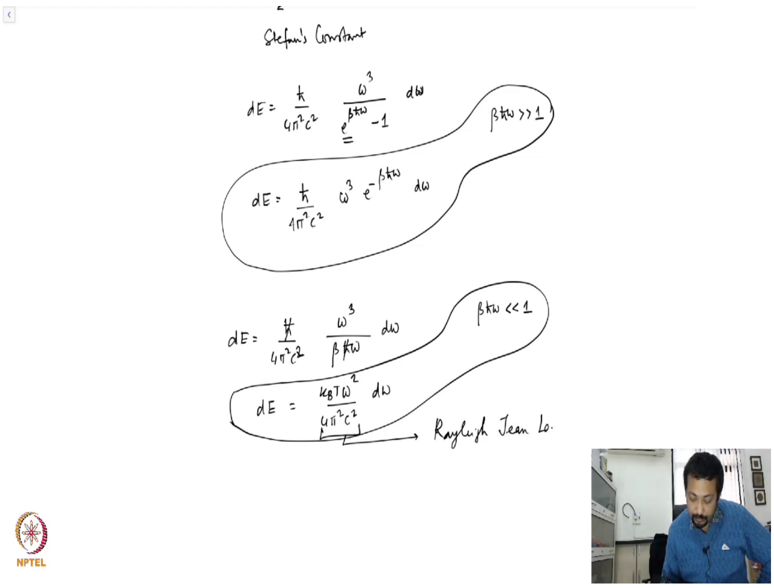

Now, if beta h bar omega is much much larger than 1, we have this as dE, which is going to be h bar over 4 pi square C square omega cube e to the power minus beta h bar omega d omega. In the opposite limit, when beta h bar omega is much much less than 1, I can expand the exponential over here and that gives me dE is going to be h bar over 4 pi square, C square omega cube divided by beta h bar omega.

To the leading order; the h bar h bar cancels out and of course, you have a d omega over here, which is going to be omega square over 4 pi square C square times k B T d omega dE. And this is the expression that you get, when beta h bar omega is going to be much much less than

1. This is what is called a, what is known as the Rayleigh Jeans law and this is called Wien’s law in radiation. So, with this we conclude this section.